Research Article

Sharp Integral Inequalities Based on a General Four-Point Quadrature Formula via a Generalization of the Montgomery Identity

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We consider families of general four-point quadrature formulae using a generalization of the Montgomery identity via Taylor’s formula. The results are applied to obtain some sharp inequalities for functions whose derivatives belong to $L_p$ spaces. Generalizations of Simpson’s 3/8 formula and the Lobatto four-point formula with related inequalities are considered as special cases.

1. Introduction

The most elementary quadrature rules in four nodes are Simpson’s 3/8 rule based on the following four point formula

$$\int_a^b f(t) dt = \frac{b-a}{8} \left[ f(a) + 3f \left( \frac{2a+b}{3} \right) + 3f \left( \frac{a+2b}{3} \right) + f(b) \right] - \frac{(b-a)^5}{6480} f^{(4)}(\xi), \quad (1.1)$$

where $\xi \in [a, b]$, and Lobatto rule based on the following four point formula

$$\int_{-1}^1 f(t) dt = \frac{1}{6} \left[ f(-1) + 5f \left( -\frac{\sqrt{5}}{5} \right) + 5f \left( \frac{\sqrt{5}}{5} \right) + f(1) \right] - \frac{2}{23625} f^{(6)}(\eta), \quad (1.2)$$
where $\eta \in [-1,1]$. Formula (1.1) is valid for any function $f$ with a continuous fourth derivative $f^{(4)}$ on $[a,b]$ and formula (1.2) is valid for any function $f$ with a continuous sixth derivative $f^{(6)}$ on $[-1,1]$.

Let $f : [a,b] \to \mathbb{R}$ be differentiable on $[a,b]$ and $f' : [a,b] \to \mathbb{R}$ integrable on $[a,b]$. Then the Montgomery identity holds (see [1])

$$f(x) = \frac{1}{b-a} \int_a^b f(t) \, dt + \int_a^b P(x,t) f'(t) \, dt,$$

where the Peano kernel is

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \leq t \leq x, \\ \frac{t-b}{b-a}, & x < t \leq b. \end{cases}$$

In [2], Pečarić proved the following weighted Montgomery identity

$$f(x) = \int_a^b w(t) f(t) \, dt + \int_a^b P_w(x,t) f'(t) \, dt,$$

where $w : [a,b] \to [0,\infty)$ is some probability density function, that is, integrable function, satisfying $\int_a^b w(t) \, dt = 1$, and $W(t) = \int_a^t w(x) \, dx$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$ and $P_w(x,t)$ is the weighted Peano kernel defined by

$$P_w(x,t) = \begin{cases} W(t), & a \leq t \leq x, \\ W(t) - 1, & x < t \leq b. \end{cases}$$

Now, let us suppose that $I$ is an open interval in $\mathbb{R}$, $[a,b] \subset I$, $f : I \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 2$, $w : [a,b] \to [0,\infty)$ is a probability density function. Then the following generalization of the weighted Montgomery identity via Taylor’s formula states (given by Aglić Aljinović and Pečarić in [3])

$$f(x) = \int_a^b w(t) f(t) \, dt - \sum_{i=0}^{n-2} \frac{f^{(i+1)}(x)}{(i+1)!} \int_a^b w(s)(s-x)^i \, ds$$

$$+ \frac{1}{(n-1)!} \int_a^b T_{w,n}(x,s) f^{(n)}(s) \, ds,$$

where $x \in [a,b]$ and

$$T_{w,n}(x,s) = \begin{cases} \int_a^s w(u)(u-s)^{n-1} \, du, & a \leq s \leq x, \\ - \int_s^b w(u)(u-s)^{n-1} \, du, & x < s \leq b. \end{cases}$$
If we take $w(t) = 1/(b-a)$, $t \in [a,b]$, equality (1.7) reduces to

$$f(x) = \frac{1}{b-a} \int_a^b f(t)dt - \sum_{i=0}^{n-2} f^{(i+1)}(x) \frac{(b-x)^{i+2} - (a-x)^{i+2}}{(i+2)! (b-a)} + \frac{1}{(n-1)!} \int_a^b T_n(x, s)f^{(n)}(s)ds,$$

where $x \in [a,b]$ and

$$T_n(x, s) = \begin{cases} \frac{(a-s)^n}{n(b-a)}, & a \leq s \leq x, \\ \frac{(b-s)^n}{n(b-a)}, & x < s \leq b. \end{cases}$$

For $n = 1$, (1.9) reduces to the Montgomery identity (1.3).

