Research Article

Complex Hessian Equations on Some Compact Kähler Manifolds

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On a compact connected $2m$ -dimensional Kähler manifold with Kähler form ω , given a smooth function $f : M \rightarrow \mathbb{R}$ and an integer $1 < k < m$, we want to solve uniquely in $[\omega]$ the equation $\tilde{\omega}^k \wedge \omega^{m-k} = e^f \omega^m$, relying on the notion of k -positivity for $\tilde{\omega} \in [\omega]$ (the extreme cases are solved: $k = m$ by (Yau in 1978), and $k = 1$ trivially). We solve by the continuity method the corresponding complex elliptic k th Hessian equation, more difficult to solve than the Calabi-Yau equation ($k = m$), under the assumption that the holomorphic bisectional curvature of the manifold is nonnegative, required here only to derive an a priori eigenvalues pinching.

1. The Theorem

All manifolds considered in this paper are *connected*.

Let (M, J, g, ω) be a compact connected Kähler manifold of complex dimension $m \geq 3$. Fix an integer $2 \leq k \leq m - 1$. Let $\varphi : M \rightarrow \mathbb{R}$ be a smooth function, and let us consider the $(1, 1)$ -form $\tilde{\omega} = \omega + i\partial\bar{\partial}\varphi$ and the associated 2-tensor $\tilde{g}$ defined by $\tilde{g}(X, Y) = \tilde{\omega}(X, JY)$. Consider the sesquilinear forms h and $\tilde{h}$ on $T^{1,0}$ defined by $h(U, V) = g(U, \bar{V})$ and $\tilde{h}(U, V) = \tilde{g}(U, \bar{V})$. We denote by $\lambda(g^{-1}\tilde{g})$ the eigenvalues of $\tilde{h}$ with respect to the Hermitian form h . By definition, these are the eigenvalues of the unique endomorphism A of $T^{1,0}$ satisfying

$$\tilde{h}(U, V) = h(U, AV) \quad \forall U, V \in T^{1,0}. \quad (1.1)$$

Calculations infer that the endomorphism A writes

$$\begin{aligned} A : T^{1,0} &\longrightarrow T^{1,0}, \\ U^i \partial_i &\longmapsto A_i^j U^i \partial_j = g^{j\bar{\ell}} \tilde{g}_{i\bar{\ell}} U^i \partial_j. \end{aligned} \quad (1.2)$$

A is a self-adjoint/Hermitian endomorphism of the Hermitian space $(T^{1,0}, h)$, therefore $\lambda(g^{-1}\tilde{g}) \in \mathbb{R}^m$. Let us consider the following cone: $\Gamma_k = \{\lambda \in \mathbb{R}^m / \forall 1 \leq j \leq k, \sigma_j(\lambda) > 0\}$, where σ_j denotes the j th elementary symmetric function.

Definition 1.1. φ is said to be k -admissible if and only if $\lambda(g^{-1}\tilde{g}) \in \Gamma_k$.

In this paper, we prove the following theorem.

Theorem 1.2 (the σ_k equation). *Let (M, J, g, ω) be a compact connected Kähler manifold of complex dimension $m \geq 3$ with nonnegative holomorphic bisectional curvature, and let $f : M \rightarrow \mathbb{R}$ be a function of class C^∞ satisfying $\int_M e^f \omega^m = \binom{m}{k} \int_M \omega^m$. There exists a unique function $\varphi : M \rightarrow \mathbb{R}$ of class C^∞ such that*

$$(1) \quad \int_M \varphi \omega^m = 0, \quad (1.3)$$

$$(2) \quad \tilde{\omega}^k \wedge \omega^{m-k} = \left(\frac{e^f}{\binom{m}{k}} \right) \omega^m. \quad (E_k)$$

Moreover the solution φ is k -admissible.

This result was announced in a note in the Comptes Rendus de l'Académie des Sciences de Paris published online in December 2009 [1]. The curvature assumption is used, in Section 6.2 only, for an a priori estimate on $\lambda(g^{-1}\tilde{g})$ as in [2, page 408], and it should be removed (as did Aubin for the case $k = m$ in [3], see also [4] for this case). For the analogue of (E_k) on $\mathbb{C}^m$, the Dirichlet problem is solved in [5, 6], and a Bedford-Taylor type theory, for weak solutions of the corresponding degenerate equations, is addressed in [7]. Thanks to Julien Keller, we learned of an independent work [8] aiming at the same result as ours, with a different gradient estimate and a similar method to estimate $\lambda(g^{-1}\tilde{g})$, but no proofs given for the C^0 and the C^2 estimates.

Let us notice that the function f appearing in the second member of (E_k) satisfies necessarily the normalisation condition $\int_M e^f \omega^m = \binom{m}{k} \int_M \omega^m$. Indeed, this results from the following lemma.

Lemma 1.3. *Consider $\int_M \tilde{\omega}^k \wedge \omega^{m-k} = \int_M \omega^m$.*

Proof. See [9, page 44]. □

Let us write (E_k) differently.

Lemma 1.4. *Consider $\tilde{\omega}^k \wedge \omega^{m-k} = (\sigma_k(\lambda(g^{-1}\tilde{g})) / \binom{m}{k}) \omega^m$.*

Proof. Let $P \in M$. It suffices to prove the equality at P in a g -normal $\tilde{g}$ -adapted chart z centered at P . In such a chart $g_{i\bar{j}}(0) = \delta_{ij}$ and $\tilde{g}_{i\bar{j}}(0) = \delta_{ij}\lambda_i(0)$, so at $z = 0$, $\omega = idz^a \wedge dz^{\bar{a}}$ and $\tilde{\omega} = i\lambda_a(0)dz^a \wedge dz^{\bar{a}}$. Thus

$$\begin{aligned}\tilde{\omega}^k \wedge \omega^{m-k} &= \left(\sum_a i\lambda_a(0)dz^a \wedge dz^{\bar{a}} \right)^k \wedge \left(\sum_b idz^b \wedge dz^{\bar{b}} \right)^{m-k} \\ &= \sum_{\substack{(a_1, \dots, a_k) \in \{1, \dots, m\} \\ \text{distinct integers} \\ (b_1, \dots, b_{m-k}) \in \{1, \dots, m\} \setminus \{a_1, \dots, a_k\} \\ \text{distinct integers}}} i^m \lambda_{a_1}(0) \cdots \lambda_{a_k}(0) \\ &\quad \left(dz^{a_1} \wedge dz^{\bar{a}_1} \right) \wedge \cdots \wedge \left(dz^{a_k} \wedge dz^{\bar{a}_k} \right) \wedge \left(dz^{b_1} \wedge dz^{\bar{b}_1} \right) \wedge \cdots \wedge \left(dz^{b_{m-k}} \wedge dz^{\bar{b}_{m-k}} \right).\end{aligned}\tag{1.4}$$

Now $a_1, \dots, a_k, b_1, \dots, b_{m-k}$ are m distinct integers of $\{1, \dots, m\}$ and 2-forms commute therefore,

$$\begin{aligned}\tilde{\omega}^k \wedge \omega^{m-k} &= \left(\sum_{\substack{(a_1, \dots, a_k) \in \{1, \dots, m\} \\ \text{distinct integers} \\ (b_1, \dots, b_{m-k}) \in \{1, \dots, m\} \setminus \{a_1, \dots, a_k\} \\ \text{distinct integers}}} \lambda_{a_1}(0) \cdots \lambda_{a_k}(0) \right) \\ &\quad \underbrace{i^m \left(dz^1 \wedge dz^{\bar{1}} \right) \wedge \cdots \wedge \left(dz^m \wedge dz^{\bar{m}} \right)}_{= \frac{\omega^m}{m!}} \\ &= \left(\sum_{\substack{(a_1, \dots, a_k) \in \{1, \dots, m\} \\ \text{distinct integers}}} (m-k)! \lambda_{a_1}(0) \cdots \lambda_{a_k}(0) \right) \frac{\omega^m}{m!} \\ \tilde{\omega}^k \wedge \omega^{m-k} &= \frac{(m-k)!}{m!} k! \sigma_k(\lambda_1(0), \dots, \lambda_m(0)) \omega^m = \frac{\sigma_k(\lambda(g^{-1}\tilde{g}))}{\binom{m}{k}} \omega^m.\end{aligned}\tag{1.5}$$

□

Consequently, (E_k) writes:

$$\sigma_k\left(\lambda\left(g^{-1}\tilde{g}\right)\right) = e^f.\tag{E'_k}$$

Let us remark that E_m corresponds to the Calabi-Yau equation $\det(\tilde{g})/\det(g) = e^f$, when E_1 is just a linear equation in Laplacian form. Since the endomorphism A is Hermitian, the spectral theorem provides an h -orthonormal basis for $T^{1,0}$ of eigenvectors $e_1, \dots, e_m$: $Ae_i = \lambda_i e_i$, $\lambda = (\lambda_1, \dots, \lambda_m) \in \Gamma_k$. At $P \in M$ in a chart z , we have $\text{Mat}_{\partial_1, \dots, \partial_m} A_P = [A_j^i(z)]_{1 \leq i, j \leq m}$, thus

$\sigma_k(\lambda(A_P)) = \sigma_k(\lambda([A_j^i(z)]_{1 \leq i, j \leq m}))$. In addition, $A_i^j = g^{j\bar{\ell}} \tilde{g}_{i\bar{\ell}} = g^{j\bar{\ell}}(g_{i\bar{\ell}} + \partial_{i\bar{\ell}}\varphi) = \delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi$, so the equation writes locally:

$$\sigma_k\left(\lambda\left(\left[\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right]_{1 \leq i, j \leq m}\right)\right) = e^f. \quad (E''_k)$$

Let us notice that a solution of this equation (E''_k) is necessarily k -admissible [9, page 46]. Let us define $f_k(B) = \sigma_k(\lambda(B))$ and $F_k(B) = \ln \sigma_k(\lambda(B))$ where $B = [B_i^j]_{1 \leq i, j \leq m}$ is a Hermitian matrix. The function f_k is a polynomial in the variables B_i^j , specifically $f_k(B) = \sum_{|I|=k} B_{II}$ (sum of the principal minors of order k of the matrix B). Equivalently (E''_k) writes:

$$F_k\left(\left[\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi\right]_{1 \leq i, j \leq m}\right) = f. \quad (E'''_k)$$

It is a nonlinear elliptic second order PDE of complex Monge-Ampère type. We prove the existence of a k -admissible solution by the continuity method.

2. Derivatives and Concavity of F_k

2.1. Calculation of the Derivatives at a Diagonal Matrix

The first derivatives of the symmetric polynomial σ_k are given by the following: for all $1 \leq i \leq m$, $(\partial\sigma_k/\partial\lambda_i)(\lambda) = \sigma_{k-1,i}(\lambda)$ where $\sigma_{k-1,i}(\lambda) := \sigma_{k-1}|_{\lambda_i=0}$. For $1 \leq i \neq j \leq m$, let us denote $\sigma_{k-2,ij}(\lambda) := \sigma_{k-2}|_{\lambda_i=\lambda_j=0}$ and $\sigma_{k-2,ii}(\lambda) = 0$. The second derivatives of the polynomial σ_k are given by $(\partial^2\sigma_k/\partial\lambda_i\partial\lambda_j)(\lambda) = \sigma_{k-2,ij}(\lambda)$. We calculate the derivatives of the function $f_k : \mathcal{H}_m(\mathbb{C}) \rightarrow \mathbb{R}$, where $\mathcal{H}_m(\mathbb{C})$ denotes the set of Hermitian matrices, at diagonal matrices using the formula:

$$\begin{aligned} f_k(B) &= \sum_{1 \leq i_1 < \dots < i_k \leq m} \sum_{\sigma \in S_k} \varepsilon(\sigma) B_{i_1}^{i_{\sigma(1)}} \dots B_{i_k}^{i_{\sigma(k)}} \\ &= \frac{1}{k!} \sum_{1 \leq i_1, \dots, i_k, j_1, \dots, j_k \leq m} \varepsilon_{j_1 \dots j_k}^{i_1 \dots i_k} B_{i_1}^{j_1} \dots B_{i_k}^{j_k}, \end{aligned} \quad (2.1)$$

where

$$\varepsilon_{j_1 \dots j_k}^{i_1 \dots i_k} = \begin{cases} 1 & \text{if } i_1, \dots, i_k \text{ distinct and } j_1, \dots, j_k \text{ even permutation of } i_1, \dots, i_k, \\ -1 & \text{if } i_1, \dots, i_k \text{ distinct and } j_1, \dots, j_k \text{ odd permutation of } i_1, \dots, i_k, \\ 0 & \text{else.} \end{cases} \quad (2.2)$$

These derivatives are given by [9, page 48]

$$\begin{aligned} \frac{\partial f_k}{\partial B_i^j}(\text{diag}(b_1, \dots, b_m)) &= \begin{cases} 0 & \text{if } i \neq j, \\ \sigma_{k-1,i}(b_1, \dots, b_m) & \text{if } i = j, \end{cases} \\ \text{if } i \neq j \quad \frac{\partial^2 f_k}{\partial B_j^i \partial B_i^j}(\text{diag}(b_1, \dots, b_m)) &= \sigma_{k-2,ij}(b_1, \dots, b_m) \\ \frac{\partial^2 f_k}{\partial B_j^i \partial B_i^j}(\text{diag}(b_1, \dots, b_m)) &= -\sigma_{k-2,ij}(b_1, \dots, b_m), \end{aligned} \quad (2.3)$$

and all the other second derivatives of f_k at $\text{diag}(b_1, \dots, b_m)$ vanish.

Consequently, the derivatives of the function $F_k = \ln f_k : \lambda^{-1}(\Gamma_k) \subset \mathcal{H}_m(\mathbb{C}) \rightarrow \mathbb{R}$ at diagonal matrices $\text{diag}(\lambda_1, \dots, \lambda_m)$ with $\lambda = (\lambda_1, \dots, \lambda_m) \in \Gamma_k$, where $\lambda^{-1}(\Gamma_k) = \{B \in \mathcal{H}_m(\mathbb{C}) / \lambda(B) \in \Gamma_k\}$, are given by

$$\frac{\partial F_k}{\partial B_i^j}(\text{diag}(\lambda_1, \dots, \lambda_m)) = \begin{cases} 0 & \text{if } i \neq j, \\ \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} & \text{if } i = j, \end{cases} \quad (2.4)$$

$$\begin{aligned} \text{if } i \neq j \quad \frac{\partial^2 F_k}{\partial B_j^i \partial B_i^j}(\text{diag}(\lambda_1, \dots, \lambda_m)) &= -\frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)} \\ \frac{\partial^2 F_k}{\partial B_j^i \partial B_i^j}(\text{diag}(\lambda_1, \dots, \lambda_m)) &= \frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)} - \frac{\sigma_{k-1,i}(\lambda)\sigma_{k-1,j}(\lambda)}{(\sigma_k(\lambda))^2} \\ \frac{\partial^2 F_k}{\partial B_i^i \partial B_i^i}(\text{diag}(\lambda_1, \dots, \lambda_m)) &= -\frac{(\sigma_{k-1,i}(\lambda))^2}{(\sigma_k(\lambda))^2} \end{aligned} \quad (2.5)$$

and all the other second derivatives of F_k at $\text{diag}(\lambda_1, \dots, \lambda_m)$ vanish.

2.2. The Invariance of F_k and of Its First and Second Differentials

The function $F_k : \lambda^{-1}(\Gamma_k) \rightarrow \mathbb{R}$ is invariant under unitary similitudes:

$$\forall B \in \lambda^{-1}(\Gamma_k), \quad \forall U \in U_m(\mathbb{C}), \quad F_k(B) = F_k({}^t \bar{U} B U). \quad (2.6)$$

Differentiating the previous invariance formula (2.6), we show that the first and second differentials of F_k are also invariant under unitary similitudes:

$$\begin{aligned} \forall B \in \lambda^{-1}(\Gamma_k), \quad \forall \zeta \in \mathcal{H}_m(\mathbb{C}), \quad \forall U \in U_m(\mathbb{C}), \\ (dF_k)_B \cdot \zeta = (dF_k)_{{}^t\overline{U}BU} \cdot ({}^t\overline{U}\zeta U), \end{aligned} \quad (2.7)$$

$$\begin{aligned} \forall B \in \lambda^{-1}(\Gamma_k), \quad \forall \zeta \in \mathcal{H}_m(\mathbb{C}), \quad \forall \Theta \in \mathcal{H}_m(\mathbb{C}), \quad \forall U \in U_m(\mathbb{C}), \\ (d^2F_k)_B \cdot (\zeta, \Theta) = (d^2F_k)_{{}^t\overline{U}BU} \cdot ({}^t\overline{U}\zeta U, {}^t\overline{U}\Theta U). \end{aligned} \quad (2.8)$$

These invariance formulas are allowed to come down to the diagonal case, when it is useful.

2.3. Concavity of F_k

We prove in [9] the concavity of the functions $u \circ \lambda$ and more generally $u \circ \lambda_B$ when $u \in \Gamma_0(\mathbb{R}^m)$ and is symmetric [9, Theorem VII.4.2], which in particular gives the concavity of the functions $F_k = \ln \sigma_k \lambda$ [9, Corollary VII.4.30] and more generally $\ln \sigma_k \lambda_B$ [9, Theorem VII.4.29]. In this section, let us show by an elementary calculation the concavity of the function F_k .

