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Research Article

Tripled Coincidence Point Theorems for Nonlinear Contractions in Partially Ordered Metric Spaces

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Tripled fixed points are extensions of the idea of coupled fixed points introduced in a recent paper by Berinde and Borcut, 2011. Here using a separate methodology we extend this result to a triple coincidence point theorem in partially ordered metric spaces. We have defined several concepts pertaining to our results. The main results have several corollaries and an illustrative example. The example shows that the extension proved here is actual and also the main theorem properly contains all its corollaries.

1. Introduction and Preliminaries

In recent times coupled fixed point theory has experienced a rapid growth in partially ordered metric spaces. The speciality of this line of research is that the problems herein utilize both order theoretic and analytic methods. References [1–19] are some instances of these works.

Definition 1.1 (see [14]). A function $g : R \to R$ is said to be monotone nondecreasing (or increasing) if $x \geq y$ implies $g(x) \geq g(y)$.

Definition 1.2 (see [14]). Let X be a nonempty set. Let $F : X \times X \to X$ be a mapping. An element (x, y) is called a coupled fixed point of F if

$$F(x, y) = x, \qquad F(y, x) = y. \tag{1.1}$$

Recently, Berinde and Borcut [20] extended the idea of coupled fixed points to tripled fixed points. The definition is as follows.

Definition 1.3 (see [20]). Let X be a nonempty set. Let $F : X \times X \times X \to X$ be a mapping. An element (x, y, z) is called a tripled fixed point of F if

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z. \tag{1.2}$$

They also extended the mixed monotone property to functions with three arguments.

Definition 1.4 (see [20]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \times X \to X$. The mapping F is said to have the mixed monotone property if for any $x, y, z \in X$

$$\begin{aligned} x_1, x_2 \in X, \quad & x_1 \preceq x_2 \Longrightarrow F(x_1, y, z) \preceq F(x_2, y, z), \\ y_1, y_2 \in X, \quad & y_1 \preceq y_2 \Longrightarrow F(x, y_1, z) \succeq F(x, y_2, z), \\ z_1, z_2 \in X, \quad & z_1 \preceq z_2 \Longrightarrow F(x, y, z_1) \preceq F(x, y, z_2). \end{aligned} \tag{1.3}$$

Our purpose here is to establish tripled coincidence point results in metric spaces with partial ordering. For that purpose we define mixed g-monotone property in the following. Mixed g-monotone property was already defined in the context of coupled fixed points [14]. Here in the spirit of Definition 1.4 we have made an extension of that.

Definition 1.5. Let $(X, \preceq)$ be a partially ordered set. Let $g : X \to X$ and $F : X \times X \times X \to X$. The mapping F is said to have the mixed g-monotone property if for any $x, y, z \in X$.

$$\begin{aligned} x_1, x_2 \in X, \quad & gx_1 \preceq gx_2 \Longrightarrow F(x_1, y, z) \preceq F(x_2, y, z), \\ y_1, y_2 \in X, \quad & gy_1 \preceq gy_2 \Longrightarrow F(x, y_1, z) \succeq F(x, y_2, z), \\ z_1, z_2 \in X, \quad & gz_1 \preceq gz_2 \Longrightarrow F(x, y, z_1) \preceq F(x, y, z_2). \end{aligned} \tag{1.4}$$

Coupled coincidence point was defined by Lakshmikantham and Ćirić [14]. We also extend the concept of coupled coincidence point to tripled coincidence point in the following.

Definition 1.6. Let X be any nonempty set. Let $g : X \to X$ and $F : X \times X \times X \to X$. An element (x, y, z) is called a tripled coincidence point of g and F if

$$F(x, y, z) = gx, \qquad F(y, x, y) = gy, \qquad F(z, y, x) = gz. \tag{1.5}$$

We extend the concept of commuting mappings given by Lakshmikantham and Ćirić [14], in the following definition.

Definition 1.7. Let X be a nonempty set. Then one says that the mappings $g : X \to X$ and $F : X \times X \times X \to X$ are commuting if for all $x, y, z \in X$

$$g(F(x, y, z)) = F(gx, gy, gz). \tag{1.6}$$

The following is the definition of compatible mappings which is an extension of the compatibility defined by Choudhury and Kundu in [8].