In this paper, we generalize the results from [4]. Namely, we use identities (1.7) and (1.9) to establish for each number $x \in (a, (a+b)/2]$ a general four-point quadrature formula of the type

$$\int_a^b w(t)f(t)dt = \left(\frac{1}{2} - A(x)\right) \left[f(a) + f(b)\right] + A(x) \left[f(x) + f(a+b-x)\right] + R(f, w; x),$$

where $R(f, w; x)$ is the remainder and $A : (a, (a+b)/2] \to \mathbb{R}$ is a real function. The obtained formula is used to prove a number of inequalities which give error estimates for the general four-point formula for functions whose derivatives are from $L_p$-spaces. These inequalities are generally sharp. As special cases of the general non-weighted four-point quadrature formula, we obtain generalizations of the well-known Simpson’s 3/8 formula and Lobatto four-point formula with related inequalities.

### 2. General Weighted Four-Point Formula

Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n-1)}$ exists on $[a,b]$ for some $n \geq 2$. We introduce the following notation for each $x \in (a, (a+b)/2]$: $D(x) = \left(\frac{1}{2} - A(x)\right) \left[f(a) + f(b)\right] + A(x) \left[f(x) + f(a+b-x)\right]$. 


Remark 2.2.

Proof.

We put \( x = \frac{a + b}{2} \) in (1.7) to obtain four new formulae. After multiplying these four formulae by \( \frac{1}{2} - A(x) \), \( A(x) \), \( 1/2 - A(x) \), respectively, and adding, we get (2.2).

\[
\int_a^b \omega(t) f(t) dt = D(x) + t_{w,n}(x) + \frac{1}{(n - 1)!} \int_a^b \tilde{T}_{w,n}(x,s) f^{(n)}(s) ds. \tag{2.2}
\]

**Theorem 2.1.** Let \( I \) be an open interval in \( \mathbb{R} \), \( [a,b] \subset I \), and let \( \omega : [a,b] \rightarrow [0,\infty) \) be some probability density function. Let \( f : I \rightarrow \mathbb{R} \) be such that \( f^{(n-1)} \) is absolutely continuous for some \( n \geq 2 \). Then for each \( x \in (a,(a+b)/2] \) the following identity holds

\[
\int_a^b \omega(t) f(t) dt = D(x) + t_{w,n}(x) + \frac{1}{(n - 1)!} \int_a^b \tilde{T}_{w,n}(x,s) f^{(n)}(s) ds.
\]

**Proof.** We put \( x = a, x = \frac{a+b-x}{2} \) and \( x = b \) in (1.7) to obtain four new formulae. After multiplying these four formulae by \( \frac{1}{2} - A(x) \), \( A(x) \), \( 1/2 - A(x) \), respectively, and adding, we get (2.2).

**Remark 2.2.** Identity (2.2) holds true in the case \( n = 1 \). It can also be obtained by taking \( x = a, x = \frac{a+b-x}{2} \), and \( x = b \) in (1.5), multiplying these four formulae by \( \frac{1}{2} - A(x) \), \( A(x) \), \( 1/2 - A(x) \), respectively, and adding. In this special case we have

\[
\int_a^b \omega(t) f(t) dt = D(x) + \int_a^b \tilde{T}_{w,1}(x,s) f'(s) ds,
\]

\[
\int_a^b \omega(t) f(t) dt = D(x) + \int_a^b \tilde{T}_{w,1}(x,s) f'(s) ds.
\tag{2.3}
\]
where

\[
\tilde{T}_{w,1}(x, s) = -\left(\frac{1}{2} - A(x)\right)[T_{w,1}(a, s) + T_{w,1}(b, s)] - A(x)[T_{w,1}(x, s) + T_{w,1}(a + b - x, s)]
\]

\[
= -\left(\frac{1}{2} - A(x)\right)[P_w(a, s) + P_w(b, s)] - A(x)[P_w(x, s) + P_w(a + b - x, s)]
\]

\[
= \begin{cases} 
\frac{1}{2} - A(x) - W(s), & a \leq s \leq x, \\
\frac{1}{2} - W(s), & x < s \leq a + b - x, \\
\frac{1}{2} + A(x) - W(s), & a + b - x < s \leq b.
\end{cases}
\] (2.4)