Proposition 2.1. *The function $F_k : \lambda^{-1}(\Gamma_k) \rightarrow \mathbb{R}$, $B \mapsto F_k(B) = \ln \sigma_k(\lambda(B))$ is concave (this holds for all $k \in \{1, \dots, m\}$).*

Proof. The function F_k is of class C^2 , so its concavity is equivalent to the following inequality:

$$\forall B \in \lambda^{-1}(\Gamma_k), \quad \forall \zeta \in \mathcal{H}_m(\mathbb{C}) \quad \sum_{i,j,r,s=1}^m \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j}(B) \zeta_i^j \zeta_r^s \leq 0. \quad (2.9)$$

Let $B \in \lambda^{-1}(\Gamma_k)$, $\zeta \in \mathcal{H}_m(\mathbb{C})$, and $U \in U_m(\mathbb{C})$ such that ${}^t\overline{U}BU = \text{diag}(\lambda_1, \dots, \lambda_m)$. We have $\lambda = (\lambda_1, \dots, \lambda_m) \in \Gamma_k$. Let us denote $\tilde{\zeta} = {}^t\overline{U}\zeta U \in \mathcal{H}_m(\mathbb{C})$:

$$\begin{aligned} S &:= \sum_{i,j,r,s=1}^m \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j}(B) \zeta_i^j \zeta_r^s \\ &= (d^2F_k)_B \cdot (\zeta, \zeta) \quad \text{so by the invariance formula (2.8)} \\ &= (d^2F_k)_{{}^t\overline{U}BU} \cdot ({}^t\overline{U}\zeta U, {}^t\overline{U}\zeta U) \\ &= \sum_{i,j,r,s=1}^m \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j}(\text{diag}(\lambda_1, \dots, \lambda_m)) \tilde{\zeta}_i^j \tilde{\zeta}_r^s \\ &= \sum_{i \neq j=1}^m -\frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)} \tilde{\zeta}_i^j \underbrace{\tilde{\zeta}_j^i}_{=\tilde{\zeta}_i^j} \end{aligned}$$

$$\begin{aligned}
& + \sum_{i \neq j=1}^m \underbrace{\left(\frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)} - \frac{\sigma_{k-1,i}(\lambda)\sigma_{k-1,j}(\lambda)}{(\sigma_k(\lambda))^2} \right)}_{=:c_{ij}} \tilde{\zeta}_i^i \tilde{\zeta}_j^j + \sum_{i=1}^m -\frac{(\sigma_{k-1,i}(\lambda))^2}{(\sigma_k(\lambda))^2} (\tilde{\zeta}_i^i)^2 \\
& = \sum_{i,j=1}^m -\frac{\sigma_{k-2,ij}(\lambda)}{\sigma_k(\lambda)} |\tilde{\zeta}_i^j|^2 + \sum_{i,j=1}^m c_{ij} \tilde{\zeta}_i^i \tilde{\zeta}_j^j.
\end{aligned} \tag{2.10}$$

But $c_{ij} = (\partial^2(\ln \sigma_k)/\partial \lambda_i \partial \lambda_j)(\lambda)$, and $\tilde{\zeta}_i^i \in \mathbb{R}$, so $\sum_{i,j=1}^m c_{ij} \tilde{\zeta}_i^i \tilde{\zeta}_j^j \leq 0$ by concavity of $\ln \sigma_k$ at $\lambda \in \Gamma_k$ [10, page 269]. In addition, $\sigma_{k-2,ij}(\lambda) > 0$ since $\lambda \in \Gamma_k$ [11], consequently $\sum_{i,j=1}^m -(\sigma_{k-2,ij}(\lambda)/\sigma_k(\lambda)) |\tilde{\zeta}_i^j|^2 \leq 0$, which shows that $S \leq 0$ and achieves the proof. $\square$

3. The Proof of Uniqueness

Let φ_0 and φ_1 be two smooth k -admissible solutions of (E_k''') such that $\int_M \varphi_0 \omega^m = \int_M \varphi_1 \omega^m = 0$. For all $t \in [0, 1]$, let us consider the function $\varphi_t = t\varphi_1 + (1-t)\varphi_0 = \varphi_0 + t\varphi$ with $\varphi = \varphi_1 - \varphi_0$. Let $P \in M$, and let us denote $h_k^P(t) = f_k([\delta_i^j + g^{j\bar{\ell}}(P)\partial_{i\bar{\ell}}\varphi_t(P)])$. We have $h_k^P(1) - h_k^P(0) = 0$ which is equivalent to $\int_0^1 h_k^{P'}(t) dt = 0$. But

$$h_k^{P'}(t) = \sum_{i,j=1}^m \underbrace{\left(\sum_{\ell=1}^m \frac{\partial f_k}{\partial B_i^{\ell}}([\delta_i^j + g^{j\bar{\ell}}(P)\partial_{i\bar{\ell}}\varphi_t(P)]) g^{\ell\bar{j}}(P) \right)}_{=: \alpha_{ij}^t(P)} \partial_{i\bar{j}}\varphi(P). \tag{3.1}$$

Therefore we obtain

$$\mathcal{L}\varphi(P) := \sum_{i,j=1}^m a_{ij}(P) \partial_{i\bar{j}}\varphi(P) = 0 \quad \text{with } a_{ij}(P) = \int_0^1 \alpha_{ij}^t(P) dt. \tag{3.2}$$

We show easily that the matrix $[a_{ij}(P)]_{1 \leq i, j \leq m}$ is Hermitian [9, page 53]. Besides the function φ is continuous on the compact manifold M so it assumes its minimum at a point $m_0 \in M$, so that the complex Hessian matrix of φ at the point m_0 , namely, $[\partial_{i\bar{j}}\varphi(m_0)]_{1 \leq i, j \leq 2m'}$ is positive-semidefinite.

Lemma 3.1. *For all $t \in [0, 1]$, $\lambda(g^{-1}\tilde{g}_{\varphi_t})(m_0) \in \Gamma_k$; namely, the functions $(\varphi_t)_{t \in [0,1]}$ are k -admissible at m_0 .*

Proof. Let us denote $\mathcal{W} := \{t \in [0, 1] / \lambda(g^{-1}\tilde{g}_{\varphi_t})(m_0) \in \Gamma_k\}$. The set $\mathcal{W}$ is nonempty, it contains 0, and it is an open subset of $[0, 1]$. Let t be the largest number of $[0, 1]$ such that $[0, t] \subset \mathcal{W}$. Let us suppose that $t < 1$ and show that we get a contradiction. Let $1 \leq q \leq k$, we have $\sigma_q(\lambda(g^{-1}\tilde{g}_{\varphi_t})(m_0)) - \sigma_q(\lambda(g^{-1}\tilde{g}_{\varphi_0})(m_0)) = h_q^{m_0}(t) - h_q^{m_0}(0) = \int_0^t h_q^{m_0'}(s) ds$. Let us prove that

$h_q^{m_0}(s) \geq 0$ for all $s \in [0, t[$. Fix $s \in [0, t[$; the quantity $h_q^{m_0}(s)$ is intrinsic so it suffices to prove the assertion in a particular chart at m_0 . Now at m_0 in a g -unitary $\tilde{g}_{\varphi_s}$ -adapted chart at m_0

$$\begin{aligned} h_q^{m_0}(s) &= \sum_{i,j,\ell=1}^m \frac{\partial f_q}{\partial B_i^j} \left([\delta_i^j + g^{j\bar{\ell}}(m_0) \partial_{i\bar{\ell}} \varphi_s(m_0)] \right) g^{j\bar{\ell}}(m_0) \partial_{i\bar{\ell}} \varphi(m_0) \\ &= \sum_{i=1}^m \frac{\partial \sigma_q}{\partial \lambda_i} \left(\lambda(g^{-1} \tilde{g}_{\varphi_s})(m_0) \right) \partial_{i\bar{i}} \varphi(m_0). \end{aligned} \quad (3.3)$$

But $\lambda(g^{-1} \tilde{g}_{\varphi_s})(m_0) \in \Gamma_k \subset \Gamma_q$ since $s \in [0, t[\subset \mathcal{W}$, then $(\partial \sigma_q / \partial \lambda_i)(\lambda(g^{-1} \tilde{g}_{\varphi_s})(m_0)) > 0$ for all $1 \leq i \leq m$. Besides, $\partial_{i\bar{i}} \varphi(m_0) \geq 0$ since the matrix $[\partial_{i\bar{j}} \varphi(m_0)]_{1 \leq i, j \leq m}$ is positive-semi-definite. Therefore, we infer that $h_q^{m_0}(s) \geq 0$. Consequently, we obtain that $\sigma_q(\lambda(g^{-1} \tilde{g}_{\varphi_t})(m_0)) \geq \sigma_q(\lambda(g^{-1} \tilde{g}_{\varphi_0})(m_0)) > 0$ (since φ_0 is k -admissible). This holds for all $1 \leq q \leq k$; we deduce then that $\lambda(g^{-1} \tilde{g}_{\varphi_t})(m_0) \in \Gamma_k$ which proves that $t \in \mathcal{W}$. This is a contradiction; we infer then that $\mathcal{W} = [0, 1]$. $\square$

We check easily that the Hermitian matrix $[a_{ij}(m_0)]_{1 \leq i, j \leq m}$ is positive definite [9, page 54] and deduce then the following lemma since the map $P \mapsto a_{ij}(P) = \int_0^1 (\sum_{\ell=1}^m (\partial f_k / \partial B_i^\ell) ([\delta_i^j + g^{j\bar{\ell}}(P) \partial_{i\bar{\ell}} \varphi_t(P)] g^{\ell\bar{j}}(P)) dt$ is continuous on a neighbourhood of m_0 .

Lemma 3.2. *There exists an open ball B_{m_0} centered at m_0 such that for all $P \in B_{m_0}$ the Hermitian matrix $[a_{ij}(P)]_{1 \leq i, j \leq m}$ is positive definite.*

Consequently, the operator $\mathcal{L}$ is elliptic on the open set B_{m_0} . But the map φ is C^∞ , assumes its minimum at $m_0 \in B_{m_0}$, and satisfies $\mathcal{L}\varphi = 0$; then by the Hopf maximum principle [12], we deduce that $\varphi(P) = \varphi(m_0)$ for all $P \in B_{m_0}$. Let us denote $\mathcal{S} := \{P \in M / \varphi(P) = \varphi(m_0)\}$. This set is nonempty and it is a closed set. Let us prove that $\mathcal{S}$ is an open set: let m be a point of $\mathcal{S}$, so $\varphi(m) = \varphi(m_0)$, then the map φ assumes its minimum at the point m . Therefore, by the same proof as for the point m_0 , we infer that there exists an open ball B_m centered at m such that for all $P \in B_m$ $\varphi(P) = \varphi(m)$ so for all $P \in B_m$ $\varphi(P) = \varphi(m_0)$ then $B_m \subset \mathcal{S}$, which proves that $\mathcal{S}$ is an open set. But the manifold M is connected; then $\mathcal{S} = M$, namely, $\varphi(P) = \varphi(m_0)$ for all $P \in M$. Besides $\int_M \varphi \omega^m = 0$, therefore we deduce that $\varphi \equiv 0$ on M namely that $\varphi_1 \equiv \varphi_0$ on M , which achieves the proof of uniqueness.

4. The Continuity Method

Let us consider the one parameter family of $(E_{k,t})$, $t \in [0, 1]$

$$\mathcal{F}_k[\varphi_t] := F_k \left([\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi_t]_{1 \leq i, j \leq m} \right) = t f + \ln \left(\frac{\binom{m}{k} \int_M \omega^m}{\underbrace{\int_M e^{t f} \omega^m}_{A_t}} \right). \quad (E_{k,t})$$

The function $\varphi_0 \equiv 0$ is a k -admissible solution of $(E_{k,0})$: $\sigma_k(\lambda([\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi_0]_{1 \leq i, j \leq m})) = \binom{m}{k}$ and satisfies $\int_M \varphi_0 \omega^m = 0$. For $t = 1$, $A_1 = 1$ so $(E_{k,1})$ corresponds to (E_k^m) .

Let us fix $l \in \mathbb{N}$, $l \geq 5$ and $0 < \alpha < 1$, and let us consider the nonempty set (containing 0):

$$\begin{aligned} \mathcal{T}_{l,\alpha} := \left\{ t \in [0, 1] / (E_{k,t}) \text{ have a } k\text{-admissible solution } \varphi \in C^{l,\alpha}(M) \right. \\ \left. \text{such that } \int_M \varphi \omega^m = 0 \right\}. \end{aligned} \quad (4.1)$$

The aim is to prove that $1 \in \mathcal{T}_{l,\alpha}$. For this we prove, using the connectedness of $[0, 1]$, that $\mathcal{T}_{l,\alpha} = [0, 1]$.

4.1. $\mathcal{T}_{l,\alpha}$ Is an Open Set of $[0, 1]$

This arises from the local inverse mapping theorem and from solving a linear problem. Let us consider the following sets:

$$\begin{aligned} \tilde{S}_{l,\alpha} &:= \left\{ \varphi \in C^{l,\alpha}(M), \int_M \varphi \omega^m = 0 \right\}, \\ S_{l,\alpha} &:= \left\{ \varphi \in \tilde{S}_{l,\alpha}, k\text{-admissible for } g \right\}, \end{aligned} \quad (4.2)$$

where $\tilde{S}_{l,\alpha}$ is a vector space and $S_{l,\alpha}$ is an open set of $\tilde{S}_{l,\alpha}$. Using these notations, the set $\mathcal{T}_{l,\alpha}$ writes $\mathcal{T}_{l,\alpha} := \{ t \in [0, 1] / \exists \varphi \in S_{l,\alpha} \text{ solution of } (E_{k,t}) \}$.

Lemma 4.1. *The operator $\mathcal{F}_k : S_{l,\alpha} \rightarrow C^{l-2,\alpha}(M)$, $\varphi \mapsto \mathcal{F}_k[\varphi] = F_k([\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi]_{1 \leq i, j \leq m})$, is differentiable, and its differential at a point $\varphi \in S_{l,\alpha}$, $d\mathcal{F}_k \varphi \in \mathcal{L}(\tilde{S}_{l,\alpha}, C^{l-2,\alpha}(M))$ is equal to*

$$d\mathcal{F}_k \varphi \cdot \psi = \sum_{i,j=1}^m \frac{\partial F_k}{\partial B_i^j} \left([\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi] \right) g^{j\bar{\ell}} \partial_{i\bar{\ell}} \psi \quad \forall \psi \in \tilde{S}_{l,\alpha}. \quad (4.3)$$

Proof. See [9, page 60]. □

Proposition 4.2. *The nonlinear operator $\mathcal{F}_k$ is elliptic on $S_{l,\alpha}$.*

Proof. Let us fix a function $\varphi \in S_{l,\alpha}$ and check that the nonlinear operator $\mathcal{F}_k$ is elliptic for this function φ . This goes back to show that the linearization at φ of the nonlinear operator $\mathcal{F}_k$ is elliptic. By Lemma 4.1, this linearization is the following linear operator:

$$d\mathcal{F}_k \varphi \cdot v = \sum_{i,o=1}^m \left(\sum_{j=1}^m \frac{\partial F_k}{\partial B_i^j} [\delta_i^j + g^{j\bar{o}} \partial_{i\bar{o}} \varphi]_{1 \leq i, j \leq m} \times g^{j\bar{o}} \right) \partial_{i\bar{o}} v. \quad (4.4)$$

In order to prove that this linear operator is elliptic, it suffices to check the ellipticity in a particular chart, for example, at the center of a g -normal $\tilde{g}_\varphi$ -adapted chart. At the center of such a chart,

$$d\mathcal{F}_{k\varphi} \cdot v = \sum_{i,o=1}^m \left(\frac{\partial F_k}{\partial B_i^o} \left(\text{diag } \lambda(g^{-1}\tilde{g}) \right) \right) \partial_{i\bar{o}} v = \sum_{i=1}^m \frac{\sigma_{k-1,i} \lambda(g^{-1}\tilde{g})}{\sigma_k \lambda(g^{-1}\tilde{g})} \partial_{i\bar{i}} v. \quad (4.5)$$

But for all $i \in \{1, \dots, m\}$ we have $\sigma_{k-1,i} \lambda(g^{-1}\tilde{g}) / \sigma_k \lambda(g^{-1}\tilde{g}) > 0$ on M since $\lambda(g^{-1}\tilde{g}) \in \Gamma_k$ [11], which proves that the linearization is elliptic and achieves the proof. $\square$

Let us denote $\mathfrak{F}_k$ the operator

$$\mathfrak{F}_k[\varphi] := f_k \left(\left[\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi \right]_{1 \leq i, j \leq m} \right). \quad (4.6)$$

As $\mathcal{F}_k$, the operator $\mathfrak{F}_k : S_{l,\alpha} \rightarrow C^{l-2,\alpha}(M)$ is differentiable and elliptic on $S_{l,\alpha}$ of differential

$$d\mathfrak{F}_{k\varphi} \cdot \psi = \sum_{i,j=1}^m \frac{\partial f_k}{\partial B_i^j} \left(\left[\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi \right] \right) g^{i\bar{\ell}} \partial_{i\bar{\ell}} \psi \quad \forall \psi \in \tilde{S}_{l,\alpha}. \quad (4.7)$$

Let us denote a_φ the matrix $[\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi]_{1 \leq i, j \leq m}$ and calculate this linearization in a different way, by using the expression (2.1) of f_k :

$$\mathfrak{F}_k[\varphi] = f_k(a_\varphi) = \frac{1}{k!} \sum_{1 \leq i_1, \dots, i_k, j_1, \dots, j_k \leq m} \varepsilon_{j_1 \dots j_k}^{i_1 \dots i_k} (a_\varphi)_{i_1}^{j_1} \dots (a_\varphi)_{i_k}^{j_k}. \quad (4.8)$$

Thus

$$\begin{aligned} d\mathfrak{F}_{k\varphi} \cdot v &= \frac{d}{dt} (\mathfrak{F}_k[\varphi + tv])|_{t=0} \\ &= \frac{d}{dt} \left(\frac{1}{k!} \sum_{1 \leq i_1, \dots, i_k, j_1, \dots, j_k \leq m} \varepsilon_{j_1 \dots j_k}^{i_1 \dots i_k} (a_{\varphi+tv})_{i_1}^{j_1} \dots (a_{\varphi+tv})_{i_k}^{j_k} \right) \Big|_{t=0} \\ &= \frac{1}{k!} \sum_{1 \leq i_1, \dots, i_k, j_1, \dots, j_k \leq m} \varepsilon_{j_1 \dots j_k}^{i_1 \dots i_k} \left(g^{j_1 \bar{s}} \partial_{i_1 \bar{s}} v \right) (a_\varphi)_{i_2}^{j_2} \dots (a_\varphi)_{i_k}^{j_k} \\ &\quad + \frac{1}{k!} \sum_{1 \leq i_1, \dots, i_k, j_1, \dots, j_k \leq m} \varepsilon_{j_1 \dots j_k}^{i_1 \dots i_k} (a_\varphi)_{i_1}^{j_1} \left(g^{j_2 \bar{s}} \partial_{i_2 \bar{s}} v \right) \dots (a_\varphi)_{i_k}^{j_k} \\ &\quad + \dots + \frac{1}{k!} \sum_{1 \leq i_1, \dots, i_k, j_1, \dots, j_k \leq m} \varepsilon_{j_1 \dots j_k}^{i_1 \dots i_k} (a_\varphi)_{i_1}^{j_1} \dots (a_\varphi)_{i_{k-1}}^{j_{k-1}} \left(g^{j_k \bar{s}} \partial_{i_k \bar{s}} v \right) \\ &= \frac{1}{(k-1)!} \sum_{1 \leq i_1, \dots, i_k, j_1, \dots, j_k \leq m} \varepsilon_{j_1 \dots j_k}^{i_1 \dots i_k} (a_\varphi)_{i_1}^{j_1} \dots (a_\varphi)_{i_{k-1}}^{j_{k-1}} \left(g^{j_k \bar{s}} \partial_{i_k \bar{s}} v \right) \end{aligned}$$

by symmetry

$$= \sum_{i,j=1}^m \underbrace{\left(\frac{1}{(k-1)!} \sum_{1 \leq i_1, \dots, i_{k-1}, j_1, \dots, j_{k-1} \leq m} \varepsilon_{j_1 \dots j_{k-1} j}^{i_1 \dots i_{k-1} i} (a_\varphi)_{i_1}^{j_1} \dots (a_\varphi)_{i_{k-1}}^{j_{k-1}} \right)}_{=: C_j^i(a_\varphi)} \nabla_i^j v. \quad (4.9)$$

We infer then the following proposition.