Definition 1.8 (see [8]). Let (X, d) be a metric space. The mappings g and F, where $g : X \to X$ and $F : X \times X \times X \to X$ are said to be compatible if

$$\begin{aligned} &\lim_{n\to\infty} d(gF(x_n, y_n, z_n), F(gx_n, gy_n, gz_n)) = 0, \\ &\lim_{n\to\infty} d(gF(y_n, x_n, y_n), F(gy_n, gx_n, gy_n)) = 0, \\ &\lim_{n\to\infty} d(gF(z_n, y_n, x_n), F(gz_n, gy_n, gx_n)) = 0, \end{aligned} \tag{1.7}$$

whenever $\{x_n\}, \{y_n\}, \{z_n\}$ are sequences in X such that

$$\begin{aligned} &\lim_{n\to\infty} F(x_n, y_n, z_n) = gx_n = x, \\ &\lim_{n\to\infty} F(y_n, x_n, y_n) = gy_n = y, \\ &\lim_{n\to\infty} F(z_n, y_n, x_n) = gz_n = z. \end{aligned} \tag{1.8}$$

2. Main Results

Theorem 2.1. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose $F : X \times X \times X \to X$ and $g : X \to X$ are such that, g is monotone increasing, F has the mixed g-monotone property and*

$$d(F(x, y, z), F(u, v, w)) \le \psi(\max\{d(gx, gu), d(gy, gv), d(gz, gw)\}) \tag{2.1}$$

for all $x, y, z \in X$ for which $gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$, where $\psi : [0, +\infty) \to [0, +\infty)$ is such that $\psi(t)$ is monotone, $\psi(t) < t$ and $\lim_{r\to t^+}\psi(r) < t$ for all $t > 0$. Suppose $F(X \times X \times X) \subseteq g(X)$, g is continuous, and $\{g, F\}$ is a compatible pair. Suppose either

(a) *F is continuous or*

(b) *X has the following properties:*

(i) *if a nondecreasing sequence $\{\alpha_n\} \to \alpha$, then $\alpha_n \preceq \alpha$ for all n,*

(ii) *if a nonincreasing sequence $\{\beta_n\} \to \beta$, then $\beta_n \succeq \beta$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, y_0)$, and $gz_0 \preceq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = gx, \qquad F(y, x, y) = gy, \qquad F(z, y, x) = gz, \tag{2.2}$$

that is, g and F have a tripled coincidence point.

Proof. By a condition of the theorem, there exist $x_0, y_0, z_0 \in X$ such that $gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, y_0)$, and $gz_0 \preceq F(z_0, y_0, x_0)$. Since $F(X \times X \times X) \subseteq g(X)$, we can choose $x_1, y_1, z_1 \in X$ such that

$$gx_1 = F(x_0, y_0, z_0), \qquad gy_1 = F(y_0, x_0, y_0), \qquad gz_1 = F(z_0, y_0, x_0). \tag{2.3}$$

Continuing this process, we can construct sequences $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n, z_n), \qquad gy_{n+1} = F(y_n, x_n, y_n), \qquad gz_{n+1} = F(z_n, y_n, x_n). \tag{2.4}$$

Next we will show that, for $n \geq 0$,

$$gx_n \preceq gx_{n+1}, \qquad gy_n \succeq gy_{n+1}, \qquad gz_n \preceq gz_{n+1}. \tag{2.5}$$

Since, $gx_0 \preceq F(x_0, y_0, z_0)$, $gy_0 \succeq F(y_0, x_0, y_0)$, and $gz_0 \preceq F(z_0, y_0, x_0)$, by (2.3), we get

$$gx_0 \preceq gx_1, \qquad gy_0 \succeq gy_1, \qquad gz_0 \preceq gz_1, \tag{2.6}$$

that is, (2.5) holds for $n = 0$.