**Theorem 2.3.** Suppose that all assumptions of Theorem 2.1 hold. Additionally, assume that \((p, q)\) is a pair of conjugate exponents, that is, \(1 \leq p, q \leq \infty, 1/p + 1/q = 1\), let \(f^{(n)} \in L^p[a, b]\) for some \(n \geq 1\). Then for each \(x \in (a, (a + b)/2]\) we have

\[
\left|\int_a^b w(t)f(t)dt - D(x) - t_{w,n}(x)\right| \leq \frac{1}{(n-1)!} \left\|\tilde{T}_{w,n}(x, \cdot)\right\|_q \left\|f^{(n)}\right\|_p.
\] (2.5)

Inequality (2.5) is sharp for \(1 < p \leq \infty\).

**Proof.** By applying the Hölder inequality we have

\[
\left|\int_a^b \frac{1}{(n-1)!} \tilde{T}_{w,n}(x, s)f^{(n)}(s)ds\right| \leq \frac{1}{(n-1)!} \left\|\tilde{T}_{w,n}(x, \cdot)\right\|_q \left\|f^{(n)}\right\|_p.
\] (2.6)

By using the above inequality from (2.2) we obtain estimate (2.5). Let us denote \(U^x_n(s) = \tilde{T}_{w,n}(x, s)\). For the proof of sharpness, we will find a function \(f\) such that

\[
\left|\int_a^b U^x_n(s)f^{(n)}(s)ds\right| = \left\|U^x_n\right\|_q \left\|f^{(n)}\right\|_p.
\] (2.7)

For \(1 < p < \infty\), take \(f\) to be such that

\[
f^{(n)}(s) = \text{sign} U^x_n(s) \cdot \left|U^x_n(s)\right|^{1/(p-1)},
\] (2.8)

where for \(p = \infty\) we put

\[
f^{(n)}(s) = \text{sign} U^x_n(s).
\] (2.9)

**Remark 2.4.** Inequality (2.5) for \(A(x) = 1/4\) was proved by Aglić Aljinović et al. in [4].
3. Non-Weighted Four-Point Formula and Applications

Here we define

\[
\tilde{t}_n(x) = A(x) \sum_{i=0}^{n-2} f^{(i+1)}(x) + (-1)^{i+1} f^{(i+1)}(a + b - x) \frac{(b - x)^{i+2} - (a - x)^{i+2}}{(i + 2)! (b - a)}
\]

\[\tilde{\mathcal{T}}_n(x, s) = -n \left\{ \left( \frac{1}{2} - A(x) \right) \left[ T_n(a, s) + T_n(b, s) \right] + A(x) \left[ T_n(x, s) + T_n(a + b - x, s) \right] \right\} \]

\[= \begin{cases} \\
\left( \frac{1}{2} + A(x) \right) \frac{(a - s)^n}{(b - a)} + \left( \frac{1}{2} - A(x) \right) \frac{(b - s)^n}{(b - a)}, & a \leq s \leq x, \\
\frac{(a - s)^n + (b - s)^n}{2(b - a)}, & x < s \leq a + b - x, \\
\left( \frac{1}{2} - A(x) \right) \frac{(a - s)^n}{(b - a)} + \left( \frac{1}{2} + A(x) \right) \frac{(b - s)^n}{(b - a)}, & a + b - x < s \leq b. 
\end{cases} \]  

**Theorem 3.1.** Let I be an open interval in \( \mathbb{R} \), \([a, b] \subset I\), and let \( f : I \to \mathbb{R} \) be such that \( f^{(n-1)} \) is absolutely continuous for some \( n \geq 1 \). Then for each \( x \in (a, (a + b) / 2) \) the following identity holds

\[
\frac{1}{b - a} \int_a^b f(t) dt = D(x) + \tilde{t}_n(x) + \frac{1}{n!} \int_a^b \tilde{\mathcal{T}}_n(x, s) f^{(n)}(s) ds.
\]