Proposition 4.3. *The linearization $d\mathfrak{F}_k$ of the operator $\mathfrak{F}_k$ is of divergence type:*

$$d\mathfrak{F}_{k\varphi} = \nabla_i (C_j^i(a_\varphi) \nabla^j). \quad (4.10)$$

Proof. By (4.9) we have

$$\begin{aligned} d\mathfrak{F}_{k\varphi} \cdot v &= \sum_{i,j=1}^m C_j^i(a_\varphi) \nabla_i^j v \\ &= \sum_{i=1}^m \nabla_i \left(\sum_{j=1}^m C_j^i(a_\varphi) \nabla^j v \right) - \sum_{j=1}^m \left(\sum_{i=1}^m \nabla_i (C_j^i(a_\varphi)) \right) \nabla^j v. \end{aligned} \quad (4.11)$$

Moreover

$$\sum_{i=1}^m \nabla_i (C_j^i(a_\varphi)) = \frac{1}{(k-2)!} \sum_{i=1}^m \sum_{1 \leq i_1, \dots, i_{k-1}, j_1, \dots, j_{k-1} \leq m} \varepsilon_{j_1 \dots j_{k-1} j}^{i_1 \dots i_{k-1} i} (a_\varphi)_{i_1}^{j_1} \dots (a_\varphi)_{i_{k-2}}^{j_{k-2}} \nabla_i ((a_\varphi)_{i_{k-1}}^{j_{k-1}}). \quad (4.12)$$

But $\nabla_i ((a_\varphi)_{i_{k-1}}^{j_{k-1}}) = \nabla_i (\delta_{i_{k-1}}^{j_{k-1}} + \nabla_{i_{k-1}}^{j_{k-1}} \varphi) = \nabla_{ii_{k-1}}^{j_{k-1}} \varphi$, then

$$\sum_{i=1}^m \nabla_i (C_j^i(a_\varphi)) = \frac{1}{(k-2)!} \sum_{i=1}^m \sum_{1 \leq i_1, \dots, i_{k-1}, j_1, \dots, j_{k-1} \leq m} \varepsilon_{j_1 \dots j_{k-1} j}^{i_1 \dots i_{k-1} i} (a_\varphi)_{i_1}^{j_1} \dots (a_\varphi)_{i_{k-2}}^{j_{k-2}} \nabla_{ii_{k-1}}^{j_{k-1}} \varphi. \quad (4.13)$$

Besides, the quantity $\nabla_{ii_{k-1}}^{j_{k-1}} \varphi$ is symmetric in i, i_{k-1} (indeed, $\nabla_{ii_{k-1}}^{j_{k-1}} \varphi - \nabla_{i_{k-1}i}^{j_{k-1}} \varphi = R_{sii_{k-1}}^{j_{k-1}} \nabla^s \varphi$ and $R_{sii_{k-1}}^{j_{k-1}} = 0$ since g is Kähler), and $\varepsilon_{j_1 \dots j_{k-1} j}^{i_1 \dots i_{k-1} i}$ is antisymmetric in i, i_{k-1} ; it follows then that $\sum_{i=1}^m \nabla_i (C_j^i(a_\varphi)) = 0$, consequently $d\mathfrak{F}_{k\varphi} \cdot v = \sum_{i=1}^m \nabla_i (\sum_{j=1}^m C_j^i(a_\varphi) \nabla^j v)$. $\square$

From Proposition 4.3, we infer easily [9, page 62] the following corollary.

Corollary 4.4. *The map $F : S_{l,\alpha} \rightarrow \tilde{S}_{l-2,\alpha}, \varphi \mapsto F(\varphi) = \mathfrak{F}_k[\varphi] - \binom{m}{k}$ is well defined and differentiable and its differential equals $dF_\varphi = d\mathfrak{F}_{k\varphi} = \nabla_i (C_j^i(a_\varphi) \nabla^j) \in \mathcal{L}(\tilde{S}_{l,\alpha}, \tilde{S}_{l-2,\alpha})$.*

Now, let $t_0 \in \mathcal{T}_{l,\alpha}$ and let $\varphi_0 \in S_{l,\alpha}$ be a solution of the corresponding equation $(E_{k,t_0}) : F(\varphi_0) = e^{t_0 f} A_{t_0} - \binom{m}{k}$.

Lemma 4.5. $dF_{\varphi_0} : \tilde{S}_{l,\alpha} \rightarrow \tilde{S}_{l-2,\alpha}$ is an isomorphism.

Proof. Let $\psi \in C^{l-2,\alpha}(M)$ with $\int_M \psi v_g = 0$. Let us consider the equation

$$\nabla_i \left(C_j^i(a_{\varphi_0}) \nabla^j u \right) = \psi. \quad (4.14)$$

We have $C_j^i(a_{\varphi_0}) \in C^{l-2,\alpha}(M)$ and the matrix $[C_j^i(a_{\varphi_0})]_{1 \leq i, j \leq m} = [(\partial f_k / \partial B_i^j)([\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi_0])]_{1 \leq i, j \leq m}$ is positive definite (since $\mathfrak{F}_k$ is elliptic at φ_0); then by Theorem 4.7 of [13, p. 104] on the operators of divergence type, we deduce that there exists a unique function $u \in C^{l,\alpha}(M)$ satisfying $\int_M u v_g = 0$ which is solution of (4.14) and then solution of $dF_{\varphi_0} u = \psi$. Thus, the linear continuous map $dF_{\varphi_0} : \tilde{S}_{l,\alpha} \rightarrow \tilde{S}_{l-2,\alpha}$ is bijective, and its inverse is continuous by the open map theorem, which achieves the proof. $\square$

We deduce then by the local inverse mapping theorem that there exists an open set U of $S_{l,\alpha}$ containing φ_0 and an open set V of $\tilde{S}_{l-2,\alpha}$ containing $F(\varphi_0)$ such that $F : U \rightarrow V$ is a diffeomorphism. Now, let us consider a real number $t \in [0, 1]$ very close to t_0 and let us check that it belongs also to $\mathcal{T}_{l,\alpha}$: if $|t - t_0| \leq \varepsilon$ is sufficiently small then $\|(e^{tf} A_t - (\frac{m}{k})) - (e^{t_0 f} A_{t_0} - (\frac{m}{k}))\|_{C^{l-2,\alpha}(M)}$ is small enough so that $e^{tf} A_t - (\frac{m}{k}) \in V$, thus there exists $\varphi \in U \subset S_{l,\alpha}$ such that $F(\varphi) = e^{tf} A_t - (\frac{m}{k})$ and consequently there exists $\varphi \in C^{l,\alpha}(M)$ of vanishing integral for g which is solution of $(E_{k,t})$. Hence $t \in \mathcal{T}_{l,\alpha}$. We conclude therefore that $\mathcal{T}_{l,\alpha}$ is an open set of $[0, 1]$.

4.2. $\mathcal{T}_{l,\alpha}$ Is a Closed Set of $[0, 1]$: The Scheme of the Proof

This section is based on a priori estimates. Finding these estimates is the most difficult step of the proof. Let $(t_s)_{s \in \mathbb{N}}$ be a sequence of elements of $\mathcal{T}_{l,\alpha}$ that converges to $\tau \in [0, 1]$, and let $(\varphi_{t_s})_{s \in \mathbb{N}}$ be the corresponding sequence of functions: φ_{t_s} is $C^{l,\alpha}$, k -admissible, has a vanishing integral, and is a solution of

$$F_k \left(\left[\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi_{t_s} \right]_{1 \leq i, j \leq m} \right) = t_s f + \ln(A_{t_s}). \quad (E_{k,t_s})$$

Let us prove that $\tau \in \mathcal{T}_{l,\alpha}$. Here is the scheme of the proof.

- (1) Reduction to a $C^{2,\beta}(M)$ estimate: if $(\varphi_{t_s})_{s \in \mathbb{N}}$ is bounded in a $C^{2,\beta}(M)$ with $0 < \beta < 1$, the inclusion $C^{2,\beta}(M) \subset C^2(M, \mathbb{R})$ being compact, we deduce that after extraction $(\varphi_{t_s})_{s \in \mathbb{N}}$ converges in $C^2(M, \mathbb{R})$ to $\varphi_\tau \in C^2(M, \mathbb{R})$. We show by tending to the limit that φ_τ is a solution of $(E_{k,\tau})$ (it is then necessarily k -admissible) and of vanishing integral for g . We check finally by a nonlinear regularity theorem [14, page 467] that $\varphi_\tau \in C^\infty(M, \mathbb{R})$, which allows us to deduce that $\tau \in \mathcal{T}_{l,\alpha}$ (see [9, pages 64–67] for details).
- (2) We show that $(\varphi_{t_s})_{s \in \mathbb{N}}$ is bounded in $C^0(M, \mathbb{R})$: first of all we prove a positivity Lemma 5.4 for $(E_{k,t})$, inspired by the ones of [15, page 843] (for $k = m$), but in a very different way, required since the k -positivity of $\tilde{\omega}_{t_s}$ is weaker with $k < m$ (in this case, some eigenvalues can be nonpositive, which complicates the proof), using a polarization method of [7, page 1740] (cf. 5.2) and a Gårding inequality 5.3; we

infer then from this lemma a fundamental inequality 5.5 as Proposition 7.18 of [13, page 262]. We conclude the proof using the Moser's iteration technique exactly as for the equation of Calabi-Yau. We deal with this C^0 estimate in Section 5.

- (3) We establish the key point of the proof, namely, a C^2 a priori estimate (Section 6).
- (4) With the uniform ellipticity at hand (consequence of the previous step), we obtain the needed $C^{2,\beta}(M)$ estimate by the Evans-Trudinger theory (Section 7).

5. The C^0 A Priori Estimate

5.1. The Positivity Lemma

Our first three lemmas are based on the ideas of [7, Section 2].

Lemma 5.1. *Let π be a real $(1-1)$ -form, it then writes $\pi = ip_{a\bar{b}}dz^a \wedge d\bar{z}^b$, with $p_{a\bar{b}} = p(\partial_a, \partial_{\bar{b}})$ where p is the symmetric tensor $p(\mathcal{U}, V) = \pi(\mathcal{U}, JV)$; hence*

$$\forall \ell \leq m \quad \pi^\ell \wedge \omega^{m-\ell} = \frac{\ell!(m-\ell)!}{m!} \sigma_\ell \left(\lambda \left[g^{-1} p \right] \right) \omega^m. \quad (5.1)$$

Proof. The same proof as Lemma 1.4. □

We consider for $1 \leq \ell \leq m$ the map $f_\ell = \sigma_\ell \circ \lambda : \mathcal{H}_m \rightarrow \mathbb{R}$ where $\mathcal{H}_m$ denotes the $\mathbb{R}$ -vector space of Hermitian square matrices of size m . f_ℓ is a real polynomial of degree ℓ and in m^2 real variables. Moreover, it is I hyperbolic (cf. [16] for the proof) and it satisfies $f_\ell(I) = \sigma_\ell(1, \dots, 1) = \binom{m}{\ell} > 0$. Let $\tilde{f}_\ell$ be the totally polarized form of f_ℓ . This polarized form $\tilde{f}_\ell : \underbrace{\mathcal{H}_m \times \dots \times \mathcal{H}_m}_{\ell \text{ times}} \rightarrow \mathbb{R}$ is characterized by the following properties:

- (i) $\tilde{f}_\ell$ is ℓ -linear.
- (ii) $\tilde{f}_\ell$ is symmetric.
- (iii) For all $B \in \mathcal{H}_m$, $\tilde{f}_\ell(B, \dots, B) = f_\ell(B)$.

Using these notations, we infer from Lemma 5.1 that at the center of a g -unitary chart (this guarantees that the matrix $g^{-1}p$ is Hermitian), we have

$$\pi^\ell \wedge \omega^{m-\ell} = \frac{\ell!(m-\ell)!}{m!} f_\ell \left(g^{-1} p \right) \omega^m. \quad (5.2)$$

By transition to the polarized form in this equality we obtain the following lemma.

Lemma 5.2. *Let $1 \leq \ell \leq m$ and $\pi_1, \dots, \pi_\ell$ be real $(1-1)$ -forms. These forms write $\pi_\alpha = i(p_\alpha)_{a\bar{b}}dz^a \wedge d\bar{z}^b$, with $(p_\alpha)_{a\bar{b}} = p_\alpha(\partial_a, \partial_{\bar{b}})$ where p_α is the symmetric tensor $p_\alpha(\mathcal{U}, V) = \pi_\alpha(\mathcal{U}, JV)$. Then, at the center of a g -unitary chart we have*

$$\pi_1 \wedge \dots \wedge \pi_\ell \wedge \omega^{m-\ell} = \frac{\ell!(m-\ell)!}{m!} \tilde{f}_\ell \left(g^{-1} p_1, \dots, g^{-1} p_\ell \right) \omega^m. \quad (5.3)$$

Proof. See [9, page 71]. $\square$

Theorem 5 of Gårding [16] applies to f_ℓ with $2 \leq \ell \leq m$.

Lemma 5.3 (the Gårding inequality for f_ℓ). *Let $2 \leq \ell \leq m$, for all $y^1, \dots, y^\ell \in \Gamma(f_\ell, I)$,*

$$\tilde{f}_\ell(y^1, \dots, y^\ell) \geq f_\ell(y^1)^{1/\ell} \cdots f_\ell(y^\ell)^{1/\ell}. \quad (5.4)$$

Let us recall that $\Gamma(f_\ell, I)$ is the connected component of $\{y \in \mathcal{H}_m / f_\ell(y) > 0\}$ containing I . The same proof as [17, pages 129, 130] implies that

$$\Gamma(f_\ell, I) = \{y \in \mathcal{H}_m / \forall 1 \leq i \leq \ell \ f_i(y) > 0\} = \{y \in \mathcal{H}_m / \lambda(y) \in \Gamma_\ell\} = \lambda^{-1}(\Gamma_\ell). \quad (5.5)$$

Note that the Gårding inequality (Lemma 5.3) holds for $\tilde{\Gamma}(f_\ell, I) = \{y \in \mathcal{H}_m / \forall 1 \leq i \leq \ell \ f_i(y) \geq 0\}$.

Let us now apply the previous lemmas in order to prove the following positivity lemma inspired by the ones of [15, page 843] (for $k = m$); let us emphasize that the proof is very different since the k -positivity is weaker.

Lemma 5.4 (positivity lemma). *Let α be a real 1-form on M and $j \in \{1, \dots, k-1\}$, then the function $f : M \rightarrow \mathbb{R}$ defined by $f\omega^m = {}^tJ\alpha \wedge \alpha \wedge \omega^{m-1-j} \wedge \tilde{\omega}^j$ is nonnegative.*

Proof. Let $1 \leq j \leq k-1$, then $2 \leq \ell = j+1 \leq k$. Let α be a real 1-form, it then writes $\alpha = \alpha_a dz^a + \bar{\alpha}_{\bar{a}} d\bar{z}^{\bar{a}}$. Let $\pi_1 = {}^tJ\alpha \wedge \alpha$, hence $\pi_1(\partial_a, \partial_{\bar{b}}) = \alpha(J\partial_a)\alpha(\partial_{\bar{b}}) - \alpha(J\partial_{\bar{b}})\alpha(\partial_a) = i\alpha_a \bar{\alpha}_{\bar{b}} - (-i)\bar{\alpha}_{\bar{b}} \alpha_a = 2i\alpha_a \bar{\alpha}_{\bar{b}}$. Similarly, we prove that $\pi_1(\partial_a, \partial_b) = \pi_1(\partial_{\bar{a}}, \partial_{\bar{b}}) = 0$, consequently $\pi_1 = \underbrace{i2\alpha_a \bar{\alpha}_{\bar{b}}}_{=: p_{a\bar{b}}} dz^a \wedge d\bar{z}^{\bar{b}}$. Besides, set $\pi_2 = \dots = \pi_{j+1} = \tilde{\omega} = i\tilde{g}_{\bar{a}b} dz^a \wedge d\bar{z}^{\bar{b}}$. Now, let $x \in M$ and ϕ

be a g -unitary chart centered at x . Using Lemma 5.2, we infer that at x in the chart ϕ :

$$\begin{aligned} {}^tJ\alpha \wedge \alpha \wedge \tilde{\omega}^j \wedge \omega^{m-(j+1)} &= \pi_1 \wedge \dots \wedge \pi_{j+1} \wedge \omega^{m-(j+1)} \\ &= \frac{(m-j-1)!(j+1)!}{m!} \tilde{f}_{j+1}(g^{-1}p, g^{-1}\tilde{g}, \dots, g^{-1}\tilde{g}) \omega^m. \end{aligned} \quad (5.6)$$

But at x , $g^{-1}\tilde{g} = \tilde{g} \in \Gamma(f_{j+1}, I)$ and $g^{-1}p = p \in \tilde{\Gamma}(f_{j+1}, I)$. Indeed, $\lambda(g^{-1}\tilde{g}) \in \Gamma_k$ since φ is k -admissible and $\Gamma_k \subset \Gamma_{j+1}$. Moreover, the Hermitian matrix $[2\alpha_a \bar{\alpha}_{\bar{b}}]_{1 \leq a, b \leq m}$ is positive-semidefinite since for all $\xi \in \mathbb{C}^m$, we have $\sum_{a,b=1}^m 2\alpha_a \bar{\alpha}_{\bar{b}} \xi_a \bar{\xi}_b = 2|\sum_{a=1}^m \alpha_a \xi_a|^2 \geq 0$; we then deduce that for all $1 \leq i \leq j+1$, we have at x , $f_i(g^{-1}p) = \sigma_i(\lambda(g^{-1}p)) \geq 0$. Finally, we infer by the Gårding inequality that at x in the chart ϕ we have

$$\tilde{f}_{j+1}(g^{-1}p, g^{-1}\tilde{g}, \dots, g^{-1}\tilde{g}) \geq f_{j+1}(g^{-1}p)^{1/(j+1)} f_{j+1}(g^{-1}\tilde{g})^{j/(j+1)} \geq 0 \quad (5.7)$$

which proves the positivity lemma. $\square$

5.2. The Fundamental Inequality

The C^0 a priori estimate is based on the following crucial proposition which is a generalization of the Proposition 7.18 of [13, page 262].