We presume that (2.5) holds for some $n = m > 0$. As F has the mixed g-monotone property and $gx_m \preceq gx_{m+1}$, $gy_m \succeq gy_{m+1}$ and $gz_m \preceq gz_{m+1}$, we obtain

$$\begin{aligned} gx_{m+1} &= F(x_m, y_m, z_m) \\ &\preceq F(x_{m+1}, y_m, z_m) \\ &\preceq F(x_{m+1}, y_m, z_{m+1}) \\ &\preceq F(x_{m+1}, y_{m+1}, z_{m+1}) = gx_{m+2}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} gy_{m+1} &= F(y_m, x_m, y_m) \\ &\succeq F(y_m, x_m, y_{m+1}) \\ &\succeq F(y_{m+1}, x_m, y_{m+1}) \\ &\succeq F(y_{m+1}, x_{m+1}, y_{m+1}) = gy_{m+2}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} gz_{m+1} &= F(z_m, y_m, x_m) \preceq F(z_{m+1}, y_m, x_m) \\ &\preceq F(z_{m+1}, y_{m+1}, x_m) \\ &\preceq F(z_{m+1}, y_{m+1}, x_{m+1}) = gz_{m+2}. \end{aligned} \tag{2.9}$$

Thus, (2.5) holds for $n = m + 1$. Then, by induction, we conclude that (2.5) holds for $n \geq 1$.

If for some $n \in \mathbb{N}$,

$$gx_n = gx_{n+1}, \qquad gy_n = gy_{n+1}, \qquad gz_n = gz_{n+1}, \tag{2.10}$$

then, by (2.4), (x_n, y_n, z_n) is a tripled coincidence point of g and F. Therefore we assume, for any $n \in \mathbb{N}$,

$$(gx_n, gy_n, gz_n) \neq (gx_{n+1}, gy_{n+1}, gz_{n+1}). \tag{2.11}$$

Set $\delta_n = \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\}$.

Then

$$\delta_n > 0 \quad \forall n \geq 0. \tag{2.12}$$

Then, by (2.1), (2.4) and (2.5), we have

$$\begin{aligned}
d(gx_n, gx_{n+1}) &= d(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\
&\leq \psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\}), \\
d(gy_n, gy_{n+1}) &= d(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n)) \\
&\leq \psi(\max\{d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n)\}), \\
d(gz_n, gz_{n+1}) &= d(F(z_{n-1}, y_{n-1}, x_{n-1}), F(z_n, y_n, x_n)) \\
&\leq \psi(\max\{d(gz_{n-1}, gz_n), d(gy_{n-1}, gy_n), d(gx_{n-1}, gx_n)\}).
\end{aligned} \tag{2.13}$$

Thus, from (2.13) we obtain that

$$\begin{aligned}
\delta_n &= \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1}), d(gz_n, gz_{n+1})\} \\
&\leq \psi(\max\{d(gx_{n-1}, gx_n), d(gy_{n-1}, gy_n), d(gz_{n-1}, gz_n)\}).
\end{aligned} \tag{2.14}$$

It then follows from (2.12) and a property ψ, that for all $n \geq 1$,

$$\delta_n \leq \psi(\delta_{n-1}) < \delta_{n-1}. \tag{2.15}$$

Thus, $\{\delta_n\}$ is a monotone decreasing sequence of nonnegative real numbers. So, there exist a $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \delta. \tag{2.16}$$

Suppose $\delta > 0$. Letting $n \to \infty$ in (2.14), using (2.15), (2.16), and a property of ψ, we get

$$\delta \leq \psi(\delta) < \delta, \tag{2.17}$$

which is a contradiction. Thus $\delta = 0$, or

$$\lim_{n\to\infty} \delta_n = 0, \tag{2.18}$$

or

$$\begin{aligned}
\lim_{n\to\infty} d(gx_{n+1}, gx_n) &= 0,\\
\lim_{n\to\infty} d(gy_{n+1}, gy_n) &= 0,\\
\lim_{n\to\infty} d(gz_{n+1}, gz_n) &= 0.
\end{aligned} \tag{2.19}$$

Now, we will prove that $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$ is not a Cauchy sequence. So, there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$ of $\{gx_n\}$, $\{gy_{n(k)}\}$ of $\{gy_n\}$, and $\{gz_{n(k)}\}$ of $\{gz_n\}$ with $n(k) > m(k) \geq k$ such that

$$\alpha_k = \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} \geq \varepsilon. \tag{2.20}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.20). Then, for all $k \geq 0$,

$$\max\{d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)-1}, gy_{m(k)}), d(gz_{n(k)-1}, gz_{m(k)})\} < \varepsilon. \tag{2.21}$$

