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Research Article

Sheffer and Non-Sheffer Polynomial Families

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By using the integral transform method, we introduce some non-Sheffer polynomial sets. Furthermore, we show how to compute the connection coefficients for particular expressions of Appell polynomials.

1. Introduction

The Sheffer polynomials are defined through the generating function [1] as follows:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} s_n(x) = A(t) e^{xB(t)}, \tag{1.1}$$

where $A(t)$ and $B(t)$ are two analytic functions of the parameter t.

The above families of polynomials have been shown to be quasimonomials under the action of the operators:

$$\begin{aligned} \widehat{P} &= B^{-1}(\partial_x), \\ \widehat{M} &= xB'(\partial_x) + \frac{A'(\partial_x)}{A(\partial_x)}, \end{aligned} \tag{1.2}$$

indeed, being $\widehat{D}_x$ the ordinary derivative, we easily prove that

$$\begin{aligned}\widehat{P}s_n(x) &= ns_{n-1}(x),\\ \widehat{M}s_n(x) &= s_{n+1}(x).\end{aligned} \tag{1.3}$$

The polynomials, $\sigma_{n,\alpha}(x)$, specified by the generating function:

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\sigma_{n,\alpha}(x) = \frac{A(t)}{[1-xB(t)]^{\alpha}} \tag{1.4}$$

do not belong to the Sheffer family. We can, however, take advantage from the identity:

$$a^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} e^{-\xi a}\xi^{\alpha-1}d\xi \tag{1.5}$$

to prove that they are linked to the $s_n(x)$ by the integral transform:

$$\sigma_{n,\alpha}(x) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} e^{-\xi}s_n(\xi x)\ \xi^{\alpha-1}d\xi. \tag{1.6}$$

Before discussing the problem in general terms, we consider the case in which

$$\begin{aligned}A(t) &= e^{yt^2},\\ B(t) &= t.\end{aligned} \tag{1.7}$$

The use of the previous procedure yields

$$\eta_{n,\alpha}(x,y) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} e^{-\xi}H_n(\xi x,y)\xi^{\alpha-1}d\xi. \tag{1.8}$$

Being the Hermite-Kampé de Fériet polynomials defined by

$$H_n(x,y) = n!\sum_{r=0}^{[n/2]}\frac{x^{n-2r}y^r}{(n-2r)!r!}, \tag{1.9}$$

we find the explicit expression:

$$\eta_{n,\alpha}(x,y) = \frac{n!}{\Gamma(\alpha)}\sum_{r=0}^{[n/2]}\frac{\Gamma(n-2r+\alpha)\ x^{n-2r}y^r}{(n-2r)!r!}. \tag{1.10}$$

By noting that the monomiality operators associated with the $H_n(x,y)$ are simply given by [2] as

$$\widehat{P} = \partial_x, \qquad \widehat{M} = x + 2y\partial_x, \tag{1.11}$$

we obtain the following recurrences:

$$\begin{aligned} \partial_x\ \eta_{n,\alpha}(x,y) &= \alpha\ n\ \eta_{n-1,\alpha+1}(x,y), \\ \partial_y\ \eta_{n,\alpha}(x,y) &= n(n-1)\eta_{n-2,\alpha}(x,y), \\ \eta_{n+1,\alpha}(x,y) &= \alpha\ x\ \eta_{n,\alpha+1}(x,y) + 2n\ y\ \eta_{n-1,\alpha}(x,y). \end{aligned} \tag{1.12}$$

We can further extend the previous definition by adding an extra variable such that

$$\begin{aligned} \eta_{n,\alpha}(x,y \mid \beta) &= \frac{1}{\Gamma(\alpha)}\int_0^\infty e^{-\beta\xi}H_n(\xi\ x,y)\xi^{\alpha-1}d\xi \\ &= \frac{n!}{\Gamma(\alpha)\beta^\alpha}\sum_{r=0}^{[n/2]}\frac{\Gamma(n-2r+\alpha)(x/\beta)^{n-2r}y^r}{(n-2r)!\ r!}, \end{aligned} \tag{1.13}$$

thus finding that

$$\partial_y\partial_\beta^2\eta_{n,\alpha}(x,y \mid \beta) = \partial_x^2\eta_{n,\alpha}(x,y \mid \beta). \tag{1.14}$$

According to the previous example, we have introduced anew family of polynomials, with nontrivial properties, and applications just starting from the corresponding Sheffer family and by exploiting the wealth of properties that such a family possesses.

In the second part of this paper we show how to construct particular connection coefficients relevant to several Sheffer polynomial sets, including multivariable Hermite, Legendre, and Laguerre polynomials, and, in general, Appell-type polynomial sets.