Proposition 5.5. *Let $h(t)$ be an increasing function of class C^1 defined on $\mathbb{R}$, and let φ be a C^2 k -admissible function defined on M , then the following inequality is satisfied:*

$$\int_M \left[\binom{m}{k} - f_k(g^{-1}\tilde{g}) \right] h(\varphi) \omega^m \geq \frac{1}{2m} \binom{m}{k} \int_M h'(\varphi) |\nabla \varphi|_g^2 \omega^m. \quad (5.8)$$

Proof. We have the equality $\int_M [(\binom{m}{k} - f_k(g^{-1}\tilde{g}))h(\varphi)\omega^m = (\binom{m}{k}) \int_M h(\varphi)(\omega^m - \tilde{\omega}^k \wedge \omega^{m-k})$. Besides, since $\Lambda^2 M$ is commutative $\omega^m - \tilde{\omega}^k \wedge \omega^{m-k} = (\omega - \tilde{\omega}) \wedge (\omega^{m-1} + \omega^{m-2} \wedge \tilde{\omega} + \dots + \omega^{m-k} \wedge \tilde{\omega}^{k-1})$, namely, $\omega^m - \tilde{\omega}^k \wedge \omega^{m-k} = -(1/2)dd^c\varphi \wedge \Omega$, then

$\int_M [(\binom{m}{k} - f_k(g^{-1}\tilde{g}))h(\varphi)\omega^m = -(1/2)(\binom{m}{k}) \int_M dd^c\varphi \wedge (h(\varphi)\Omega)$. But $d(d^c\varphi \wedge h(\varphi)\Omega) = dd^c\varphi \wedge h(\varphi)\Omega + (-1)^1 d^c\varphi \wedge d(h(\varphi)\Omega)$, and $d(h(\varphi)\Omega) = h'(\varphi)d\varphi \wedge \Omega + (-1)^0 h(\varphi) \underbrace{d\Omega}_{=0 \text{ since } \omega \text{ and } \bar{\omega} \text{ are closed}}$ so $dd^c\varphi \wedge h(\varphi)\Omega = d^c\varphi \wedge h'(\varphi)d\varphi \wedge \Omega + d(\text{something})$. In addition

by Stokes' theorem, $\int_M d(\text{something}) = 0$; therefore,

$$\begin{aligned} \int_M \left[\binom{m}{k} - f_k(g^{-1}\tilde{g}) \right] h(\varphi) \omega^m &= -\frac{1}{2} \binom{m}{k} \int_M h'(\varphi) d^c\varphi \wedge d\varphi \wedge \Omega \\ &= \frac{1}{2} \binom{m}{k} \left(\underbrace{\int_M h'(\varphi) (-d^c\varphi) \wedge d\varphi \wedge \omega^{m-1}}_{T_1} \right. \\ &\quad \left. + \sum_{j=1}^{k-1} \underbrace{\int_M h'(\varphi) (-d^c\varphi) \wedge d\varphi \wedge \omega^{m-1-j} \wedge \tilde{\omega}^j}_{T_2} \right). \end{aligned} \quad (5.9)$$

Let us prove that $T_2 \geq 0$ (using the positivity lemma) and that $T_1 = (1/m) \int_M h'(\varphi) |\nabla \varphi|_g^2 \omega^m$. Let us apply the positivity lemma to $d\varphi$: the function $f : M \rightarrow \mathbb{R}$ defined by $f\omega^m = {}^t J d\varphi \wedge d\varphi \wedge \omega^{m-1-j} \wedge \tilde{\omega}^j$ is nonnegative for all $1 \leq j \leq k-1$. But ${}^t J d\varphi = -d^c\varphi$ and h is an increasing function; then $T_2 \geq 0$. Let us now calculate T_1 . Fix $x \in M$, and let us work in a g -unitary chart

centered at x and satisfying $d\varphi/|d\varphi|_g = (dz^m + dz^{\bar{m}})/\sqrt{2}$ at x . We have then $\omega = idz^a \wedge dz^{\bar{a}}$ at x and ${}^tJd\varphi \wedge d\varphi = i|d\varphi|_g^2 dz^m \wedge dz^{\bar{m}}$; therefore,

$$\begin{aligned}
 & {}^tJd\varphi \wedge d\varphi \wedge \omega^{m-1} \\
 &= \sum_{\substack{a_1, \dots, a_{m-1} \in \{1, \dots, m-1\} \\ 2 \text{ by } 2 \neq}} i^m |d\varphi|_g^2 (dz^m \wedge dz^{\bar{m}}) \wedge (dz^{a_1} \wedge dz^{\bar{a}_1}) \wedge \dots \wedge (dz^{a_{m-1}} \wedge dz^{\bar{a}_{m-1}}) \\
 &= \left(\sum_{\substack{a_1, \dots, a_{m-1} \in \{1, \dots, m-1\} \\ 2 \text{ by } 2 \neq}} 1 \right) |d\varphi|_g^2 i^m (dz^1 \wedge dz^{\bar{1}}) \wedge \dots \wedge (dz^m \wedge dz^{\bar{m}}) \\
 &= (m-1)! |d\varphi|_g^2 \frac{\omega^m}{m!} = \frac{1}{m} |\nabla \varphi|_g^2 \omega^m.
 \end{aligned} \tag{5.10}$$

Thus $T_1 = (1/m) \int_M h'(\varphi) |\nabla \varphi|_g^2 \omega^m$, consequently $\int_M [(\binom{m}{k} - f_k(g^{-1}\tilde{g})] h(\varphi) \omega^m \geq (1/2) (\binom{m}{k}) T_1 = (1/2m) (\binom{m}{k}) \int_M h'(\varphi) |\nabla \varphi|_g^2 \omega^m$, which achieves the proof of the proposition. $\square$

5.3. The Moser Iteration Technique

We conclude the proof using the Moser's iteration technique exactly as for the equation of Calabi-Yau. Let us apply the proposition to φ_{t_s} in order to obtain a crucial inequality (the inequality (IN1)) from which we will infer the a priori estimate of $\|\varphi_{t_s}\|_{C^0}$. Let $p \geq 2$ be a real number. The function φ_{t_s} is C^2 admissible. Let us consider the function $h(u) := u|u|^{p-2} : \mathbb{R} \rightarrow \mathbb{R}$. This function is of class C^1 and $h'(u) = |u|^{p-2} + u(p-2)|u|^{p-4} = (p-1)|u|^{p-2} \geq 0$, so h is increasing. Therefore we infer by the previous proposition that

$$\frac{p-1}{2m} \binom{m}{k} \int_M |\varphi_{t_s}|^{p-2} |\nabla \varphi_{t_s}|^2 v_g \leq \int_M \left[\binom{m}{k} - f_k(g^{-1}\tilde{g}) \right] \varphi_{t_s} |\varphi_{t_s}|^{p-2} v_g. \tag{5.11}$$

Besides, $|\nabla |\varphi_{t_s}|^{p/2}|^2 = 2g^{a\bar{b}} \partial_a |\varphi_{t_s}|^{p/2} \partial_{\bar{b}} |\varphi_{t_s}|^{p/2} = 2g^{a\bar{b}} ((p/2) \varphi_{t_s} |\varphi_{t_s}|^{p/2-2})^2 \partial_a \varphi_{t_s} \partial_{\bar{b}} \varphi_{t_s} = (p^2/4) |\varphi_{t_s}|^{p-2} |\nabla \varphi_{t_s}|^2$, so the previous inequality writes:

$$\int_M |\nabla |\varphi_{t_s}|^{p/2}|^2 v_g \leq \frac{mp^2}{2(p-1) \binom{m}{k}} \int_M \left[\binom{m}{k} - f_k(g^{-1}\tilde{g}) \right] \varphi_{t_s} |\varphi_{t_s}|^{p-2} v_g. \tag{IN1}$$

Let us infer from the inequality (IN1) another inequality (the inequality (IN4)) that is required for the proof. It follows from the continuous Sobolev embedding $H_1^2(M) \subset L^{2m/(m-1)}(M)$ that

$$\| |\varphi_{t_s}|^p \|_{m/(m-1)} = \| |\varphi_{t_s}|^{p/2} \|_{2m/(m-1)}^2 \leq Cste \left(\int_M |\nabla |\varphi_{t_s}|^{p/2}|^2 + \int_M |\varphi_{t_s}|^{(p/2) \cdot 2} \right), \tag{IN2}$$

where $Cste$ is independent of p . Besides, $f_k(g^{-1}\tilde{g})$ is uniformly bounded; indeed,

$$\left| f_k(g^{-1}\tilde{g}) \right| = e^{t_s f} \frac{\binom{m}{k} \text{Vol}(M)}{\int_M e^{t_s f} v_g} \leq \binom{m}{k} e^{2t_s \|f\|_\infty} \leq \binom{m}{k} e^{2\|f\|_\infty}. \quad (\text{IN3})$$

Using the inequalities (IN1), (IN2), (IN3), and $p^2/2(p-1) \leq p$ we obtain

$$\| |\varphi_{t_s}|^p \|_{m/(m-1)} \leq Cste' \times p \left(\int_M |\varphi_{t_s}|^{p-1} + \int_M |\varphi_{t_s}|^p \right) \quad (p \geq 2), \quad (\text{IN4})$$

where $Cste'$ is independent of p . Suppose that $Cste' \geq 1$.

Using the Green's formula and the inequalities of Sobolev-Poincaré (IN2) and of Hölder, we prove following [13] these L_q estimates.

Lemma 5.6. *There exists a constant μ such that for all $1 \leq q \leq 2m/(m-1)$,*

$$\|\varphi_{t_s}\|_q \leq \mu. \quad (5.12)$$

Proof. M is a compact Riemannian manifold and $\varphi_{t_s} \in C^2$, so by the Green's formula $\varphi_{t_s}(x) = (1/\text{Vol}(M)) \underbrace{\int_M \varphi_{t_s} dv}_{=0} + \int_M G(x, y) \Delta \varphi_{t_s}(y) dv(y)$, where $G(x, y) \geq 0$ and $\int_M G(x, y) dv(y)$ is

independent of x . Here $\Delta \varphi_{t_s}$ denotes the real Laplacian. Then, we infer that $\|\varphi_{t_s}\|_1 \leq C \|\Delta \varphi_{t_s}\|_1$. But $\|\Delta \varphi_{t_s}\|_1 = \int_M \Delta \varphi_{t_s}^+ + \Delta \varphi_{t_s}^-$ and $\int_M \Delta \varphi_{t_s} = \int_M \Delta \varphi_{t_s}^+ - \Delta \varphi_{t_s}^- = 0$; then $\|\Delta \varphi_{t_s}\|_1 = 2 \int_M \Delta \varphi_{t_s}^+$. Besides $\Delta \varphi_{t_s} < 2m$ since φ_{t_s} is k -admissible: indeed, at x in a g -normal $\tilde{g}$ -adapted chart, namely, a chart satisfying $g_{a\bar{b}} = \delta_{ab}$, $\tilde{g}_{a\bar{b}} = \delta_{ab} \lambda_a$ and $\partial_v g_{a\bar{b}} = 0$ for all $1 \leq a, b \leq m$, $v \in \{1, \dots, m, \bar{1}, \dots, \bar{m}\}$, we have $\lambda(g^{-1}\tilde{g}) = (\lambda_1, \dots, \lambda_m)$ so $\lambda = (\lambda_1, \dots, \lambda_m) \in \Gamma_k$ since φ_{t_s} is k -admissible; consequently $\Delta \varphi_{t_s} = -2g^{a\bar{b}} \partial_{a\bar{b}} \varphi_{t_s} = -2 \sum_a \partial_{a\bar{a}} \varphi_{t_s} = 2 \sum_a (1 - \lambda_a) = 2m - 2\sigma_1(\lambda)$, but $\sigma_1(\lambda) > 0$ since $\lambda \in \Gamma_k$ which proves that $\Delta \varphi_{t_s} < 2m$. Therefore $\Delta \varphi_{t_s}^+ < 2m$ and $\|\Delta \varphi_{t_s}\|_1 \leq 4m \text{Vol}(M)$. We infer then that $\|\varphi_{t_s}\|_1 \leq 4mC \text{Vol}(M)$. Now let us take $p = 2$ in the inequality (IN2): $\|\varphi_{t_s}\|_{2m/(m-1)}^2 \leq Cste(\int_M |\nabla \varphi_{t_s}|^2 + \int_M |\varphi_{t_s}|^2)$. Besides, $\varphi_{t_s} \in H_1^2(M)$ and has a vanishing integral; then by the Sobolev-Poincaré inequality we infer $\|\varphi_{t_s}\|_2 \leq \mathcal{A} \|\nabla \varphi_{t_s}\|_2$. But $|\nabla \varphi_{t_s}| = |\nabla \varphi_{t_s}|$ almost everywhere; therefore $\|\varphi_{t_s}\|_{2m/(m-1)} \leq Cste \|\nabla \varphi_{t_s}\|_2$. Using the inequality (IN1) with $p = 2$ and the fact that $f_k(g^{-1}\tilde{g})$ is uniformly bounded, we obtain that $\|\nabla \varphi_{t_s}\|_2^2 \leq Cste \|\varphi_{t_s}\|_1 \leq Cste'$. Consequently, we infer that $\|\varphi_{t_s}\|_{2m/(m-1)} \leq Cste$.

Let $1 \leq q \leq 2m/(m-1) =: 2\delta$. By the Hölder inequality we have $\|\varphi_{t_s}\|_q^q = \int_M |\varphi_{t_s}|^q \cdot 1 \leq (\int_M |\varphi_{t_s}|^{q \cdot (2\delta/q)})^{q/2\delta} \text{Vol}(M)^{1-q/2\delta}$. Therefore $\|\varphi_{t_s}\|_q \leq \text{Vol}(M)^{(1/q)-(1/2\delta)} \|\varphi_{t_s}\|_{2\delta}$. But

$$\text{Vol}(M)^{1/q-1/2\delta} = e^{(1/q-1/2\delta) \ln(\text{Vol}(M))} \leq \begin{cases} 1 & \text{if } \text{Vol}(M) \leq 1, \\ \text{Vol}(M)^{1-1/2\delta} & \text{if } \text{Vol}(M) \geq 1 \end{cases} \quad (5.13)$$

and $\|\varphi_{t_s}\|_{2\delta} \leq Cste$, thus $\|\varphi_{t_s}\|_q \leq \mu := Cste \times \text{Max}(1, \text{Vol}(M)^{1-1/2\delta})$. $\square$

Suppose without limitation of generality that $\mu \geq 1$. Now, we deduce from the previous lemma and the inequality (IN4), by induction, these more general L_p estimates using the same method as [13].

Lemma 5.7. *There exists a constant C_0 such that for all $p \geq 2$,*

$$\|\varphi_{t_s}\|_p \leq C_0 (\delta^{m-1} Cp)^{-m/p}, \quad (5.14)$$

with $\delta = m/(m-1)$ and $C = Cste'(1 + \text{Max}(1, \text{Vol}(M)^{1/2})) \geq 1$ where $Cste'$ is the constant of the inequality (IN4).

Proof. We prove this lemma by induction: first we check that the inequality is satisfied for $2 \leq p \leq 2\delta = 2m/(m-1)$; afterwards we show that if the inequality is true for p , then it is satisfied for δp too. Denote $C_0 = \mu \delta^{m(m-1)} C^m e^{m/e}$. For $2 \leq p \leq 2\delta$ we have $\|\varphi_{t_s}\|_p \leq \mu$, so it suffices to check that $\mu \leq C_0 (\delta^{m-1} Cp)^{-m/p}$. This inequality is equivalent to $\delta^{m(m-1)} C^m e^{m/e} (\delta^{m-1} Cp)^{-m/p} \geq 1$; then $(\delta^{m(m-1)} C^m) e^{m/e} \geq (\delta^{m(m-1)} C^m)^{1/p} p^{m/p}$. But if $x \geq 1$, then $x \geq x^{1/p}$ (since $p \geq 1$), and $\delta^{m(m-1)} C^m \geq 1$ (since $C \geq 1, m \geq 1$ and $\delta \geq 1$); therefore $\delta^{m(m-1)} C^m \geq (\delta^{m(m-1)} C^m)^{1/p}$. Besides, $p^{m/p} = e^{m(\ln p/p)} \leq e^{m/e}$, which proves the inequality for $2 \leq p \leq 2\delta$. Now let us fix $p \geq 2$. Suppose that $\|\varphi_{t_s}\|_p \leq C_0 (\delta^{m-1} Cp)^{-m/p}$ and prove that $\|\varphi_{t_s}\|_{\delta p} \leq C_0 (\delta^{m-1} C \delta p)^{-m/\delta p}$. The inequality (IN4) proved previously writes:

$$\|\varphi_{t_s}\|_p^p \leq Cste' \times p \left(\int_M |\varphi_{t_s}|^{p-1} + \int_M |\varphi_{t_s}|^p \right) \quad (p \geq 2), \quad (\text{IN4}')$$

where $Cste'$ is independent of p , namely, $\|\varphi_{t_s}\|_{\delta p}^p \leq Cste' \times p (\|\varphi_{t_s}\|_{p-1}^{p-1} + \|\varphi_{t_s}\|_p^p)$. But since $1 \leq p-1 \leq p$, we have by the Hölder inequality that $\|\varphi_{t_s}\|_{p-1} \leq \text{Vol}(M)^{1/(p-1)-1/p} \|\varphi_{t_s}\|_p$; therefore $\|\varphi_{t_s}\|_{\delta p}^p \leq Cste' \times p (\text{Vol}(M)^{1/p} \|\varphi_{t_s}\|_p^{p-1} + \|\varphi_{t_s}\|_p^p)$.