By using (2.20) and (2.21) we have for $k \geq 0$,

$$\begin{aligned}
\varepsilon \leq \alpha_k &= \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\}\\
&\leq \max\{d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)}, gy_{n(k)-1})\\
&\qquad + d(gy_{n(k)-1}, gy_{m(k)}), d(gz_{n(k)}, gz_{n(k)-1}) + d(gz_{n(k)-1}, gz_{m(k)})\}\\
&\leq \max\{d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1}), d(gz_{n(k)}, gz_{n(k)-1})\} + \varepsilon\\
&\leq \delta_{n(k)-1} + \varepsilon.
\end{aligned} \tag{2.22}$$

Letting $k \to \infty$ in (2.22), and using (2.19), we get

$$\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} = \varepsilon. \tag{2.23}$$

Let, for $k \geq 0$,

$$\beta_k = \max\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), d(gz_{n(k)+1}, gz_{m(k)+1})\}. \tag{2.24}$$

Again, for all $k \geq 0$,

$$\begin{aligned}
\alpha_k &= \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} \\
&\leq \max\{d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)}), \\
&\qquad (gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)}), \\
&\qquad d(gz_{n(k)}, gz_{n(k)+1}) + d(gz_{n(k)+1}, gz_{m(k)+1}) + d(gz_{m(k)+1}, gz_{m(k)})\} \\
&\leq \max\{d(gx_{n(k)}, gx_{n(k)+1}), d(gy_{n(k)}, gy_{n(k)+1}), d(gz_{n(k)}, gz_{n(k)+1})\} \\
&\quad + \max\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), d(gz_{n(k)+1}, gz_{m(k)+1})\} \\
&\quad + \max\{d(gx_{m(k)}, gx_{m(k)+1}), d(gy_{m(k)}, gy_{m(k)+1}), d(gz_{m(k)}, gz_{m(k)+1})\} \\
&\leq \delta_{n(k)+1} + \beta_k + \delta_{m(k)+1}.
\end{aligned} \tag{2.25}$$

Analogously we have for $k \geq 0$,

$$\begin{aligned}
\beta_k &= \max\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+}, gy_{m(k)+1}), d(gz_{n(k)+1}, gz_{m(k)+1})\} \\
&\leq \max\{d(gx_{n(k)+1}, gx_{n(k)}) + d(gx_{n(k)}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)+1}), \\
&\qquad d(gy_{n(k)+1}, gy_{n(k)}) + d(gy_{n(k)}, gy_{m(k)}) + d(gy_{m(k)}, gy_{m(k)+1}), \\
&\qquad d(gz_{n(k)+1}, gz_{n(k)}) + d(gz_{n(k)}, gz_{m(k)}) + d(gz_{m(k)}, gz_{m(k)+1})\} \\
&\leq \max\{d(gx_{n(k)}, gx_{n(k)+1}), d(gy_{n(k)}, gy_{n(k)+1}), d(gz_{n(k)}, gz_{n(k)+1})\} \\
&\quad + \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\} \\
&\quad + \max\{d(gx_{m(k)}, gx_{m(k)+1}), d(gy_{m(k)}, gy_{m(k)+1}), d(gz_{m(k)}, gz_{m(k)+1})\} \\
&\leq \delta_{n(k)+1} + \alpha_k + \delta_{m(k)+1}.
\end{aligned} \tag{2.26}$$

Letting $k \to \infty$ in (2.25) and (2.26), we get that

$$\begin{aligned}
&\lim_{k\to\infty} \max\{d(gx_{n(k)+1}, gx_{m(k)+1}), d(gy_{n(k)+1}, gy_{m(k)+1}), d(gz_{n(k)+1}, gz_{m(k)+1})\} \\
&\qquad = \lim_{k\to\infty} \beta_k = \varepsilon = \lim_{k\to\infty} \alpha_k.
\end{aligned} \tag{2.27}$$