**Proof.** We take \( w(t) = 1/(b - a) \), \( t \in [a, b] \) in (2.2). \( \square \)

**Theorem 3.2.** Suppose that all assumptions of Theorem 3.1 hold. Additionally, assume that \( (p, q) \) is a pair of conjugate exponents, that is, \( 1 \leq p, q \leq \infty, 1/p + 1/q = 1 \) and \( f^{(n)} \in L^p[a, b] \) for some \( n \geq 1 \). Then for each \( x \in (a, (a + b) / 2) \) we have

\[
\left| \frac{1}{b - a} \int_a^b f(t) dt - D(x) - \tilde{t}_n(x) \right| \leq \frac{1}{n!} \| \tilde{\mathcal{T}}_n(x, \cdot) \|_q \| f^{(n)} \|_p.
\]

**Inequality (3.4) is sharp for 1 < p \leq \infty.**

**Proof.** We take \( w(t) = 1/(b - a) \), \( t \in [a, b] \) in (2.5). \( \square \)

Now, we set

\[
A(x) = \frac{(b - a)^2}{12(x - a)(b - x)}, \quad x \in \left( a, \frac{a + b}{2} \right).
\]

This special choice of the function \( A \) enables us to consider generalizations of the well-known Simpson’s 3/8 formula (1.1) and Lobatto formula (1.2)
Suppose that all assumptions of Theorem 3.1 hold. Then the following generalization of Simpson’s 3/8 formula reads

\[
\frac{1}{b-a} \int_a^b f(t)dt = D\left(\frac{2a+b}{3}\right) + i_a\left(\frac{2a+b}{3}\right) + \frac{1}{n!} \int_a^b T_a\left(\frac{2a+b}{3}, s\right)f^{(n)}(s)ds, \tag{3.6}
\]

where

\[
D\left(\frac{2a+b}{3}\right) = \frac{1}{8} \left(f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right),
\]

\[
i_a\left(\frac{2a+b}{3}\right) = \frac{1}{8} \sum_{i=0}^{n-2} \left[f^{(i+1)}\left(\frac{2a+b}{3}\right) + (-1)^{i+1}f^{(i+1)}\left(\frac{a+2b}{3}\right)\right] \frac{2^{i+2} + (-1)^{i+1}(b-a)^{i+1}}{3^{i+1}(i+2)!},
\]

\[
T_a\left(\frac{2a+b}{3}, s\right) = -\frac{n}{8} \left[T_n(a, s) + 3T_n\left(\frac{2a+b}{3}, s\right) + 3T_n\left(\frac{a+2b}{3}, s\right) + T_n(b, s)\right]
\]

\[
= \begin{cases} 
\frac{7(a-s)^n + (b-s)^n}{8(b-a)}, & a \leq s \leq \frac{2a+b}{3}, \\
\frac{(a-s)^n + (b-s)^n}{2(b-a)}, & \frac{2a+b}{3} \leq s \leq \frac{a+2b}{3}, \\
\frac{(a-s)^n + 7(b-s)^n}{8(b-a)}, & \frac{a+2b}{3} \leq s \leq b.
\end{cases}
\]

In the next corollaries we will use the beta function and the incomplete beta function of Euler type defined by

\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad B_r(x, y) = \int_0^r t^{x-1}(1-t)^{y-1}dt, \quad x, y > 0. \tag{3.8}
\]

**Corollary 3.3.** Suppose that all assumptions of Theorem 3.1 hold. Additionally, assume that \((p, q)\) is a pair of conjugate exponents and \(n \in \mathbb{N}\).