- (i) If $\|\varphi_{t_s}\|_p \leq 1$, then $\|\varphi_{t_s}\|_{\delta p}^p \leq C \times p$; therefore $\|\varphi_{t_s}\|_{\delta p} \leq (Cp)^{1/p}$. Let us check that $(Cp)^{1/p} \leq C_0 (\delta^{m-1} C \delta p)^{-m/\delta p}$. This inequality is equivalent to $p^{(1/p)(1+m/\delta)} \leq \mu \delta^{m(m-1)(1-1/p)} e^{m/e} \times C^{m-m/\delta p-1/p}$, but $1 + m/\delta = m$ so it is equivalent to $p^{m/p} \leq \mu \delta^{m(m-1)(1-1/p)} e^{m/e} \times C^{m(1-1/p)}$. Besides $p^{m/p} \leq e^{m/e}$ and $\mu \delta^{m(m-1)(1-1/p)} \geq 1$, then it suffices to have $C^{m(1-1/p)} \geq 1$, and this is satisfied since $C \geq 1$.
- (ii) If $\|\varphi_{t_s}\|_p \geq 1$, we infer that $\|\varphi_{t_s}\|_{\delta p}^p \leq C \times p \|\varphi_{t_s}\|_p^p$, therefore $\|\varphi_{t_s}\|_{\delta p} \leq C^{1/p} \times p^{1/p} \|\varphi_{t_s}\|_p \leq (Cp)^{1/p} C_0 (\delta^{m-1} Cp)^{-m/p}$ by the induction hypothesis. But $(1-m)/p = -m/\delta p$; then we obtain the required inequality $\|\varphi_{t_s}\|_{\delta p} \leq C_0 \delta^{-m^2/\delta p} (Cp)^{-m/\delta p} = C_0 (\delta^{m-1} C \delta p)^{-m/\delta p}$. $\square$

By tending to the limit $p \rightarrow +\infty$ in the inequality of the previous lemma, we obtain the needed C^0 a priori estimate.

Corollary 5.8. *Consider*

$$\|\varphi_{t_s}\|_{C^0} \leq C_0. \quad (5.15)$$

6. The C^2 A Priori Estimate

6.1. Strategy for a C^2 Estimate

First, we will look for a uniform upper bound on the eigenvalues $\lambda([\delta_i^j + g^{j\bar{i}}\partial_{i\bar{i}}\varphi_t]_{1 \leq i, j \leq m})$. Secondly, we will infer from it the uniform ellipticity of the continuity equation $(E_{k,t})$ and a uniform gradient bound. Thirdly, with the uniform ellipticity at hand, we will derive a one-sided estimate on pure second derivatives and finally get the needed C^2 bound.

6.2. Eigenvalues Upper Bound

6.2.1. The Functional

Let $t \in \mathcal{T}_{l,\alpha}$, and let $\varphi_t : M \rightarrow \mathbb{R}$ be a $C^{l,\alpha}$ k -admissible solution of $(E_{k,t})$ satisfying $\int_M \varphi_t \omega^m = 0$. Consider the following functional:

$$\begin{aligned} B : UT^{1,0} &\longrightarrow \mathbb{R} \\ (P, \xi) &\longmapsto B(P, \xi) = \tilde{h}_P(\xi, \xi) - \varphi_t(P), \end{aligned} \quad (6.1)$$

where $UT^{1,0}$ is the unit sphere bundle associated to $(T^{1,0}, h)$ and $\tilde{g}$ is related to g by: $\tilde{\omega} = \omega + i\partial\bar{\partial}\varphi_t$. B is continuous on the compact set $UT^{1,0}$, so it assumes its maximum at a point $(P_0, \xi_0) \in UT^{1,0}$. In addition, for fixed $P \in M$, $\xi \in UT_P^{1,0} \mapsto \tilde{h}_P(\xi, \xi)$ is continuous on the compact subset $UT_P^{1,0}$ (the fiber); therefore it attains its maximum at a unit vector $\xi_P \in UT_P^{1,0}$, and by the min-max principle we can choose ξ_P as the direction of the largest eigenvalue of A_P , $\lambda_{\max}(A_P)$. Specifically, we have the following.

Lemma 6.1 (min-max principle). *Consider*

$$\tilde{h}_P(\xi_P, \xi_P) = \max_{\xi \in T_P^{1,0}, h_P(\xi, \xi)=1} \tilde{h}_P(\xi, \xi) = \lambda_{\max}(A_P). \quad (6.2)$$

For fixed P , we have $\max_{h_P(\xi, \xi)=1} B(P, \xi) = B(P, \xi_P) = \lambda_{\max}(A_P) - \varphi_t(P)$; therefore $\max_{(P, \xi) \in UT^{1,0}} B(P, \xi) = \max_{P \in M} B(P, \xi_P) = B(P_0, \xi_0) \leq B(P_0, \xi_{P_0})$; hence,

$$\max_{(P, \xi) \in UT^{1,0}} B(P, \xi) = B(P_0, \xi_{P_0}) = \lambda_{\max}(A_{P_0}) - \varphi_t(P_0). \quad (6.3)$$

At the point P_0 , consider $e_1^{P_0}, \dots, e_m^{P_0}$ an h_{P_0} -orthonormal basis of $(T_{P_0}^{1,0}, h_{P_0})$ made of eigenvectors of A_{P_0} that satisfies the following properties:

- (1) h_{P_0} -orthonormal: $[h_{ij}(P_0)]_{1 \leq i, j \leq m} = I_m$.
- (2) $\tilde{h}_{P_0}$ -diagonal: $[\tilde{h}_{ij}(P_0)]_{1 \leq i, j \leq m} = \text{Mat} A_{P_0} = \text{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda \in \Gamma_k$.
- (3) $\lambda_{\max}(A_{P_0})$ is achieved in the direction $e_1^{P_0} = \xi_{P_0}$: $A_{P_0}(\xi_{P_0}) = \lambda_{\max}(A_{P_0})\xi_{P_0} = \lambda_1 \xi_{P_0}$ and $\lambda_1 \geq \dots \geq \lambda_m$.

In other words, it is a basis satisfying

- (1) $[g_{i\bar{j}}^-(P_0)]_{1 \leq i, j \leq m} = I_m$,
- (2) $[\tilde{g}_{i\bar{j}}^-(P_0)]_{1 \leq i, j \leq m} = \text{Mat} A_{P_0} = \text{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda \in \Gamma_k$,
- (3) $\lambda_{\max}(A_{P_0}) = \lambda_1 \geq \dots \geq \lambda_m$.

Let us consider a holomorphic normal chart (U_0, φ_0) centered at P_0 such that $\varphi_0(P_0) = 0$ and $\partial_i|_{P_0} = e_i^{P_0}$ for all $i \in \{1 \dots m\}$.

6.2.2. Auxiliary Local Functional

From now on, we work in the chart (U_0, φ_0) constructed at P_0 . The map $P \mapsto g_{1\bar{1}}(P)$ is continuous on U_0 and is equal to 1 at P_0 , so there exists an open subset $U_1 \subset U_0$ such that $g_{1\bar{1}}(P) > 0$ for all $P \in U_1$. Let B_1 be the functional

$$B_1 : U_1 \longrightarrow \mathbb{R}$$

$$P \longmapsto B_1(P) = \frac{\tilde{g}_{1\bar{1}}(P)}{g_{1\bar{1}}(P)} - \varphi_t(P). \quad (6.4)$$

We claim that B_1 assumes a local maximum at P_0 . Indeed, we have at each $P \in U_1$: $\tilde{g}_{1\bar{1}}(P)/g_{1\bar{1}}(P) = \tilde{g}_P(\partial_1, \partial_{\bar{1}})/g_P(\partial_1, \partial_{\bar{1}}) = \tilde{h}_P(\partial_1, \partial_1)/h_P(\partial_1, \partial_1) = \tilde{h}_P(\partial_1/|\partial_1|_{h_P}, \partial_1/|\partial_1|_{h_P}) \leq \lambda_{\max}(A_P)$ (see Lemma 6.1); thus $B_1(P) \leq \lambda_{\max}(A_P) - \varphi_t(P) \leq \lambda_{\max}(A_{P_0}) - \varphi_t(P_0) = B_1(P_0)$.

6.2.3. Differentiating the Equation

For short, we drop henceforth the subscript t of φ_t . Let us differentiate $(E_{k,t})$ at P , in a chart z :

$$t\partial_{\bar{1}}f = dF_k[\delta_i^j + g^{j\bar{\ell}}(P)\partial_{i\bar{\ell}}\varphi(P)]_{1 \leq i, j \leq m} \cdot \left[\partial_{\bar{1}}(g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi) \right]_{1 \leq i, j \leq m}$$

$$= \sum_{i,j=1}^m \frac{\partial F_k}{\partial B_i^j} [\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi] \left(\partial_{\bar{1}}g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi + g^{j\bar{\ell}}\partial_{1\bar{i}\bar{\ell}}\varphi \right). \quad (6.5)$$

Differentiating once again, we find

$$t\partial_{1\bar{1}}f = \sum_{i,j,r,s=1}^m \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j} [\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi] \left(\partial_1 g^{s\bar{o}} \partial_{r\bar{o}}\varphi + g^{s\bar{o}} \partial_{1r\bar{o}}\varphi \right)$$

$$\times \left(\partial_{\bar{1}}g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi + g^{j\bar{\ell}}\partial_{1\bar{i}\bar{\ell}}\varphi \right) + \sum_{i,j=1}^m \frac{\partial F_k}{\partial B_i^j} [\delta_i^j + g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi]$$

$$\times \left(\partial_{1\bar{1}}g^{j\bar{\ell}}\partial_{i\bar{\ell}}\varphi + \partial_{\bar{1}}g^{j\bar{\ell}}\partial_{1\bar{i}\bar{\ell}}\varphi + \partial_1 g^{j\bar{\ell}}\partial_{1\bar{i}\bar{\ell}}\varphi + g^{j\bar{\ell}}\partial_{11\bar{i}\bar{\ell}}\varphi \right). \quad (6.6)$$

Using the above chart (U_1, φ_0) at the point P_0 , normality yields $g^{j\bar{\ell}} = \delta^{j\ell}$, $\partial_\alpha g_{i\bar{\ell}} = 0$ and $\partial_\alpha g^{i\bar{\ell}} = 0$. Furthermore $[\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi] = [\delta_i^j + \partial_{i\bar{j}} \varphi] = [\tilde{g}_{i\bar{j}}] = \text{diag}(\lambda_1, \dots, \lambda_m)$. In this chart, we can simplify the previous expression; we get then at P_0 ,

$$\begin{aligned} t\partial_{1\bar{1}}f &= \sum_{i,j,r,s=1}^m \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j} (\text{diag}(\lambda_1, \dots, \lambda_m)) \partial_{1r\bar{s}} \varphi \partial_{1\bar{i}j} \varphi \\ &+ \sum_{i,j=1}^m \frac{\partial F_k}{\partial B_i^j} (\text{diag}(\lambda_1, \dots, \lambda_m)) (\partial_{1\bar{1}} g^{j\bar{i}} \partial_{i\bar{i}} \varphi + \partial_{1\bar{1}i\bar{j}} \varphi). \end{aligned} \quad (6.7)$$

Besides, $\partial_{1\bar{1}} g^{j\bar{i}} = \partial_{1\bar{1}} (-g^{j\bar{s}} g^{o\bar{i}} \partial_{1s\bar{o}} \varphi)$, so still by normality, we obtain at P_0 that $\partial_{1\bar{1}} g^{j\bar{i}} = -g^{j\bar{s}} g^{o\bar{i}} \partial_{1\bar{1}s\bar{o}} \varphi = -\partial_{1\bar{1}} g_{i\bar{j}} - R_{1\bar{1}i\bar{j}}$. Therefore we get

$$\begin{aligned} t\partial_{1\bar{1}}f &= \sum_{i,j=1}^m \frac{\partial F_k}{\partial B_i^j} (\text{diag}(\lambda_1, \dots, \lambda_m)) (\partial_{1\bar{1}i\bar{j}} \varphi - R_{1\bar{1}i\bar{j}} \partial_{i\bar{i}} \varphi) \\ &+ \sum_{i,j,r,s=1}^m \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j} (\text{diag}(\lambda_1, \dots, \lambda_m)) \partial_{1r\bar{s}} \varphi \partial_{1\bar{i}j} \varphi. \end{aligned} \quad (6.8)$$

6.2.4. Using Concavity

Now, using the concavity of $\ln \sigma_k$ [10], we prove for Proposition 2.1 that the second sum of (6.8) is negative [9, page 84]. This is not a direct consequence of the concavity of the function F_k since the matrix $[\partial_{1\bar{i}j} \varphi]_{1 \leq i, j \leq m}$ is not Hermitian.

Lemma 6.2. *Consider*

$$S := \sum_{i,j,r,s=1}^m \frac{\partial^2 F_k}{\partial B_r^s \partial B_i^j} (\text{diag}(\lambda_1, \dots, \lambda_m)) \partial_{1r\bar{s}} \varphi \partial_{1\bar{i}j} \varphi \leq 0. \quad (6.9)$$

Hence, from (6.8) combined with Lemma 6.2 we infer

$$t\partial_{1\bar{1}}f \leq \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} (\partial_{1\bar{1}i\bar{i}} \varphi - R_{1\bar{1}i\bar{i}} \partial_{i\bar{i}} \varphi). \quad (6.10)$$

6.2.5. Differentiation of the Functional B_1

Let us differentiate twice the functional B_1 :

$$\begin{aligned}
 B_1(P) &= \frac{\tilde{g}_{1\bar{1}}(P)}{g_{1\bar{1}}(P)} - \varphi(P), \\
 \partial_{\bar{i}} B_1 &= \frac{\partial_{\bar{i}} \tilde{g}_{1\bar{1}}}{g_{1\bar{1}}} - \frac{\tilde{g}_{1\bar{1}} \partial_{\bar{i}} g_{1\bar{1}}}{(g_{1\bar{1}})^2} - \partial_{\bar{i}} \varphi, \\
 \partial_{\bar{i}\bar{i}} B_1 &= \frac{\partial_{\bar{i}\bar{i}} \tilde{g}_{1\bar{1}}}{g_{1\bar{1}}} - \frac{\partial_i g_{1\bar{1}} \partial_{\bar{i}} \tilde{g}_{1\bar{1}} + \partial_i \tilde{g}_{1\bar{1}} \partial_{\bar{i}} g_{1\bar{1}} + \tilde{g}_{1\bar{1}} \partial_{\bar{i}\bar{i}} g_{1\bar{1}}}{(g_{1\bar{1}})^2} \\
 &\quad + \frac{2\tilde{g}_{1\bar{1}} \partial_i g_{1\bar{1}} \partial_{\bar{i}} g_{1\bar{1}}}{(g_{1\bar{1}})^3} - \partial_{\bar{i}\bar{i}} \varphi.
 \end{aligned} \tag{6.11}$$

Therefore at P_0 , in the above chart (U_1, ψ_0) we find $\partial_{\bar{i}\bar{i}} B_1 = \partial_{\bar{i}\bar{i}}(g_{1\bar{1}} + \partial_{1\bar{1}}\varphi) - \lambda_1 \partial_{\bar{i}\bar{i}} g_{1\bar{1}} - \partial_{\bar{i}\bar{i}} \varphi = R_{1\bar{1}\bar{i}\bar{i}} + \partial_{1\bar{1}\bar{i}\bar{i}}\varphi - \lambda_1 R_{1\bar{1}\bar{i}\bar{i}} - \partial_{\bar{i}\bar{i}} \varphi$. Let us define the operator:

$$L := \sum_{i,j=1}^m \frac{\partial F_k}{\partial B_i^j} \left(\left[\delta_i^j + g^{j\bar{\ell}} \partial_{\bar{\ell}} \varphi \right]_{1 \leq i, j \leq m} \right) \nabla_i^j. \tag{6.12}$$

Thus, we have at P_0

$$L(B_1) = \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} (\partial_{1\bar{1}\bar{i}\bar{i}}\varphi + (1 - \lambda_1) R_{1\bar{1}\bar{i}\bar{i}} - \partial_{\bar{i}\bar{i}} \varphi). \tag{6.13}$$

Combining (6.13) with (6.10), we get rid of the fourth derivatives:

$$\begin{aligned}
 t\partial_{1\bar{1}} f - L(B_1) &\leq \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} R_{1\bar{1}\bar{i}\bar{i}} (\lambda_1 - 1 - \lambda_i + 1) \\
 &\quad + \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} (\lambda_i - 1).
 \end{aligned} \tag{6.14}$$

Since B_1 assumes its maximum at P_0 , we have at P_0 that $L(B_1) \leq 0$. So we are left with the following inequality at P_0 :

$$0 \geq \sum_{i=2}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} (-R_{1\bar{1}\bar{i}\bar{i}}) (\lambda_1 - \lambda_i) - \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} \lambda_i + \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} + t\partial_{1\bar{1}} f. \tag{6.15}$$

Curvature Assumption

Henceforth, we will suppose that the holomorphic bisectional curvature is nonnegative at any $P \in M$. Thus in a holomorphic normal chart centered at P we have $R_{a\bar{a}b\bar{b}}(P) \leq 0$ for all

$1 \leq a, b \leq m$. This holds in particular at P_0 in the previous chart φ_0 . This assumption will be used only to derive an a priori eigenvalues pinching and is not required in the other sections.