2. Further Non-Sheffer Polynomial Sets

It is quite straightforward to see that, inasmuch $B(t) = t$, the Sheffer polynomials are essentially Appell polynomials [3], since it always happens that the monomial operator $\widehat{P}$ coincides with the ordinary derivative and therefore the associated polynomials satisfy the recurrence reported in the first of equations (1.12). In the more general case, the situation is more interesting. For example, if

$$A(t) = 1, \qquad B(t) = e^t - 1, \tag{2.1}$$

the monomiality operators, associated with the relevant Sheffer form, namely, the Bell polynomials $be_n(x)$, are

$$\hat{P} = \ln(1+\partial_x),$$
$$\widehat{M} = x\ (1+\partial_x), \tag{2.2}$$
$$be_n(x) = \sum_{k=1}^{n} S_2(n,k)\ x^k,$$

with $S_2(n,k)$ being the Stirling numbers of second kind.

The corresponding $\beta_{n,\alpha}(x)$ polynomials are therefore

$$\beta_{n,\alpha}(x) = \frac{1}{\Gamma(\alpha)} \sum_{k=1}^{n} S_2(n,k)\Gamma(k+\alpha)\ x^k, \tag{2.3}$$

and a recurrence satisfied by the above family is given by

$$\beta_{n+1,\alpha}(x) = x\ \alpha\ \beta_{n,\alpha+1}(x) + x\ \partial_x \beta_{n,\alpha}(x). \tag{2.4}$$

The Lagrange polynomials are characterized by the generating function (strictly speaking the Lagrange polynomials, as currently defined in the literature, are $g_{n,\alpha,\beta}(x,y) = (1/n!)\lambda_n^{(\alpha,\beta)}(x,y)$) in [4]:

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\ \lambda_n^{(\alpha,\beta)}(x,y) = \frac{1}{[1-xt]^{\alpha}[1-yt]^{\beta}}. \tag{2.5}$$

They belong therefore to the family (1.4) with

$$A(t) = \frac{1}{[1-yt]^{\beta}},$$
$$B(t) = t. \tag{2.6}$$

It is, therefore, evident that the technique we have proposed allows a general tool to frame the theory of polynomial sets in a straightforward context, which permits a natural understanding of their properties.

3. Special Connection Coefficients for Hermite, Gould-Hopper, and Laguerre Polynomials

In the previous section, we have discussed different families of polynomials which can be framed within a common thread. The point to be clarified is whether they can be exploited to obtain convenient expansions.

We introduce the topics we will discuss in the second part of this paper, by considering the following expansion, involving the two variable Hermite polynomials $H_n(x,y)$:

$$H_n(px,qy)=\sum_{m=0}^{n}h_m(n,p,q,y)H_m(x,y). \tag{3.1}$$

The problem to be solved is that of deriving the coefficients $h_m(n,p,q,y)$ of this expansion.

The use of the identities:

$$\begin{aligned}e^{y\partial_x^2}x^n&=H_n(x,y),\\ e^{z\partial_x^2}H_n(x,y)&=H_n(x,y+z)\end{aligned} \tag{3.2}$$

allows to recast (3.1) in the following form:

$$H_n\left(px,\left(q-p^2\right)y\right)=\sum_{m=0}^{n}h_m(n,p,q,y)\ x^m. \tag{3.3}$$

Let $x=e^{i\vartheta}$ so that we find from (3.3):

$$\begin{aligned}h_m(n,p,q,y)&=\frac{1}{2\pi}\int_0^{2\pi}e^{-im\vartheta}H_n\left(pe^{i\vartheta},\left(q-p^2\right)y\right)\,d\vartheta\\ &=n!\,\frac{p^m\left[\left(q-p^2\right)y\right]^{(n-m)/2}}{m!\,((n-m)/2)!},\end{aligned} \tag{3.4}$$

where $n-m\equiv$ even.

The whole procedure is based on the possibility of connecting different forms of Hermite polynomials using an exponential differential operator. According to the above result, we can consider now the Gould-Hopper polynomials in [2, 5, 6]:

$$\begin{aligned}e^{y\partial_x^d}x^n&=H_n^{(d)}(x,y),\\ H_n^{(d)}(x,y)&=n!\sum_{r=0}^{[n/d]}\frac{x^{n-dr}y^r}{(n-dr)!\,r!}.\end{aligned} \tag{3.5}$$

In this case, for the connection coefficients $h_m^{(d)}(n,p,q,y)$, such that

$$H_n^{(d)}(px,qy)=\sum_{m=0}^{n}h_m^{(d)}(n,p,q,y)H_m^{(d)}(x,y), \tag{3.6}$$

we obtain the expression

$$h_m^{(d)}(n,p,q,y) = n!\,\frac{p^m\left[\left(q-p^d\right)y\right]^{(n-m)/d}}{m!\,((n-m)/d)!}, \tag{3.7}$$

where $n-m \equiv dk$, $k \equiv$ integer.