Since $n(k) > m(k)$, for $k \geq 0$, we have

$$\begin{gathered}
gx_{n(k)} \succeq gx_{m(k)}, \qquad gy_{n(k)} \preceq gy_{m(k)}, \\
gz_{n(k)} \succeq gz_{m(k)}.
\end{gathered} \tag{2.28}$$

Then from (2.1), (2.4), and (2.28), we have for $k \geq 0$,

$$\begin{aligned}
d(gx_{n(k)+1}, gx_{m(k)+1}) &= d(F(x_{n(k)}, y_{n(k)}, z_{n(k)}), F(x_{m(k)}, y_{m(k)}, z_{m(k)})) \\
&\leq \psi(\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\}), \\
d(gy_{n(k)+1}, gy_{m(k)+1}) &= d(F(y_{n(k)}, x_{n(k)}, y_{n(k)}), F(y_{m(k)}, x_{m(k)}, y_{m(k)})) \\
&\leq \psi(\max\{d(gy_{n(k)}, gy_{m(k)}), d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}), \\
d(gz_{n(k)+1}, gz_{m(k)+1}) &= d(F(z_{n(k)}, y_{n(k)}, x_{n(k)}), F(z_{m(k)}, y_{m(k)}, x_{m(k)})) \\
&\leq \psi(\max\{d(gz_{n(k)}, gz_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gx_{n(k)}, gx_{m(k)})\}).
\end{aligned} \tag{2.29}$$

From (2.29) for $k \geq 0$, we get

$$\beta_k \leq \psi(\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)}), d(gz_{n(k)}, gz_{m(k)})\}) = \psi(\alpha_k). \tag{2.30}$$

Letting $k \to \infty$ in (2.30), using (2.20), (2.27), and a property of ψ, we get

$$\varepsilon \leq \psi(\varepsilon) < \varepsilon, \tag{2.31}$$

which is a contradiction. This shows that $\{gx_n\}$, $\{gy_n\}$, and $\{gz_n\}$ are Cauchy sequences.

Since X is complete, there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} gx_n = x, \qquad \lim_{n\to\infty} gy_n = y, \qquad \lim_{n\to\infty} gz_n = z. \tag{2.32}$$

From (2.4) and (2.32), using the continuity of g, we have

$$gx = \lim_{n\to\infty} g(gx_{n+1}) = \lim_{n\to\infty} g(F(x_n, y_n, z_n)), \tag{2.33}$$

$$gy = \lim_{n\to\infty} g(gy_{n+1}) = \lim_{n\to\infty} g(F(y_n, x_n, y_n)), \tag{2.34}$$

$$gz = \lim_{n\to\infty} g(gz_{n+1}) = \lim_{n\to\infty} g(F(z_n, y_n, x_n)). \tag{2.35}$$

Now we will show that $gx = F(x, y, z)$, $gy = F(y, x, y)$, and $gz = F(z, y, x)$.

Since g and F are compatible, in addition with (2.33), (2.34), and (2.35), respectively imply

$$\lim_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n))) = 0, \tag{2.36}$$

$$\lim_{n\to\infty} d(g(F(y_n, x_n, y_n)), F(g(y_n), g(x_n), g(y_n))) = 0, \tag{2.37}$$

$$\lim_{n\to\infty} d(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n))) = 0. \tag{2.38}$$

Suppose now the assumption (a) holds, that is, F is continuous.

For all $n \geq 0$, we have

$$d(gx, F(gx_n, gy_n, gz_n)) \leq d(gx, g(F(x_n, y_n, z_n))) + d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)). \tag{2.39}$$

Taking the limit as $n \to \infty$, using (2.32), (2.33), (2.36), and the facts that g and F are continuous, we have $d(gx, F(x, y, z)) = 0$.

Similarly, by using (2.32), (2.34), and (2.37) and (2.32), (2.35), and (2.38), respectively, and also the facts that g and F are continuous, we have $d(gy, F(y, x, y)) = 0$ and $d(gz, F(z, y, x)) = 0$.

Thus we have proved that g and F have a tripled coincidence point.