Back to the inequality (6.15), we have $\sigma_k(\lambda) > 0$ and $\sigma_{k-1,i}(\lambda) > 0$ since $\lambda \in \Gamma_k$, and under our curvature assumption $(-R_{1\bar{1}i\bar{i}}) \geq 0$ for all $i \geq 2$. Besides, $\lambda_i \leq \lambda_1$ for all i ; therefore $\sum_{i=2}^m (\sigma_{k-1,i}(\lambda)/\sigma_k(\lambda))(-R_{1\bar{1}i\bar{i}})(\lambda_1 - \lambda_i) \geq 0$. So we can get rid of the curvature terms in (6.15) and infer from it the inequality

$$0 \geq - \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} \lambda_i + \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} + t \partial_{1\bar{1}} f. \quad (6.16)$$

6.2.6. A λ_1 's Upper Bound

Here, we require elementary identities satisfied by the σ_ℓ 's [11], namely:

$$\begin{aligned} \forall 1 \leq \ell \leq m \quad \sigma_\ell(\lambda) &= \sigma_{\ell,i}(\lambda) + \lambda_i \sigma_{\ell-1,i}(\lambda), \\ \forall 1 \leq \ell \leq m \quad \sum_{i=1}^m \sigma_{\ell-1,i}(\lambda) \lambda_i &= \ell \sigma_\ell(\lambda), \\ \text{so in particular} \quad \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} \lambda_i &= k, \\ \forall 1 \leq \ell \leq m \quad \sum_{i=1}^m \sigma_{\ell,i}(\lambda) &= (m - \ell) \sigma_\ell(\lambda), \\ \text{so in particular} \quad \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda)}{\sigma_k(\lambda)} &= (m - k + 1) \frac{\sigma_{k-1}(\lambda)}{\sigma_k(\lambda)}. \end{aligned} \quad (6.17)$$

Consequently, (6.16) writes:

$$q_k := \frac{(m - k + 1)}{k} \frac{\sigma_{k-1}(\lambda)}{\sigma_k(\lambda)} \leq 1 - \frac{t}{k} \partial_{1\bar{1}} f. \quad (6.18)$$

So $q_k \leq 1 + (1/k) |\partial_{1\bar{1}} f|$. But at P_0 , $|\nabla^2 f|_g^2 = 2g^{a\bar{c}} g^{d\bar{b}} (\nabla_{a\bar{b}} f \nabla_{\bar{c}d} f + \nabla_{ad} f \nabla_{\bar{c}\bar{b}} f) = 2 \sum_{a,b=1}^m (|\partial_{a\bar{b}} f|^2 + |\partial_{ab} f|^2)$, then $|\partial_{1\bar{1}} f| \leq |\nabla^2 f|_g$, and consequently $q_k \leq 1 + (1/k) \|f\|_{C^2(M)} =: C_1$. In other words, there exists a constant C_1 independent of $t \in [0, 1]$ such that

$$q_k \leq C_1. \quad (6.19)$$

To proceed further, we recall the following

Lemma 6.3 (Newton inequalities). *For all $\ell \geq 2$, $\lambda \in \mathbb{R}^m$:*

$$\sigma_\ell(\lambda) \sigma_{\ell-2}(\lambda) \leq \frac{(\ell - 1)(m - \ell + 1)}{\ell(m - \ell + 2)} [\sigma_{\ell-1}(\lambda)]^2. \quad (6.20)$$

Let us use Newton inequalities to relate q_k to σ_1 . Since for $2 \leq \ell \leq k$ and $\lambda \in \Gamma_k$ we have $\sigma_\ell(\lambda) > 0$, $\sigma_{\ell-1}(\lambda) > 0$ and $\sigma_{\ell-2}(\lambda) > 0$ ($\sigma_0(\lambda) = 1$ by convention), Newton inequalities imply then that $((m - \ell + 2)/(\ell - 1))(\sigma_{\ell-2}(\lambda)/\sigma_{\ell-1}(\lambda)) \leq ((m - \ell + 1)/\ell)(\sigma_{\ell-1}(\lambda)/\sigma_\ell(\lambda))$, or else $q_{\ell-1} \leq q_\ell$, consequently $q_k \geq q_{k-1} \geq \dots \geq q_2 = (m - 1)\sigma_1(\lambda)/2\sigma_2(\lambda)$. By induction, we get $\sigma_1(\lambda) \leq (\ell!(m - \ell)!/(m - 1)!) \sigma_\ell(\lambda)(q_\ell)^{\ell-1}$ for all $2 \leq \ell \leq k$. In particular

$$\sigma_1(\lambda) \leq \frac{k!(m - k)!}{(m - 1)!} \sigma_k(\lambda)(q_k)^{k-1}. \quad (6.21)$$

But $\sigma_k(\lambda) \leq e^{2\|f\|_\infty} \binom{m}{k}$; combining this with (6.19) and (6.21) we obtain at P_0 that

$$\sigma_1(\lambda) \leq m e^{2\|f\|_\infty} (C_1)^{k-1} =: C_2. \quad (6.22)$$

Hence we may state the following.

Theorem 6.4. *There exists a constant $C_2 > 0$ depending only on $m, k, \|f\|_\infty$ and $\|f\|_{C^2}$ such that for all $1 \leq i \leq m$ $\lambda_i(P_0) \leq C_2$.*

Combining this result with the C^0 a priori estimate $\|\varphi_t\|_{C_0} \leq C_0$ immediately yields the following.

Theorem 6.5. *There exists a constant $C'_2 > 0$ depending only on $m, k, \|f\|_{C^2}$ and C_0 such that for all $P \in M$, for all $1 \leq i \leq m$, $\lambda_i(P) \leq C_2 + 2C_0 =: C'_2$.*

6.2.7. Uniform Pinching of the Eigenvalues

We infer automatically the following pinchings of the eigenvalues.

Proposition 6.6. *For all $1 \leq i \leq m$, $-(m - 1)C_2 \leq \lambda_i(P_0) \leq C_2$.*

Proposition 6.7. *For all $P \in M$, for all $1 \leq i \leq m$, $-(m - 1)C'_2 \leq \lambda_i(P) \leq C'_2$.*

6.3. Uniform Ellipticity of the Continuity Equation

To prove the next proposition on uniform ellipticity, we require some inequalities satisfied by the σ_ℓ 's.

Lemma 6.8 (Maclaurin inequalities). *For all $1 \leq \ell \leq s$ for all $\lambda \in \overline{\Gamma_s}$, $(\sigma_s(\lambda)/\binom{m}{s})^{1/s} \leq (\sigma_\ell(\lambda)/\binom{m}{\ell})^{1/\ell}$.*

Proposition 6.9 (uniform ellipticity). *There exist constants $E > 0$ and $F > 0$ depending only on $m, k, \|f\|_\infty$ and C_2 such that: $E \leq \sigma_{k-1,1}(\lambda) \leq \dots \leq \sigma_{k-1,m}(\lambda) \leq F$ where $\lambda = \lambda(P_0)$.*

Proof. We have $\partial\sigma_k/\partial\lambda_1 = \sigma_{k-1,1}(\lambda) \leq \dots \leq \sigma_{k-1,m}(\lambda) \leq \binom{m-1}{k-1}(C_2)^{k-1} =: F$ where, indeed, the constant F so defined depends only on m, k , and C_2 . Let us look for a uniform lower bound on $\sigma_{k-1,1}(\lambda)$, using the identity $\sigma_k(\lambda) = \lambda_1\sigma_{k-1,1}(\lambda) + \sigma_{k,1}(\lambda)$. We distinguish two cases.

Case 1. $(\sigma_{k,1}(\lambda) \leq 0)$. When so, we have $\sigma_k(\lambda) \leq \lambda_1 \sigma_{k-1,1}(\lambda)$; therefore $\sigma_{k-1,1}(\lambda) \geq \sigma_k(\lambda)/\lambda_1$. But $\sigma_k(\lambda) \geq e^{-2\|f\|_\infty} \binom{m}{k}$ and $0 < \lambda_1 \leq C_2$; hence $\sigma_{k-1,1}(\lambda) \geq e^{-2\|f\|_\infty} \binom{m}{k}/C_2$.

Case 2. $(\sigma_{k,1}(\lambda) > 0)$. For $1 \leq j \leq k-1$, $\sigma_j(\lambda_2, \dots, \lambda_m) = \sigma_{j,1}(\lambda) > 0$ since $j+1 \leq k$ and $\lambda \in \Gamma_k$. Besides $\sigma_k(\lambda_2, \dots, \lambda_m) = \sigma_{k,1}(\lambda) > 0$ by hypothesis, therefore $(\lambda_2, \dots, \lambda_m) \in \Gamma_{k,1} = \{\beta \in \mathbb{R}^{m-1} / \forall 1 \leq j \leq k, \sigma_j(\beta) > 0\}$. From the latter, we infer by Maclaurin inequalities $(\sigma_k(\lambda_2, \dots, \lambda_m) / \binom{m-1}{k})^{1/k} \leq (\sigma_{k-1}(\lambda_2, \dots, \lambda_m) / \binom{m-1}{k-1})^{1/(k-1)}$ or else $(\sigma_{k,1}(\lambda) / \binom{m-1}{k})^{1/k} \leq (\sigma_{k-1,1}(\lambda) / \binom{m-1}{k-1})^{1/(k-1)}$; thus we have $\sigma_{k,1}(\lambda) \leq \binom{m-1}{k} (\sigma_{k-1,1}(\lambda) / \binom{m-1}{k-1})^{1+1/(k-1)}$, consequently

$$\begin{aligned} \sigma_k(\lambda) &= \lambda_1 \sigma_{k-1,1}(\lambda) + \sigma_{k,1}(\lambda) \\ &\leq \lambda_1 \sigma_{k-1,1}(\lambda) + \binom{m-1}{k} \left(\frac{\sigma_{k-1,1}(\lambda)}{\binom{m-1}{k-1}} \right)^{1+1/(k-1)} \\ &\leq \sigma_{k-1,1}(\lambda) \left[\lambda_1 + \frac{\binom{m-1}{k}}{\binom{m-1}{k-1}} \left(\frac{\sigma_{k-1,1}(\lambda)}{\binom{m-1}{k-1}} \right)^{1/(k-1)} \right]. \end{aligned} \quad (6.23)$$

Here, let us distinguish two subcases of Case 2.

- (i) If $\sigma_{k-1,1}(\lambda) > \binom{m-1}{k-1}$, then we have the uniform lower bound that we look for.
- (ii) Else $\sigma_{k-1,1}(\lambda) \leq \binom{m-1}{k-1}$, thus $(\sigma_{k-1,1}(\lambda) / \binom{m-1}{k-1})^{1/(k-1)} \leq 1$, therefore $\sigma_k(\lambda) \leq \sigma_{k-1,1}(\lambda) [\lambda_1 + \binom{m-1}{k} / \binom{m-1}{k-1}] = \sigma_{k-1,1}(\lambda) (\lambda_1 + m/k - 1)$; then we get $\sigma_{k-1,1}(\lambda) \geq \sigma_k(\lambda) / (\lambda_1 + m/k - 1) \geq e^{-2\|f\|_\infty} \binom{m}{k} / (C_2 + m/k - 1)$.

Consequently $\sigma_{k-1,1}(\lambda) \geq \min(e^{-2\|f\|_\infty} \binom{m}{k} / C_2, \binom{m-1}{k-1}, e^{-2\|f\|_\infty} \binom{m}{k} / (C_2 + m/k - 1))$ or finally $\sigma_{k-1,1}(\lambda) \geq \min(\binom{m-1}{k-1}, e^{-2\|f\|_\infty} \binom{m}{k} / (C_2 + m/k - 1)) =: E$, where the constant E so defined depends only on $m, k, \|f\|_\infty$ and C_2 . $\square$

Similarly we prove the following.

Proposition 6.10 (uniform ellipticity). *There exists constants $E_0 > 0$ and $F_0 > 0$ depending only on $m, k, \|f\|_\infty$ and C'_2 such that for all $P \in M$, for all $1 \leq i \leq m$, $E_0 \leq \sigma_{k-1,i}(\lambda(P)) \leq F_0$.*

6.4. Gradient Uniform Estimate

The manifold M is Riemannian compact and $\varphi_t \in C^2(M)$, so by the Green's formula

$$\varphi_t(P) = \frac{1}{\text{Vol}(M)} \int_M \varphi_t(Q) dv_g(Q) + \int_M G(P, Q) \Delta \varphi_t(Q) dv_g(Q), \quad (6.24)$$

where $G(P, Q)$ is the Green's function of the Laplacian Δ . By differentiating locally under the integral sign we obtain $\partial_{u^i} \varphi_t(P) = \int_M \Delta \varphi_t(Q) (\partial_{u^i})_P G(P, Q) dv_g(Q)$. We infer then that at P in a holomorphic normal chart, we have

$$|(\nabla \varphi_t)_P| \leq \sqrt{2m} \int_M |\Delta \varphi_t(Q)| |\nabla_P G(P, Q)| dv_g(Q). \quad (6.25)$$

Now, using the uniform pinching of the eigenvalues, we prove easily the following estimate of the Laplacian.

Lemma 6.11. *There exists a constant $C_3 > 0$ depending on m and C'_2 such that $\|\Delta\varphi_t\|_{\infty, M} \leq C_3$.*

Combining Lemma 6.11 with (6.25), we deduce that $|(\nabla\varphi_t)_P| \leq \sqrt{2m}C_3 \int_M |\nabla_P G(P, Q)| dv_g(Q)$. Besides, classically [13, page 109], there exists constants $\mathcal{C}$ and $\mathcal{C}'$ such that

$$|\nabla_P G(P, Q)| \leq \frac{\mathcal{C}}{d_g(P, Q)^{2m-1}}, \quad \int_M \frac{1}{d_g(P, Q)^{2m-1}} dv_g(Q) \leq \mathcal{C}'. \quad (6.26)$$

We thus obtain the following result.

Proposition 6.12. *There exists a constant $C_5 > 0$ depending on m, C'_2 , and (M, g) such that for all $P \in M$ $|(\nabla\varphi_t)_P| \leq C_5$.*

Specifically, we can choose $C_5 = \sqrt{2m}C_3\mathcal{C}\mathcal{C}'$.

6.5. Second Derivatives Estimate

Our equation is of type:

$$F\left(P, [\partial_{u^i u^j} \varphi]_{1 \leq i, j \leq 2m}\right) = v, \quad P \in M. \quad (E)$$

6.5.1. The Functional

Consider the following functional:

$$\begin{aligned} \Phi : UT &\longrightarrow \mathbb{R} \\ (P, \xi) &\longmapsto \left(\nabla^2 \varphi_t\right)_P(\xi, \xi) + \frac{1}{2} |(\nabla \varphi_t)_P|_{g'}^2 \end{aligned} \quad (6.27)$$

where UT is the real unit sphere bundle associated to (TM, g) . Φ is continuous on the compact set UT , so it assumes its maximum at a point $(P_1, \xi_1) \in UT$.

6.5.2. Reduction to Finding a One-Sided Estimate for $(\nabla^2 \varphi_t)_{P_1}(\xi_1, \xi_1)$

If we find a uniform upper bound for $(\nabla^2 \varphi_t)_{P_1}(\xi_1, \xi_1)$, since $|\nabla \varphi_t|_{\infty} \leq C_5$, we readily deduce that there exists a constant $C_6 > 0$ such that

$$\left(\nabla^2 \varphi_t\right)_P(\xi, \xi) \leq C_6 \quad \forall (P, \xi) \in UT. \quad (6.28)$$

Fix $P \in M$. Let (U_P, ψ_P) be a holomorphic g -normal $\tilde{g}$ -adapted chart centered at P , namely, $[g_{ij}^-(P)]_{1 \leq i, j \leq m} = I_m$, $\partial_{\ell} g_{ij}^-(P) = 0$ and $[\tilde{g}_{ij}^-(P)]_{1 \leq i, j \leq m} = [\text{diag}(\lambda_1(P), \dots, \lambda_m(P))]$. Since $|\partial_{x^i}|_g =$

$\sqrt{2}$, we obtain $\partial_{x^j x^i} \varphi_t(P) = 2(\nabla^2 \varphi_t)_P(\partial_{x^j}/\sqrt{2}, \partial_{x^i}/\sqrt{2}) \leq 2C_6$ and similarly $\partial_{y^j y^i} \varphi_t(P) = 2(\nabla^2 \varphi_t)_P(\partial_{y^j}/\sqrt{2}, \partial_{y^i}/\sqrt{2}) \leq 2C_6$ for all $1 \leq j \leq m$. Besides, we have $\partial_{x^j x^i} \varphi_t(P) + \partial_{y^j y^i} \varphi_t(P) = 4\partial_{j\bar{j}} \varphi_t(P) = 4(\lambda_j(P) - 1) \geq -4[(m-1)C'_2 + 1]$; therefore we obtain

$$\begin{aligned}\partial_{x^j x^i} \varphi_t(P) &\geq -4[(m-1)C'_2 + 1] - 2C_6 =: -C_7, \\ \partial_{y^j y^i} \varphi_t(P) &\geq -C_7, \quad \forall 1 \leq j \leq m.\end{aligned}\tag{6.29}$$

Let us now bound second derivatives of mixed type $\partial_{u^r u^s} \varphi_t(P)$. Let $1 \leq r \neq s \leq 2m$. Since $|\partial_{x^r} \pm \partial_{x^s}|_g = 2$, we have $(\nabla^2 \varphi_t)_P((\partial_{x^r} \pm \partial_{x^s})/2, (\partial_{x^r} \pm \partial_{x^s})/2) = (1/4)\partial_{x^r x^r} \varphi_t(P) + (1/4)\partial_{x^s x^s} \varphi_t(P) \pm (1/2)\partial_{x^r x^s} \varphi_t(P) \leq C_6$, which yields $\pm \partial_{x^r x^s} \varphi_t(P) \leq 2C_6 - (1/2)\partial_{x^r x^r} \varphi_t(P) - (1/2)\partial_{x^s x^s} \varphi_t(P)$, hence as well $|\partial_{x^r x^s} \varphi_t(P)| \leq 2C_6 + C_7$. Similarly we prove that at P , in the above chart φ_P , we have $|\partial_{y^r y^s} \varphi_t(P)| \leq 2C_6 + C_7$ for all $1 \leq r \neq s \leq m$ and $|\partial_{x^r y^s} \varphi_t(P)| \leq 2C_6 + C_7$ for all $1 \leq r, s \leq m$. Consequently $|\partial_{u^i u^j} \varphi_t(P)| \leq 2C_6 + C_7$ for all $1 \leq i, j \leq 2m$. Therefore we deduce that

$$\left| \left(\nabla^2 \varphi_t \right) (P) \right|_g^2 = \frac{1}{4} \sum_{1 \leq i, j \leq 2m} (\partial_{u^i u^j} \varphi_t(P))^2 \leq m^2 (2C_6 + C_7)^2. \tag{6.30}$$

Theorem 6.13 (second derivatives uniform estimate). *There exists a constant $C_8 > 0$ depending only on m, C'_2 , and C_6 such that for all $P \in M$, $|(\nabla^2 \varphi_t)_P|_g \leq C_8$.*

This allows to deduce the needed uniform C^2 estimate:

$$\|\varphi\|_{C^2(M, \mathbb{R})} \leq C_0 + C_5 + C_8. \tag{6.31}$$

6.5.3. Chart Choice

For fixed $P \in M$, $\xi \in UT_P \mapsto (\nabla^2 \varphi_t)_P(\xi, \xi)$ is continuous on the compact subset UT_P (the fiber); therefore it assumes its maximum at a unit vector $\xi^P \in UT_P$. Besides, $(\nabla^2 \varphi_t)_P$ is a symmetric bilinear form on $T_P M$, so by the min-max principle we have $(\nabla^2 \varphi_t)_P(\xi^P, \xi^P) = \max_{\xi \in T_P M, g(\xi, \xi)=1} (\nabla^2 \varphi_t)_P(\xi, \xi) = \beta_{\max}(P)$, where $\beta_{\max}(P)$ denotes the largest eigenvalue of $(\nabla^2 \varphi_t)_P$ with respect to g_P ; furthermore we can choose ξ^P as the direction of the largest eigenvalue $\beta_{\max}(P)$. For fixed P , we now have $\max_{\xi \in T_P M, g_P(\xi, \xi)=1} \Phi(P, \xi) = \Phi(P, \xi^P) = (\nabla^2 \varphi_t)_P(\xi^P, \xi^P) + (1/2)|(\nabla \varphi_t)_P|_g^2 = \beta_{\max}(P) + (1/2)|(\nabla \varphi_t)_P|_g^2$, consequently $\max_{(P, \xi) \in UT} \Phi(P, \xi) = \max_{P \in M} \Phi(P, \xi^P) = \Phi(P_1, \xi_1) \leq \Phi(P_1, \xi^{P_1})$, hence $\max_{(P, \xi) \in UT} \Phi(P, \xi) = \Phi(P_1, \xi^{P_1}) = \beta_{\max}(P_1) + (1/2)|(\nabla \varphi_t)_{P_1}|_g^2$.