(a) If \(f^{(n)} \in L^\infty[a, b]\), then

\[
\left|\frac{1}{b-a} \int_a^b f(t)dt - D\left(\frac{2a+b}{3}\right)\right| \leq \frac{25}{288}(b-a)\|f'\|_\infty,
\]
where

\[
\left\| \int_a^b f(t)dt - D\left(\frac{2a + b}{3}\right) - \tilde{t}_n\left(\frac{2a + b}{3}\right) \right\|_\infty, \quad n \geq 2.
\]

(3.9)

(b) If \( f^{(n)} \in L^2[a, b] \), then

\[
\left\| \int_a^b f(t)dt - D\left(\frac{2a + b}{3}\right) - \tilde{t}_n\left(\frac{2a + b}{3}\right) \right\| \leq \frac{1}{n!} \left( \frac{3^{2n + 5 \cdot 2^{n+1} + 11}}{32 \cdot 3^n (2n + 1)} \right)^{1/2} + \frac{(-1)^n (b - a)^{2n-1}}{32}
\]

\[
\times [7B(n + 1, n + 1) + 9B_{2/3}(n + 1, n + 1) - 9B_{1/3}(n + 1, n + 1)]^{1/2} \left\| f^{(n)} \right\|_2.
\]

(3.10)

(c) If \( f^{(n)} \in L^1[a, b] \), then

\[
\left\| \int_a^b f(t)dt - D\left(\frac{2a + b}{3}\right) - \tilde{t}_n\left(\frac{2a + b}{3}\right) \right\| \leq \frac{1}{n!} K_n\left(\frac{2a + b}{3}\right) \left\| f^{(n)} \right\|_1.
\]

(3.11)

where \( K_1((2a + b)/3) = 5/24, K_2((2a + b)/3) = (5/18) (b - a), K_3((2a + b)/3) = (7/54)(b - a)^2 \)

and \( K_n((2a + b)/3) = (1/8) (b - a)^{n-1} \), for \( n \geq 4 \).

The first and the second inequality are sharp.

**Proof.** We apply (3.4) with \( x = (2a + b)/3 \) and \( p = \infty \)

\[
\int_a^b \tilde{t}_n\left(\frac{2a + b}{3}, s\right) \left| \frac{(a - s)^n + (b - s)^n}{8(b - a)} \right| ds = \int_a^{(2a+b)/3} \left| \frac{(a - s)^n + (b - s)^n}{8(b - a)} \right| ds
\]

\[
+ \int_{(2a+b)/3}^{(a+2b)/3} \left| \frac{(a - s)^n + (b - s)^n}{2(b - a)} \right| ds + \int_{(a+2b)/3}^b \left| \frac{(a - s)^n + 7(b - s)^n}{8(b - a)} \right| ds
\]

\[
= 2 \frac{[3^{n+1} - 2^{n+1} + 7 \cdot (-1)^n] (b - a)^n}{8 \cdot 3^{n+1}(n + 1)}.
\]
By an elementary calculation we get

\[
\int_a^b \left( \frac{2a+b}{3} \right)^2 ds = \frac{25}{288} (b-a).
\]

(3.13)

To obtain the second inequality we take \( p = 2 \)

\[
\int_a^b \left[ \tilde{T}_n \left( \frac{2a+b}{3}, s \right) \right]^2 ds = \int_a^{(2a+b)/3} \left( \frac{7(a-s)^n + (b-s)^n}{8(b-a)} \right)^2 ds + \int_{(2a+b)/3}^{(a+2b)/3} \left( \frac{(a-s)^n + (b-s)^n}{2(b-a)} \right)^2 ds + \int_{(a+2b)/3}^b \left( \frac{(a-s)^n + 7(b-s)^n}{8(b-a)} \right)^2 ds
\]

\[
\quad = \frac{[3^n + 5 \cdot 2^{n+1} + 11] (b-a)^{2n-1}}{32 \cdot 3^{2n}(2n + 1)} + \frac{(-1)^n (b-a)^{2n-1}}{32} \cdot [7B(n + 1, n + 1) + 9B_{2/3}(n + 1, n + 1) - 9B_{1/3}(n + 1, n + 1)].
\]

(3.14)

If \( p = 1 \), we have

\[
\sup_{s \in [a,b]} \left| \tilde{T}_n \left( \frac{2a+b}{3}, s \right) \right| = \max \left\{ \sup_{s \in [a,(2a+b)/3]} \left| \frac{7(a-s)^n + (b-s)^n}{8(b-a)} \right|, \sup_{s \in [(2a+b)/3,(a+2b)/3]} \left| \frac{(a-s)^n + (b-s)^n}{2(b-a)} \right|, \sup_{s \in [(a+2b)/3,b]} \left| \frac{(a-s)^n + 7(b-s)^n}{8(b-a)} \right| \right\}.
\]