Suppose that the assumption (b) holds. Since $\{gx_n\}$, $\{gz_n\}$ are nondecreasing and $gx_n \to x$ with $gz_n \to z$ and also $\{gy_n\}$ is nonincreasing with $gy_n \to y$, by assumption (b) we have for all n

$$gx_n \preceq x, \qquad gy_n \succeq y, \qquad gz_n \preceq z. \tag{2.40}$$

By virtue of monotone increasing property of g we have

$$ggx_n \preceq gx, \qquad ggy_n \succeq gy, \qquad ggz_n \preceq gz. \tag{2.41}$$

Now using (2.4) we have

$$\begin{aligned} d(gx, F(x, y, z)) &\leq d(gx, g(gx_{n+1})) + d(g(g(x_{n+1})), F(x, y, z)) \\ &\leq d(gx, g(gx_{n+1})) + d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) \\ &\quad + d(F(gx_n, gy_n, gz_n), F(x, y, z)) \\ &\leq d(gx, g(gx_{n+1})) + d(g(F(x_n, y_n, z_n)), F(gx_n, gy_n, gz_n)) \\ &\quad + \psi(\max\{d(ggx_n, gx), d(ggy_n, gy), d(ggz_n, gz)\}), \quad (\text{by}(2.1), (2.41)). \end{aligned} \tag{2.42}$$

Taking the limit as $n \to \infty$ in the above inequality, using (2.33), (2.36), and (2.41) we have

$$d(gx, F(x, y, z)) \leq \lim_{n \to \infty} \psi(\max\{d(ggx_n, gx), d(ggy_n, gy), d(ggz_n, gz)\}). \tag{2.43}$$

By (2.33), (2.34), (2.35), and the property of ψ, we have

$$d(gx, F(x, y, z)) \leq \psi(0) = 0, \tag{2.44}$$

that is

$$gx = F(x, y, z). \tag{2.45}$$

In a similar manner using (2.33), (2.34), (2.35), and (2.36), (2.37), (2.38), respectively, we obtain

$$\begin{aligned} gy &= F(y,x,y), \\ gz &= F(z,y,x). \end{aligned} \tag{2.46}$$

Thus, we proved that g and F have a tripled coincidence point.

This completes the proof of the theorem. □

Corollary 2.2. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose $F : X\times X\times X \to X$ and $g : X \to X$ are such that F has the mixed g-monotone property and*

$$d(F(x,y,z),F(u,v,w)) \le \psi(\max\{d(gx,gu),d(gy,gv),d(gz,gw)\}) \tag{2.47}$$

for any $x,y,z \in X$ for which $gx \preceq gu$, $gy \succeq gv$ and $gz \preceq gw$, where $\psi : [0,+\infty) \to [0,+\infty)$ be such that $\psi(t)$ is monotone, $\psi(t) < t$ and $\lim_{r\to t^+}\psi(r) < t$ for all $t > 0$. Suppose $F(X\times X\times X) \subseteq g(X)$, g is continuous, and F and g are commuting. Suppose either

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n,*

(ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.*

If there exist $x_0,y_0,z_0 \in X$ such that $gx_0 \preceq F(x_0,y_0,z_0)$, $gy_0 \succeq F(y_0,x_0,y_0)$, and $gz_0 \preceq F(z_0,y_0,x_0)$, then there exist $x,y,z \in X$ such that

$$F(x,y,z) = gx, \qquad F(y,x,y) = gy, \qquad F(z,y,x) = gz, \tag{2.48}$$

that is, F and g have a tripled coincidence point.

Proof. Since a commuting pair is also a compatible pair, the result of the Corollary 2.2 follows from Theorem 2.1. □

Later, by an example, we will show that the Corollary 2.2 is properly contained in Theorem 2.1.

Corollary 2.3. *Let $(X,\preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Suppose $F : X\times X\times X \to X$ be such that F has the mixed monotone property and*

$$d(F(x,y,z),F(u,v,w)) \le \psi(\max\{d(x,u),d(y,v),d(z,w)\}) \tag{2.49}$$

for any $x,y,z \in X$ for which $x \preceq u$, $y \succeq v$ and $z \preceq w$, where $\psi : [0,+\infty) \to [0,+\infty)$ be such that $\psi(t)$ is monotone, $\psi(t) < t$ and $\lim_{r\to t^+}\psi(r) < t$ for all $t > 0$. Suppose

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *if a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* n,

(ii) *if a nonincreasing sequence* $\{y_n\} \to y$, *then* $y_n \succeq y$ *for all* n.