In the following, we will discuss a general procedure, based on ideas analogous to that employed so far, allowing the derivation of the connection coefficients of the type (3.1) involving different forms of polynomials.

The two variable Laguerre polynomials are defined as

$$\begin{aligned}
L_n(x,y) &= n!\sum_{r=0}^{n}\frac{(-1)^r x^r y^{n-r}}{(r!)^2(n-r)!},\\
L_n(x,y) &= e^{y\ {}_L\widehat{D}_x}\left[\frac{(-x)^r}{r!}\right],\\
e^{z_L\widehat{D}_x}L_n(x,y) &= L_n(x,y+z),\\
{}_L\widehat{D}_x &:= -\partial_x x\partial_x,
\end{aligned} \tag{3.8}$$

where ${}_L\widehat{D}_x$ denotes the Laguerre derivative.

We can, therefore, use the same procedure outlined for the Hermite polynomials to derive the coefficients of the expansion:

$$L_n(px,qy) = \sum_{m=0}^{n}\ell_m(n,p,q,y)\ L_m(x,y), \tag{3.9}$$

by noting that, on account of (3.8), the following identity holds

$$e^{-y\ {}_L\widehat{D}_x}L_n(px,qy) = L_n(px,(q-p)y) = \sum_{m=0}^{n}\ell_m(n,p,q,y)\left[\frac{(-x)^m}{m!}\right], \tag{3.10}$$

which yields

$$\ell_m(n,p,q,y) = \binom{n}{m}\frac{(-p)^m\left[(q-p)y\right]^{n-m}}{m!}. \tag{3.11}$$

In the concluding section, we will discuss alternative derivations of (3.11).

The previous identities have been proposed as an example aimed at proving the reliability of the proposed method, which is further developed in the last section.

4. The Case of Legendre and Sheffer Polynomials

The Legendre polynomials can be written in terms of Hermite polynomials according to the identity from [7]:

$$P_n(x) = \frac{1}{n!\,\sqrt{\pi}} \int_0^{\infty} e^{-s} s^{-1/2} H_n(2sx, -s) ds, \tag{4.1}$$

and therefore they are essentially σ polynomials of the type (1.6). On account of (3.1) and of the fact that the ordinary Hermite polynomials are linked to their two variable counterpart by

$$H_n(2x, -1) = H_n(x), \tag{4.2}$$

can also be written as

$$\begin{gathered} P_n(x) = \frac{1}{n!\,\sqrt{\pi}} \sum_{r=0}^{n} p_r H_r(x), \\ p_r = \int_0^{\infty} e^{-s} s^{-1/2} h_r(n, s, s, -1)\ ds, \end{gathered} \tag{4.3}$$

the coefficients $h_r(n, s, s, -1)$ being defined by (3.4).

The above expansion holds for general forms of Appell polynomials:

$$\begin{gathered} a_n^{(+)}(px) = \sum_{m=0}^{n} C_m(n,p)\ a_m^{(+)}(x), \\ a_m^{(+)}(x) = A(\partial_x)\ x^m. \end{gathered} \tag{4.4}$$

In general, the above family of polynomials can be defined through the formal series:

$$a_n^{(+)}(x) = \sum_{r=0}^{n} \frac{\alpha_r^{(+)}}{(n-r)!}\ x^{n-r}. \tag{4.5}$$

Along with the "(+)" polynomials, we can define their "(−)" counterpart as

$$a_n^{(-)}(x) = [A(\partial_x)]^{-1}\ x^n. \tag{4.6}$$

Therefore, we get

$$[A(\partial_x)]^{-1}\ a_n^{(+)}(px) = \sum_{m=0}^{n} C_m(n,p)\ x^m. \tag{4.7}$$

By introducing the $\widehat{a}^{(-)}$ operator, defined by

$$\left[\widehat{a}^{(-)}(x)\right]^{n} = a_n^{(-)}(x), \tag{4.8}$$

and by noting that

$$[A(\partial_x)]^{-1} a_n^{(+)}(x) = a_n^{(+)}\left[\widehat{a}^{(-)}(x)\right] = \sum_{r=0}^{n} \frac{\alpha_r^{(+)}}{(n-r)!} \quad a_{n-r}^{(-)}(x), \tag{4.9}$$

we obtain

$$[A(\partial_x)]^{-1} a_n^{(+)}(px) = a_n^{(+)}\left(p\left[\widehat{a}^{(-)}(x)\right]\right), \tag{4.10}$$

and lastly

$$C_m(n,p) = \frac{1}{2\pi}\int_0^{2\pi} e^{-im\vartheta} a_n^{(+)}\left(p\left[\widehat{a}^{(-)}\left(e^{i\vartheta}\right)\right]\right) \, d\vartheta. \tag{4.11}$$

In a forthcoming more extended paper, we will reconsider the topics developed in this paper by studying new expansions of functions in terms of Sheffer and non-Sheffer polynomial families.

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