(3.15)

By an elementary calculation we get

\[
\sup_{s \in [(2a+b)/3]} \left| \frac{7(a-s) + (b-s)}{8(b-a)} \right| = \sup_{s \in [(a+2b)/3]} \left| \frac{(a-s) + 7(b-s)}{8(b-a)} \right| = \frac{5}{24} (b-a),
\]
Lobatto formula reads

\[
\sup_{s \in [a, (a+2b)/3]} \left| \frac{7(a-s)^3 + (b-s)^2}{8(b-a)} \right| = \sup_{s \in [(a+b)/3, b]} \left| \frac{(a-s)^2 + 7(b-s)^2}{8(b-a)} \right| = \frac{11}{72} (b-a),
\]

\[
\sup_{s \in [a, (2a+b)/3]} \left| \frac{7(a-s)^n + (b-s)^n}{8(b-a)} \right| = \sup_{s \in [(a+2b)/3, b]} \left| \frac{(a-s)^n + 7(b-s)^n}{8(b-a)} \right| = \frac{(b-a)^{n-1}}{8},
\]

for \( n \geq 3 \). The function \( y : [a, b] \to \mathbb{R}, y(x) = (a - x)^n + (b - x)^n \), is decreasing on \((a, (a+b)/2)\) and increasing on \(( (a+b)/2, b) \) if \( n \) is even, and decreasing on \((a, b)\) if \( n \) is odd. Thus

\[
\sup_{s \in [(2a+b)/3, (a+2b)/3]} \left| \frac{(a-s)^n + (b-s)^n}{2(b-a)} \right| = \frac{((-1)^n + 2^n)(b-a)^{n-1}}{2 \cdot 3^n}.
\]

Finally,

\[
\sup_{s \in [a,b]} \left| \widehat{T}_1 \left( \frac{2a + b}{3}, s \right) \right| = \frac{5}{24}
\]

and for \( n \geq 2 \)

\[
\sup_{s \in [a,b]} \left| \widehat{T}_n \left( \frac{2a + b}{3}, s \right) \right| = (b-a)^{n-1} \max \left\{ \frac{1}{8}, \frac{2^n + (-1)^n}{2 \cdot 3^n} \right\}.
\]

3.2. \([a, b] = [-1, 1], x = -\sqrt{5}/5\)

Suppose that all assumptions of Theorem 3.1 hold. Then the following generalization of Lobatto formula reads

\[
\frac{1}{2} \int_{-1}^{1} f(t) dt = D \left( -\frac{\sqrt{5}}{5} \right) + \hat{f}_n \left( -\frac{\sqrt{5}}{5} \right) + \frac{1}{n!} \int_{-1}^{1} \widehat{T}_n \left( -\frac{\sqrt{5}}{5}, s \right) f^{(n)}(s) ds,
\]

where

\[
D \left( -\frac{\sqrt{5}}{5} \right) = \frac{1}{12} \left( f(-1) + 5f \left( -\frac{\sqrt{5}}{5} \right) + 5f \left( \frac{\sqrt{5}}{5} \right) + f(1) \right),
\]

\[
\hat{f}_n \left( -\frac{\sqrt{5}}{5} \right) = \frac{5}{12} \sum_{i=0}^{n-2} \left[ f^{(i+1)} \left( -\frac{\sqrt{5}}{5} \right) + (-1)^i f^{(i+1)} \left( \frac{\sqrt{5}}{5} \right) \right] \times \frac{(5 + \sqrt{5})^{i+2} + (-1)^{i+1}(5 - \sqrt{5})^{i+2}}{2 \cdot 5^{i+2}(i + 2)!}
\]

\[+ \frac{1}{12} \sum_{i=0}^{n-2} \left( f^{(i+1)}(-1) + (-1)^i f^{(i+1)}(1) \right) \frac{2^{i+1}}{(i + 2)!},\]

\[
\int_{-1}^{1} \widehat{T}_n \left( -\frac{\sqrt{5}}{5}, s \right) f^{(n)}(s) ds = \left( -\frac{\sqrt{5}}{5} \right)^{n-1} \max \left\{ \frac{1}{8}, \frac{2^n + (-1)^n}{2 \cdot 3^n} \right\}.
\]
Corollary 3.4. Suppose that all assumptions of Theorem 3.1 hold. Additionally, assume that \((p, q)\) is a pair of conjugate exponents and \(n \in \mathbb{N}\).