If there exist $x_0, y_0, z_0 \in X$ *such that* $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$, *and* $z_0 \preceq F(z_0, y_0, x_0)$, *then there exist* $x, y, z \in X$ *such that*

$$F(x,y,z) = x, \qquad F(y,x,y) = y, \qquad F(z,y,x) = z, \tag{2.50}$$

that is, F has a tripled fixed point.

Proof. Taking $g(x) = x$ in Theorem 2.1 we obtain Corollary 2.3. □

Corollary 2.4. *Let* $(X, \preceq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Suppose* $F : X \times X \times X \to X$ *and* $g : X \to X$ *are such that F has the mixed monotone property and*

$$d(F(x,y,z), F(u,v,w)) \leq k \max\{d(x,u), d(y,v), d(z,w)\} \tag{2.51}$$

for any $x, y, z \in X$ *for which* $x \preceq u, y \succeq v$ *and* $z \preceq w$, *where* $0 < k < 1$. *Suppose either*

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *if a nondecreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* n,

(ii) *if a nonincreasing sequence* $\{y_n\} \to y$, *then* $y_n \succeq y$ *for all* n.

If there exist $x_0, y_0, z_0 \in X$ *such that* $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$, *and* $z_0 \preceq F(z_0, y_0, x_0)$, *then there exist* $x, y, z \in X$ *such that*

$$F(x,y,z) = x, \qquad F(y,x,y) = y, \qquad F(z,y,x) = z, \tag{2.52}$$

that is, F has a tripled coincidence point.

Proof. Taking $\psi(t) = kt$, $t > 0$ where $0 < k < 1$, in Corollary 2.3 we obtain Corollary 2.4. □

The following corollary is the result of Berinde and Borcut in [20].

Corollary 2.5. *Let* $(X, \preceq)$ *be a partially ordered set and suppose there is a metric* d *on* X *such that* (X, d) *is a complete metric space. Suppose* $F : X \times X \times X \to X$ *be such that F has the mixed monotone property and*

$$d(F(x,y,z), F(u,v,w)) \leq a_1 d(x,u) + a_2 d(y,v) + a_3 d(z,w) \tag{2.53}$$

for any $x, y, z \in X$ *for which* $x \preceq u, y \succeq v$ *and* $z \preceq w$, *where* $a_1 + a_2 + a_3 < 1$. *Suppose either*

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *if a nondecreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all n*

(ii) *if a nonincreasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all n.*

If there exist $x_0, y_0, z_0 \in X$ such that $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \succeq F(y_0, x_0, y_0)$, and $z_0 \preceq F(z_0, y_0, x_0)$, then there exist $x, y, z \in X$ such that

$$F(x, y, z) = x, \qquad F(y, x, y) = y, \qquad F(z, y, x) = z, \tag{2.54}$$

that is, F has a tripled fixed point.

Proof. The proof follows from Corollary 2.4, since the inequality in Corollary 2.5 implies that Corollary 2.4. □

Remark 2.6. The method used in the proof of Corollary 2.5 is different from that used by Berinde and Borcut [20].

Next we discuss an example.

Example 2.7. Let $X = \Re$. Then $(X, \preceq)$ is a partially ordered set with the partial ordering defined by $x \preceq y$ if and only if $|x| \leq |y|$ and $x \cdot y \geq 0$.

Let $d(x, y) = |x - y|$ for $x, y \in \Re$. Then (X, d) is a complete metric space.
Let $g : X \to X$ be defined as $g(x) = x^2/10$, for all $x \in X$.
Let $F : X \times X \times X \to X$ be defined as

$$F(x, y, z) = \frac{x^2 - y^2 + z^2}{9}, \quad \forall x, y, z \in X. \tag{2.55}$$