At the point P_1 , consider $\varepsilon_1^{P_1}, \dots, \varepsilon_{2m}^{P_1}$ a (real) basis of $(T_{P_1} M, g_{P_1})$ that satisfies the following properties:

- (i) $[g_{ij}(P_1)]_{1 \leq i, j \leq 2m} = I_{2m}$,
- (ii) $[(\nabla^2 \varphi_t)_{ij}(P_0)]_{1 \leq i, j \leq 2m} = \text{diag}(\beta_1, \dots, \beta_{2m})$,
- (iii) $\beta_1 = \beta_{\max}(P_1) \geq \beta_2 \geq \dots \geq \beta_{2m}$.

Let (U'_1, φ_1) be a C^∞ g -normal real chart at P_1 obtained from a holomorphic chart $z^1, \dots, z^m$ by setting $(u^1, \dots, u^{2m}) = (x^1, \dots, x^m, y^1, \dots, y^m)$ where $z^j = x^j + iy^j$ (namely, $[g_{ij}(P_1)]_{1 \leq i, j \leq 2m} =$

I_{2m} and $\partial_{u^\ell} g_{ij} = 0$ for all $1 \leq i, j, \ell \leq 2m$ satisfying $\varphi_1(P_1) = 0$ and $\partial_{u^i}|_{P_1} = \varepsilon_i^{P_1}$, so that $\partial_{u^i}|_{P_1}$ is the direction of the largest eigenvalue $\beta_{\max}(P_1)$.

6.5.4. Auxiliary Local Functional

From now on, we work in the *real chart* (U'_1, φ_1) so constructed at P_1 .

Let $U_2 \subset U'_1$ be an open subset such that $g_{11}(P) > 0$ for all $P \in U_2$, and let Φ_1 be the functional

$$\begin{aligned} \Phi_1 : U_2 &\longrightarrow \mathbb{R} \\ P &\longmapsto \Phi_1(P) = \frac{(\nabla^2 \varphi_t)_{11}(P)}{g_{11}(P)} + \frac{1}{2} |(\nabla \varphi_t)_P|_g^2. \end{aligned} \quad (6.32)$$

We claim that Φ_1 assumes its maximum at P_1 . Indeed, $(\nabla^2 \varphi_t)_{11}(P)/g_{11}(P) = (\nabla^2 \varphi)_P(\partial_{u^1}, \partial_{u^1})/g_P(\partial_{u^1}, \partial_{u^1}) = (\nabla^2 \varphi)_P(\partial_{u^1}/|\partial_{u^1}|_g, \partial_{u^1}/|\partial_{u^1}|_g) \leq \beta_{\max}(P)$, so $\Phi_1(P) \leq \beta_{\max}(P) + (1/2)|(\nabla \varphi_t)_P|_g^2 \leq \beta_{\max}(P_1) + (1/2)|(\nabla \varphi_t)_{P_1}|_g^2 = \Phi_1(P_1)$ proving our claim.

Let us now differentiate twice in the *real direction* ∂_{u^1} the equation

$$F\left(P, [\partial_{u^i u^j} \varphi]_{1 \leq i, j \leq 2m}\right) = v. \quad (E^*)$$

At the point P , in a chart u , we obtain

$$\partial_{u^1} v = \frac{\partial F}{\partial u^1} [\varphi] + \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \partial_{u^1 u^i u^j} \varphi. \quad (6.33)$$

Differentiating once again

$$\begin{aligned} \partial_{u^1 u^1} v &= \frac{\partial^2 F}{\partial u^1 \partial u^1} [\varphi] + \sum_{i,j=1}^{2m} \frac{\partial^2 F}{\partial r_{ij} \partial u^1} [\varphi] \partial_{u^1 u^i u^j} \varphi \\ &+ \sum_{i,j=1}^{2m} \left[\frac{\partial^2 F}{\partial u^1 \partial r_{ij}} [\varphi] + \sum_{e,s=1}^{2m} \frac{\partial^2 F}{\partial r_{es} \partial r_{ij}} [\varphi] \partial_{u^1 u^e u^s} \varphi \right] \partial_{u^1 u^i u^j} \varphi \\ &+ \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \partial_{u^1 u^1 u^i u^j} \varphi. \end{aligned} \quad (6.34)$$

But at the point P_1 , for our function $F(P, r) = F_k[\delta_i^j + (1/4)g^{j\bar{\ell}}(P)(r_{i\ell} + r_{(i+m)(\ell+m)} + ir_{i(\ell+m)} - ir_{(i+m)\ell})]_{1 \leq i, j \leq m}$, we have $(\partial^2 F / \partial r_{ij} \partial u^1) [\varphi] = 0$ since $\partial_{u^1} g^{s\bar{q}}(P_1) = 0$. Hence, we infer that

$$\begin{aligned} \partial_{u^1 u^1} v &= \frac{\partial^2 F}{\partial u^1 \partial u^1} [\varphi] + \sum_{i,j,e,s=1}^{2m} \frac{\partial^2 F}{\partial r_{es} \partial r_{ij}} [\varphi] \partial_{u^1 u^e u^s} \varphi \partial_{u^1 u^i u^j} \varphi \\ &+ \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \partial_{u^1 u^1 u^i u^j} \varphi. \end{aligned} \quad (6.35)$$

6.5.5. Using Concavity

The function F is concave with respect to the variable r . Indeed

$$\begin{aligned}
 F(P, r) &= F_k \left[\delta_i^j + \frac{1}{4} g^{j\bar{\ell}}(P) (r_{i\ell} + r_{(i+m)(\ell+m)} + i r_{i(\ell+m)} - i r_{(i+m)\ell}) \right]_{1 \leq i, j \leq m} \\
 &= F_k \left(g^{-1}(P) \tilde{r} \right), \quad \text{where} \\
 \tilde{r} &= \left[g_{i\bar{j}}(P) + \frac{1}{4} (r_{ij} + r_{(i+m)(j+m)} + i r_{i(j+m)} - i r_{(i+m)j}) \right]_{1 \leq i, j \leq m} \\
 &= F_k \left(\underbrace{g^{-1/2}(P) \tilde{r} g^{-1/2}(P)}_{\in \mathcal{H}_m(\mathbb{C})} \right) \\
 &= F_k(\rho_P(r)), \quad \text{where} \\
 \rho_P(r) &:= \left[\delta_i^j + \frac{1}{4} \sum_{\ell, s=1}^m (g^{-1/2}(P))_{i\ell} (g^{-1/2}(P))_{sj} (r_{\ell s} + r_{(\ell+m)(s+m)} + i r_{\ell(s+m)} - i r_{(\ell+m)s}) \right]_{1 \leq i, j \leq m}
 \end{aligned} \tag{6.36}$$

but for a fixed point P the function $r \in S_{2m}(\mathbb{R}) \mapsto \rho_P(r) \in \mathcal{H}_m(\mathbb{C})$ is affine (where $S_{2m}(\mathbb{R})$ denotes the set of symmetric matrices of size $2m$); we deduce then that the composition $F(P, \cdot) = F_k \circ \rho_P$ is concave on the set $\{r \in S_{2m}(\mathbb{R}) / \rho_P(r) \in \lambda^{-1}(\Gamma_k)\} = \rho_P^{-1}(\lambda^{-1}(\Gamma_k))$, which proves our claim. Hence, since the matrix $[\partial_{u^1 u^i u^j} \varphi]_{1 \leq i, j \leq m}$ is symmetric, we obtain that

$$\sum_{i, j, e, s=1}^{2m} \frac{\partial^2 F}{\partial r_{es} \partial r_{ij}} [\varphi] \partial_{u^1 u^e u^s} \varphi \partial_{u^1 u^i u^j} \varphi \leq 0. \tag{6.37}$$

Consequently

$$\partial_{u^1 u^1} v - \partial_{u^1 u^1} F[\varphi] \leq \sum_{i, j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \partial_{u^1 u^i u^j} \varphi. \tag{6.38}$$

Let us now calculate the quantity $\partial_{u^1 u^1} F[\varphi]$ (at P_1). Since $\partial_{u^1} g^{s\bar{q}}(P_1) = 0$, we have

$$\frac{\partial^2 F}{\partial u^1 \partial u^1} (P_1, D^2 \varphi(P_1)) = \sum_{s=1}^m \frac{\sigma_{k-1, s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \times \partial_{u^1 u^1} g^{s\bar{s}}(P_1) \partial_{s\bar{s}} \varphi(P_1). \tag{6.39}$$

But at P_1 , $\partial_{u^1 u^1} g^{s\bar{s}} = -g^{s\bar{o}} g^{q\bar{s}} \partial_{u^1 u^1} g_{q\bar{o}}$ and $[g^{i\bar{j}}]_{1 \leq i, j \leq m} = 2I_m$, then $\partial_{u^1 u^1} g^{s\bar{s}} = -4\partial_{u^1 u^1} g_{s\bar{s}}$ so that $\partial_{u^1 u^1} g^{s\bar{s}} = -\partial_{u^1 u^1} g_{s\bar{s}} - \partial_{u^1 u^1} g_{(s+m)(s+m)}$. Moreover $\Gamma_{u^j u^s}^{u^r} = (1/2)(\partial_{u^j} g_{os} + \partial_{u^s} g_{oj} - \partial_{u^o} g_{js}) g^{or}$, thus $\partial_{u^i} \Gamma_{u^j u^s}^{u^r} = (1/2)(\partial_{u^i u^j} g_{rs} + \partial_{u^i u^s} g_{rj} - \partial_{u^i u^r} g_{js})$. Similarly, we have at P_1 : $\partial_{u^i} \Gamma_{u^j u^r}^{u^s} = (1/2)(\partial_{u^i u^j} g_{rs} + \partial_{u^i u^r} g_{sj} - \partial_{u^i u^s} g_{jr})$. Consequently, we deduce that $\partial_{u^i u^j} g_{rs} = \partial_{u^i} \Gamma_{u^j u^s}^{u^r} + \partial_{u^i} \Gamma_{u^j u^r}^{u^s}$. Hence, we have

at P_1 : $\partial_{u^1 u^1} g^{s\bar{s}} = -2\partial_{u^1} \Gamma_{u^1 u^s}^{u^s} - 2\partial_{u^1} \Gamma_{u^1 u^{s+m}}^{u^{s+m}}$. Besides, $\partial_{s\bar{s}} \varphi = (1/4)(\partial_{u^s u^s} \varphi + \partial_{u^{s+m} u^{s+m}} \varphi)$, which infers that at P_1

$$\begin{aligned} \partial_{u^1 u^1} F[\varphi] &= \sum_{s=1}^m \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \times \left(-2\partial_{u^1} \Gamma_{u^1 u^s}^{u^s} - 2\partial_{u^1} \Gamma_{u^1 u^{s+m}}^{u^{s+m}} \right) \\ &\times \frac{1}{4} (\partial_{u^s u^s} \varphi + \partial_{u^{s+m} u^{s+m}} \varphi). \end{aligned} \quad (6.40)$$

Consequently, the inequality (6.38) becomes

$$\begin{aligned} \partial_{u^1 u^1} v &\leq \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \partial_{u^1 u^1 u^i u^j} \varphi - \frac{1}{2} \sum_{s=1}^m \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \\ &\times \left(\partial_{u^1} \Gamma_{u^1 u^s}^{u^s} + \partial_{u^1} \Gamma_{u^1 u^{s+m}}^{u^{s+m}} \right) \times (\partial_{u^s u^s} \varphi + \partial_{u^{s+m} u^{s+m}} \varphi). \end{aligned} \quad (6.41)$$

6.5.6. Differentiation of the Functional Φ_1

We differentiate twice the functional Φ_1 :

$$\begin{aligned} \Phi_1(P) &= \frac{(\nabla^2 \varphi)_{11}(P)}{g_{11}(P)} + \frac{1}{2} |(\nabla \varphi)_P|_g^2, \\ \partial_{u^i} \Phi_1(P) &= \frac{\partial_{u^i} (\nabla^2 \varphi)_{11}}{g_{11}(P)} - \frac{(\nabla^2 \varphi)_{11} \partial_{u^i} g_{11}(P)}{g_{11}(P)^2} + \frac{1}{2} \partial_{u^i} |(\nabla \varphi)_P|_g^2, \\ \partial_{u^i u^j} \Phi_1(P) &= \frac{\partial_{u^i u^j} (\nabla^2 \varphi)_{11}}{g_{11}(P)} - \frac{\partial_{u^j} (\nabla^2 \varphi)_{11} \partial_{u^i} g_{11}(P)}{g_{11}(P)^2} \\ &- \frac{\partial_{u^i} (\nabla^2 \varphi)_{11} \partial_{u^j} g_{11}(P) + (\nabla^2 \varphi)_{11}(P) \partial_{u^i u^j} g_{11}(P)}{g_{11}(P)^2} \\ &- (\nabla^2 \varphi)_{11}(P) \partial_{u^j} g_{11}(P) \partial_{u^i} \left(\frac{1}{g_{11}(P)^2} \right) + \frac{1}{2} \partial_{u^i u^j} |(\nabla \varphi)_P|_g^2. \end{aligned} \quad (6.42)$$

Hence, at P_1 in the chart φ_1 , we obtain

$$\partial_{u^i u^j} \Phi_1 = \partial_{u^i u^j} (\nabla^2 \varphi)_{11} - (\nabla^2 \varphi)_{11}(P_1) \partial_{u^i u^j} g_{11} + \frac{1}{2} \partial_{u^i u^j} |(\nabla \varphi)_P|_g^2(P_1). \quad (6.43)$$

Let us now calculate the different terms of this formula (at P_1 in the chart φ_1):

$$\begin{aligned} \partial_{u^i u^j} (\nabla^2 \varphi)_{11} &= \partial_{u^i u^j} (\partial_{u^1 u^1} \varphi - \Gamma_{u^1 u^1}^{u^s} \partial_{u^s} \varphi) \\ &= \partial_{u^i u^j} \partial_{u^1 u^1} \varphi - \partial_{u^i u^j} \Gamma_{u^1 u^1}^{u^s} \partial_{u^s} \varphi - \partial_{u^j} \Gamma_{u^1 u^1}^{u^s} \partial_{u^i u^s} \varphi - \partial_{u^i} \Gamma_{u^1 u^1}^{u^s} \partial_{u^j u^s} \varphi. \end{aligned} \quad (6.44)$$

Besides, we have $\Gamma_{u^i u^1}^{u^1} = (1/2)(\partial_{u^1} g_{s1} + \partial_{u^1} g_{sj} - \partial_{u^s} g_{j1}) g^{s1}$; therefore we deduce that $\partial_{u^i} \Gamma_{u^1 u^1}^{u^1} = (1/2)(\partial_{u^i u^1} g_{s1} + \partial_{u^i u^1} g_{sj} - \partial_{u^i u^s} g_{j1}) g^{s1} + 0 = (1/2) \partial_{u^i u^1} g_{11}$; namely, $\partial_{u^i u^1} g_{11} = 2 \partial_{u^i} \Gamma_{u^1 u^1}^{u^1}$. Moreover, we have at P_1

$$\begin{aligned} \partial_{u^i u^j} |(\nabla \varphi)_P|_g^2 &= \partial_{u^i u^j} \left(\sum_{r,s=1}^{2m} g^{rs} \partial_{u^r} \varphi \partial_{u^s} \varphi \right) \\ &= \sum_{r,s=1}^{2m} \partial_{u^i u^j} g^{rs} \partial_{u^r} \varphi \partial_{u^s} \varphi + g^{rs} \partial_{u^i u^j} \partial_{u^r} \varphi \partial_{u^s} \varphi \\ &\quad + g^{rs} \partial_{u^j u^r} \varphi \partial_{u^i u^s} \varphi + g^{rs} \partial_{u^i u^r} \varphi \partial_{u^j u^s} \varphi + g^{rs} \partial_{u^r} \varphi \partial_{u^i u^j u^s} \varphi \\ &= \sum_{r,s=1}^{2m} \partial_{u^i u^j} g^{rs} \partial_{u^r} \varphi \partial_{u^s} \varphi + 2 \sum_{s=1}^{2m} \partial_{u^i u^j u^s} \varphi \partial_{u^s} \varphi \\ &\quad + 2 \sum_{s=1}^{2m} \partial_{u^i u^s} \varphi \partial_{u^j u^s} \varphi. \end{aligned} \quad (6.45)$$

But at P_1 , $\partial_{u^i u^j} g^{rs} = -\partial_{u^i u^j} g_{rs}$, in addition at this point $\partial_{u^i u^j} g_{rs} = \partial_{u^i} \Gamma_{u^j u^s}^{u^r} + \partial_{u^i} \Gamma_{u^j u^r}^{u^s}$; therefore we obtain at P_1 in the chart φ_1

$$\begin{aligned} \partial_{u^i u^j} |(\nabla \varphi)_P|_g^2 &= -2 \sum_{r,s=1}^{2m} \partial_{u^i} \Gamma_{u^j u^s}^{u^r} \partial_{u^r} \varphi \partial_{u^s} \varphi + 2 \sum_{s=1}^{2m} \partial_{u^i u^j u^s} \varphi \partial_{u^s} \varphi \\ &\quad + 2 \sum_{s=1}^{2m} \partial_{u^i u^s} \varphi \partial_{u^j u^s} \varphi. \end{aligned} \quad (6.46)$$