(a) if \(f^{(n)} \in L^\infty[-1, 1]\), then

\[
\left| \frac{1}{2} \int_{-1}^{1} f(t) \, dt - D \left( -\frac{\sqrt{5}}{5} \right) - \tilde{T}_n \left( -\frac{\sqrt{5}}{5} \right) \right| \leq \left( \frac{101}{180} - \frac{\sqrt{5}}{6} \right) \| f^{(n)} \|_\infty,
\]

\[
\leq \frac{1}{(n+1)!} \left( \frac{2^{n+1} \cdot 5^n + (5 + \sqrt{5})^{n+1} - (5 + \sqrt{5})^{n+1}}{12 \cdot 5^n} \right) \frac{1 + (-1)^{n+1}}{2} \| f^{(n)} \|_\infty, \quad n \geq 2.
\]

(b) if \(f^{(n)} \in L^2[-1, 1]\), then

\[
\left| \frac{1}{2} \int_{-1}^{1} f(t) \, dt - D \left( -\frac{\sqrt{5}}{5} \right) - \tilde{T}_n \left( -\frac{\sqrt{5}}{5} \right) \right| \leq \frac{1}{n!} \frac{2^{n-2}}{3} \left( \frac{35 (5 + \sqrt{5})^{2n+1} + 85 (5 - \sqrt{5})^{2n+1} + 10^{2n+1}}{10^{2n+1}(2n+1)} \right)
\]

\[
+ (-1)^n \left[ 11B(n + 1, n + 1) + 25B(5 + \sqrt{5})/10(n + 1, n + 1) 
\right. 
\left. - 25B(5 - \sqrt{5})/10(n + 1, n + 1) \right]^{1/2} \| f^{(n)} \|_2,
\]

\[\hat{T}_n(-\sqrt{\frac{5}{5}}, s) = -\frac{n}{12} \left[ T_n(-1, s) + 5T_n\left(-\frac{\sqrt{5}}{5}, s\right) + 5T_n\left(\frac{\sqrt{5}}{5}, s\right) + T_n(1, s) \right]
\]

\[
= \begin{cases} 
\frac{11(-1-s)^n + (1-s)^n}{24} & -1 \leq s \leq -\frac{\sqrt{5}}{5}, \\
\frac{(-1-s)^n + (1-s)^n}{4} & -\frac{\sqrt{5}}{5} < s \leq \frac{\sqrt{5}}{5}, \\
\frac{(-1-s)^n + 11(1-s)^n}{24} & \frac{\sqrt{5}}{5} < s \leq 1.
\end{cases}
\]

(3.21)
(c) if \( f^{(n)} \in L^1[-1, 1], \) then

\[
\left| \frac{1}{2} \int_{-1}^{1} f(t) dt - D\left(-\frac{\sqrt{5}}{5}\right) - \hat{t}_n\left(-\frac{\sqrt{5}}{5}\right) \right| \leq \frac{1}{n!} K_n\left(-\frac{\sqrt{5}}{5}\right) \|f^{(n)}\|_1,
\]

where \( K_1(-\sqrt{5}/5) = 1/(2\sqrt{5}), \) \( K_2(-\sqrt{5}/5) = 3/5, \) \( K_3(-\sqrt{5}/5) = 8/(5\sqrt{5}), \) \( K_4(-\sqrt{5}/5) = 28/25, \) \( K_5(-\sqrt{5}/5) = 88/(25\sqrt{5}), \) \( K_n(-\sqrt{5}/5) = 2^{n-3}/3, \) for \( n \geq 6. \)

The first and the second inequality are sharp.

**Proof.** Applying (3.4) with \([a, b] = [-1, 1], x = -\sqrt{5}/5\) and \( p = \infty, p = 2, p = 1\) and carrying out the same analysis as in Corollary 3.3 we obtain the above inequalities.

**References**


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