Then F obeys the mixed g-monotone property.
Let $\psi : [0, \infty) \to [0, \infty)$ be defined as $\psi(t) = (1/3)t$ for all $t \in [0, \infty)$.
Let, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences in X such that

$$\begin{aligned} \lim_{n\to\infty} F(x_n, y_n, z_n) &= \lim_{n\to\infty} g(x_n) = a, \\ \lim_{n\to\infty} F(y_n, x_n, y_n) &= \lim_{n\to\infty} g(y_n) = b, \\ \lim_{n\to\infty} F(z_n, y_n, x_n) &= \lim_{n\to\infty} g(z_n) = c. \end{aligned} \tag{2.56}$$

Then explicitly,

$$\begin{gathered} \lim_{n\to\infty} \frac{x_n^2 - y_n^2 + z_n^2}{9} = \lim_{n\to\infty} \frac{x_n^2}{10}, \quad \forall x, y, z \in X, \\ \text{or,} \\ \frac{10a - 10b + 10c}{9} = a \text{ imply } a - 10b + 10c = 0. \end{gathered} \tag{2.57}$$

Again,

$$\lim_{n\to\infty}\frac{y_n^2-x_n^2+y_n^2}{9}=\lim_{n\to\infty}\frac{y_n^2}{10},\quad \forall x,y,z\in X,$$
$$\text{or,}\tag{2.58}$$
$$\frac{10b-10a+10b}{9}=b \text{ imply } 11b-10a=0.$$

And

$$\lim_{n\to\infty}\frac{z_n^2-y_n^2+x_n^2}{9}=\lim_{n\to\infty}\frac{z_n^2}{10},\quad \forall x,y,z\in X,$$
$$\text{or,}\tag{2.59}$$
$$\frac{10c-10\text{b}+10a}{9}=c \text{ imply } c-10b+10a=0.$$

Then from the above relations we have, $a=0$, $b=0$, and $c=0$.

Therefore,

$$\begin{aligned}
&d(g(F(x_n,y_n,z_n)),F(gx_n,gy_n,gz_n))\longrightarrow 0 \quad \text{as } n\longrightarrow\infty,\\
&d(g(F(y_n,x_n,y_n)),F(gy_n,gx_n,gy_n))\longrightarrow 0 \quad \text{as } n\longrightarrow\infty,\\
&d(g(F(z_n,y_n,x_n)),F(gz_n,gy_n,gx_n))\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.
\end{aligned}\tag{2.60}$$

Hence, the pair (g,F) is compatible in X.

Also, $x_0=0, z_0=c(>0)$, and $y_0=0$ are three points in X such that $g(x_0)=g(0)=0<c^2/9=F(0,0,c)=F(x_0,y_0,z_0)$, $g(y_0)=g(0)=0=F(0,0,0)=F(y_0,x_0,y_0)$, and $g(z_0)=g(c)=c^2/10<c^2/9=F(c,0,0)=F(z_0,y_0,x_0)$.

We next verify inequality (2.1) of Theorem 2.1. We take $x,y,z,u,v,w\in X$, such that $gx\preceq gu$, $gz\preceq gw$ and $gy\succeq gv$, that is, $x^2\le u^2$, $z^2\le w^2$, and $y^2\ge v^2$.

Let $A=\max\{d(gx,gu),d(gy,gv),d(gz,gw)\}=\max\{|(x^2-u^2)|,|(y^2-v^2)|,|(z^2-w^2)|\}$.

Then $d(F(x,y,z),F(u,v,w))=d((x^2-y^2+z^2)/9,(u^2-v^2+w^2)/9)=(|(x^2-u^2)-(y^2-v^2)+(z^2-w^2))/3|\le(|(x^2-u^2)|+|(y^2-v^2)|+|(z^2-w^2)|)/9\le 3A/9=A/3=\psi(A)=\psi(\max\{d(gx,gu),d(gy,gv),d(gz,gw)\})$.

Thus it is verified that the functions g, F, and ψ satisfy all the conditions of Theorem 2.1. Here $(0,0,0)$ is the tripled coincidence point of g and F in X.

Remark 2.8. It is observed that in Example 2.7 the function F and g do not commute, but they are compatible. Hence Corollary 2.2 cannot be applied to this example. This shows that Theorem 2.1 properly contains Corollary 2.2. Also $g\neq I$, so the results of Berinde and Borcut [20] cannot be applied to this example. This shows that result in [20] is effectively generalised.

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