Henceforth, and in order to lighten the notations, we use ∂_i instead of ∂_{u^i} and Γ_{ij}^s instead of $\Gamma_{u^i u^j}^{u^s}$, so we have

$$\begin{aligned} \partial_{ij} \Phi_1 &= \partial_{ij11} \varphi - \partial_{ij} \Gamma_{11}^s \partial_s \varphi - \partial_j \Gamma_{11}^s \partial_{is} \varphi - \partial_i \Gamma_{11}^s \partial_{js} \varphi - 2 \partial_i \Gamma_{j1}^1 (\nabla^2 \varphi)_{11} (P_1) \\ &\quad - \sum_{r,s=1}^{2m} \partial_i \Gamma_{js}^r \partial_r \varphi \partial_s \varphi + \sum_{s=1}^{2m} \partial_{ijs} \varphi \partial_s \varphi + \sum_{s=1}^{2m} \partial_{is} \varphi \partial_{js} \varphi. \end{aligned} \quad (6.47)$$

Let us now consider the second order linear operator:

$$\tilde{L} = \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \partial_{ij}. \quad (6.48)$$

Since the functional Φ_1 assumes its maximum at the point P_1 , we have $\tilde{L}(\Phi_1) \leq 0$ at P_1 in the chart φ_1 . Besides, combining the inequalities (6.41) and (6.47), we obtain

$$\begin{aligned}
2 \underbrace{\tilde{L}\Phi_1}_{\leq 0} - \partial_{11}v &\geq \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \left[\partial_{ij11}\varphi - \partial_{ij}\Gamma_{11}^s \partial_s \varphi - \partial_j \Gamma_{11}^i \partial_{ii}\varphi \right. \\
&\quad - \partial_i \Gamma_{11}^j \partial_{jj}\varphi - 2\partial_i \Gamma_{j1}^1 (\nabla^2 \varphi)_{11} (P_1) \\
&\quad - \sum_{r,s=1}^{2m} \partial_i \Gamma_{js}^r \partial_r \varphi \partial_s \varphi \\
&\quad \left. + \sum_{s=1}^{2m} \partial_{ijs} \varphi \partial_s \varphi + \delta_i^j (\partial_{ii}\varphi)^2 - \partial_{11ij}\varphi \right] \\
&\quad + \frac{1}{2} \sum_{s=1}^m \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} (\partial_1 \Gamma_{1s}^s + \partial_1 \Gamma_{1(s+m)}^{s+m}) (\partial_{ss}\varphi + \partial_{(s+m)(s+m)}\varphi).
\end{aligned} \tag{6.49}$$

The fourth derivatives are simplified. Moreover, we have at $P_1: \partial_s v = (\partial F / \partial u^1)[\varphi] + \sum_{i,j=1}^{2m} (\partial F / \partial r_{ij})[\varphi] \partial_{sij}\varphi$ with $(\partial F / \partial u^1)(P_1, D^2\varphi(P_1)) = 0$, consequently:

$$\begin{aligned}
0 &\geq \partial_{11}v + \sum_{s=1}^{2m} \partial_s v \partial_s \varphi \\
&\quad + \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \left[-2\partial_i \Gamma_{j1}^1 (\nabla^2 \varphi)_{11} (P_1) - \partial_i \Gamma_{11}^j \partial_{jj}\varphi - \partial_j \Gamma_{11}^i \partial_{ii}\varphi \right. \\
&\quad \left. - \sum_{s=1}^{2m} \partial_{ij}\Gamma_{11}^s \partial_s \varphi - \sum_{r,s=1}^{2m} \partial_i \Gamma_{js}^r \partial_r \varphi \partial_s \varphi + \delta_i^j (\partial_{ii}\varphi)^2 \right] \\
&\quad + \frac{1}{2} \sum_{s=1}^m \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} (\partial_1 \Gamma_{1s}^s + \partial_1 \Gamma_{1(s+m)}^{s+m}) (\partial_{ss}\varphi + \partial_{(s+m)(s+m)}\varphi).
\end{aligned} \tag{6.50}$$

Let us now express the quantities $\partial_i \Gamma_{j1}^1, \partial_i \Gamma_{11}^j, \partial_j \Gamma_{11}^i, \partial_i \Gamma_{js}^r$ and $\partial_{ij} \Gamma_{11}^s$ using the components of the Riemann curvature (at the point P_1 in the normal chart φ_1):

$$\begin{aligned}
\partial_i \Gamma_{j1}^1 &= \frac{1}{3} \left(R_{j11i} + \underbrace{R_{ji11}}_{=0} \right) = \frac{1}{3} R_{j11i}, \\
\partial_i \Gamma_{11}^j &= \frac{1}{3} (R_{1j1i} + R_{1i1j}) = \frac{2}{3} R_{1j1i}, \\
\partial_j \Gamma_{11}^i &= \frac{2}{3} R_{1i1j}, \\
\partial_i \Gamma_{js}^r &= \frac{1}{3} (R_{jr si} + R_{jisr}),
\end{aligned}$$

$$\begin{aligned}
\partial_1 \Gamma_{1s}^s &= \frac{1}{3} \left(R_{1ss1} + \underbrace{R_{11ss}}_{=0} \right) = \frac{1}{3} R_{1ss1}, \\
\partial_1 \Gamma_{1(s+m)}^{s+m} &= \frac{1}{3} R_{1(s+m)(s+m)1}, \\
\partial_{ij} \Gamma_{11}^s &= \frac{1}{4} (\nabla_i R_{1j1s} + \nabla_i R_{1s1j} + \nabla_j R_{1s1i} + \nabla_j R_{1i1s}) \\
&\quad - \frac{1}{12} (\nabla_s R_{1i1j} + \nabla_s R_{1j1i}) = \frac{1}{2} (\nabla_i R_{1s1j} + \nabla_j R_{1s1i}) - \frac{1}{6} \nabla_s R_{1i1j}.
\end{aligned} \tag{6.51}$$

We then obtain

$$\begin{aligned}
0 &\geq \partial_{11} v + \sum_{s=1}^{2m} \partial_s v \partial_s \varphi \\
&\quad + \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \left[\frac{-2}{3} R_{j11i} (\nabla^2 \varphi)_{11} (P_1) - \frac{2}{3} R_{1j1i} \partial_{jj} \varphi - \frac{2}{3} R_{1i1j} \partial_{ii} \varphi \right. \\
&\quad \left. - \sum_{s=1}^{2m} \left(\frac{1}{2} \nabla_i R_{1s1j} + \frac{1}{2} \nabla_j R_{1s1i} - \frac{1}{6} \nabla_s R_{1i1j} \right) \partial_s \varphi \right. \\
&\quad \left. - \sum_{r,s=1}^{2m} \frac{1}{3} (R_{jr si} + R_{jisr}) \partial_r \varphi \partial_s \varphi + \delta_i^j (\partial_{ii} \varphi)^2 \right] \\
&\quad + \frac{1}{2} \sum_{s=1}^m \frac{\sigma_{k-1,s}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \frac{1}{3} (R_{1ss1} + R_{1(s+m)(s+m)1}) (\partial_{ss} \varphi + \partial_{(s+m)(s+m)} \varphi).
\end{aligned} \tag{6.52}$$

6.5.7. The Uniform Upper Bound of $\beta_1 = (\nabla^2 \varphi)_{P_1}(\xi_1, \xi_1)$

By the uniform estimate of the gradient we have $|\partial_j \varphi_t| \leq C_5$ for all $1 \leq j \leq 2m$. Moreover, at P_1 in the chart φ_1 : $[(\nabla^2 \varphi)_{ij}(P_1)]_{1 \leq i, j \leq 2m} = [\partial_{ij} \varphi(P_1)]_{1 \leq i, j \leq 2m} = \text{diag}(\beta_1, \dots, \beta_{2m})$. Consequently

$$\begin{aligned}
0 &\geq \partial_{11} v + \sum_{s=1}^{2m} \partial_s v \partial_s \varphi \\
&\quad + \sum_{i,j=1}^{2m} \frac{\partial F}{\partial r_{ij}} [\varphi] \left[\delta_{ij} (\beta_i)^2 - \frac{2}{3} R_{j11i} \beta_1 - \frac{2}{3} R_{1j1i} \beta_j - \frac{2}{3} R_{1i1j} \beta_i - \frac{1}{3} \sum_{r,s=1}^{2m} (R_{jr si} + R_{jisr}) \partial_r \varphi \partial_s \varphi \right. \\
&\quad \left. - \frac{1}{2} \sum_{s=1}^{2m} \left(\nabla_i R_{1s1j} + \nabla_j R_{1s1i} - \frac{1}{3} \nabla_s R_{1i1j} \right) \partial_s \varphi \right] \\
&\quad + \frac{1}{6} \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \times (R_{1i i1} + R_{1(i+m)(i+m)1}) (\beta_i + \beta_{i+m}).
\end{aligned} \tag{6.53}$$

But for $F[\varphi] = F_k([\delta_i^j + g^{j\bar{\ell}} \partial_{i\bar{\ell}} \varphi]_{1 \leq i, j \leq m})$ since $\partial_{s\bar{s}} \varphi = (1/4)(\partial_{u^s u^{\bar{s}}} + \partial_{u^{s+m} u^{\bar{s}+m}})$, we obtain at P_1 in the chart φ_1 that

$$\frac{\partial F}{\partial r_{ij}}[\varphi] = \sum_{s=1}^m \frac{\partial F_k}{\partial B_s^s}(\text{diag}(\lambda(P_1))) \frac{1}{4} \frac{\partial(r_{ss} + r_{(s+m)(s+m)})}{\partial r_{ij}}. \quad (6.54)$$

Then

$$\begin{aligned} \forall 1 \leq i \neq j \leq 2m \quad \frac{\partial F}{\partial r_{ij}}[\varphi] &= 0, \\ \forall 1 \leq i \leq m \quad \frac{\partial F}{\partial r_{ii}}[\varphi] &= \frac{\partial F}{\partial r_{(i+m)(i+m)}}[\varphi] = \frac{1}{4} \frac{\partial F_k}{\partial B_i^i}(\text{diag}(\lambda(P_1))) \\ &= \frac{1}{4} \frac{\sigma_{k-1,i}(\lambda(P_1))}{\underbrace{\sigma_k(\lambda(P_1))}_{>0 \text{ since } \lambda(P_1) \in \Gamma_k}}. \end{aligned} \quad (6.55)$$

Hence

$$\begin{aligned} 0 &\geq \partial_{11} v + \sum_{s=1}^{2m} \partial_s v \partial_s \varphi + \sum_{i=1}^{2m} \frac{\partial F}{\partial r_{ii}}[\varphi] \\ &\times \left[(\beta_i)^2 + \frac{2}{3} R_{1i1i} (\beta_1 - 2\beta_i) + \frac{1}{3} \sum_{r,s=1}^{2m} R_{iris} \partial_r \varphi \partial_s \varphi - \sum_{s=1}^{2m} \left(\nabla_i R_{1s1i} - \frac{1}{6} \nabla_s R_{1i1i} \right) \partial_s \varphi \right] \\ &+ \frac{1}{6} \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} (R_{1i1i} + R_{1(i+m)(i+m)1}) (\beta_i + \beta_{i+m}). \end{aligned} \quad (6.56)$$

But at P_1 in the chart φ_1 , $\|R\|_g^2 = g^{ai} g^{bj} g^{cr} g^{ds} R_{abcd} R_{ijrs} = \sum_{a,b,c,d=1}^{2m} (R_{abcd})^2$; then $|R_{abcd}| \leq \|R\|_g$ for all $a, b, c, d \in \{1, \dots, 2m\}$, consequently

$$\begin{aligned} \left| \frac{1}{3} \sum_{r,s=1}^{2m} R_{iris} \partial_r \varphi \partial_s \varphi \right| &\leq \frac{1}{3} \sum_{r,s=1}^{2m} \|R\|_g (C_5)^2 = \frac{1}{3} (2m)^2 \|R\|_g (C_5)^2 \\ &= \frac{4}{3} m^2 (C_5)^2 \|R\|_g. \end{aligned} \quad (6.57)$$

Besides, at P_1 in the chart φ_1 , we have $\|\nabla R\|_g^2 = g^{el} g^{ai} g^{bj} g^{cr} g^{ds} \nabla_e R_{abcd} \nabla_l R_{ijrs} = \sum_{e,a,b,c,d=1}^{2m} (\nabla_e R_{abcd})^2$, so $|\nabla_e R_{abcd}| \leq \|\nabla R\|_g$ for all $e, a, b, c, d \in \{1, \dots, 2m\}$, therefore

$$\begin{aligned} \left| - \sum_{s=1}^{2m} \left(\nabla_i R_{1s1i} - \frac{1}{6} \nabla_s R_{1i1i} \right) \partial_s \varphi \right| &\leq \sum_{s=1}^{2m} \frac{7}{6} \|\nabla R\|_g C_5 = 2m \frac{7}{6} \|\nabla R\|_g C_5 \\ &= \frac{7}{3} m C_5 \|\nabla R\|_g. \end{aligned} \quad (6.58)$$

Hence at P_1 in the chart φ_1 , we obtain

$$\begin{aligned} -t\partial_{11}f - t \sum_{s=1}^{2m} \partial_s f \partial_s \varphi &\geq \sum_{i=1}^{2m} \frac{\partial F}{\partial r_{ii}} [\varphi] \left[(\beta_i)^2 + \frac{2}{3} R_{1i1i} (\beta_1 - 2\beta_i) \right] \\ &\quad + \frac{1}{6} \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \times (R_{1i1i} + R_{1(i+m)(i+m)1}) (\beta_i + \beta_{i+m}) \\ &\quad + \left(\sum_{i=1}^{2m} \frac{\partial F}{\partial r_{ii}} [\varphi] \right) \times \left[-\frac{4}{3} m^2 (C_5)^2 \|R\|_g - \frac{7}{3} m C_5 \|\nabla R\|_g \right]. \end{aligned} \quad (6.59)$$

But $|\partial_{11}f(P_1)| \leq \|f\|_{C^2(M)}$, $|\partial_s f(P_1)| \leq \|f\|_{C^2(M)}$ and $|\partial_s \varphi| \leq C_5$ for all s then

$$-t\partial_{11}f - t \sum_{s=1}^{2m} \partial_s f \partial_s \varphi \leq \|f\|_{C^2(M)} (1 + 2m C_5). \quad (6.60)$$

Besides

$$\sum_{i=1}^{2m} \frac{\partial F}{\partial r_{ii}} [\varphi] = \sum_{i=1}^m \frac{\partial F}{\partial r_{ii}} [\varphi] + \frac{\partial F}{\partial r_{(i+m)(i+m)}} [\varphi] = \frac{1}{2} \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))}. \quad (6.61)$$

Consequently, we obtain

$$\begin{aligned} \|f\|_{C^2(M)} (1 + 2m C_5) &\geq \frac{\partial F}{\partial r_{11}} [\varphi] (\beta_1)^2 + \frac{2}{3} \sum_{i=1}^{2m} \frac{\partial F}{\partial r_{ii}} [\varphi] R_{1i1i} (\beta_1 - 2\beta_i) \\ &\quad + \frac{1}{6} \sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \times (R_{1i1i} + R_{1(i+m)(i+m)1}) (\beta_i + \beta_{i+m}) \\ &\quad + \frac{1}{2} \left(\sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \right) \times \left[-\frac{4}{3} m^2 (C_5)^2 \|R\|_g - \frac{7}{3} m C_5 \|\nabla R\|_g \right]. \end{aligned} \quad (6.62)$$

Let us now estimate $|\beta_i|$ for $1 \leq i \leq m$ using β_1 . We follow the same method as for the proof of Theorem 6.13. For all $(P, \xi) \in UT$, we have the inequality $(\nabla^2 \varphi_t)_P(\xi, \xi) \leq \beta_1 + (1/2)(C_5)^2$; then at P in a holomorphic g -normal $\tilde{g}$ -adapted chart φ_P , namely, a chart such that $[g_{i\bar{j}}(P)]_{1 \leq i, j \leq m} = I_m$, $\partial_{\ell} g_{i\bar{j}}(P) = 0$ and $[\tilde{g}_{i\bar{j}}(P)]_{1 \leq i, j \leq m} = \text{diag}(\lambda_1(P), \dots, \lambda_m(P))$, we deduce that for all $j \in \{1, \dots, m\}$

$$\begin{aligned} \partial_{x^j x^j} \varphi_t(P) &= 2 \left(\nabla^2 \varphi_t \right)_P \left(\frac{\partial_{x^j}}{\sqrt{2}}, \frac{\partial_{x^j}}{\sqrt{2}} \right) \leq 2\beta_1 + (C_5)^2, \\ \partial_{y^j y^j} \varphi_t(P) &\leq 2\beta_1 + (C_5)^2. \end{aligned} \quad (6.63)$$

Since $\lambda_j(P) \geq -(m-1)C'_2$, we infer the following inequalities:

$$\begin{aligned} \forall j \in \{1, \dots, m\} \quad \partial_{x^j x^j} \varphi_t(P) &\geq -4[(m-1)C'_2 + 1] - 2\beta_1 - (C_5)^2, \\ \partial_{y^j y^j} \varphi_t(P) &\geq -4[(m-1)C'_2 + 1] - 2\beta_1 - (C_5)^2. \end{aligned} \quad (6.64)$$

Consequently

$$\forall 1 \leq i, j \leq 2m \quad |\partial_{u^i u^j} \varphi_t(P)| \leq 4\beta_1 + 2(C_5)^2 + \underbrace{4[(m-1)C'_2 + 1]}_{=: C_9}, \quad (6.65)$$

in the chart φ_P .

Hence we infer that

$$\left| \left(\nabla^2 \varphi_t \right)_P \right|_g^2 = \frac{1}{4} \sum_{i,j=1}^{2m} (\partial_{u^i u^j} \varphi_t(P))^2 \leq m^2 [4\beta_1 + 2(C_5)^2 + C_9]^2 \quad \forall P. \quad (6.66)$$

But at P_1 in the chart φ_1 , $|\nabla^2 \varphi_t|_{P_1}|_g^2 = \sum_{i=1}^{2m} (\partial_{u^i u^i} \varphi_t(P_1))^2 = \sum_{i=1}^{2m} (\beta_i)^2$; consequently we obtain

$$\forall 1 \leq i \leq 2m \quad |\beta_i| \leq m(4\beta_1 + 2(C_5)^2 + C_9). \quad (6.67)$$

Thus

$$\begin{aligned} |(R_{1i1i})(\beta_1 - 2\beta_i)| &\leq |R_{1i1i}|(|\beta_1| + 2|\beta_i|) \\ &\leq 3m\|R\|_g(4\beta_1 + 2(C_5)^2 + C_9). \end{aligned} \quad (6.68)$$

Besides

$$\begin{aligned} |(R_{1i1i} + R_{1(i+m)(i+m)1})(\beta_i + \beta_{i+m})| &\leq (|R_{1i1i}| + |R_{1(i+m)(i+m)1}|)(|\beta_i| + |\beta_{i+m}|) \\ &\leq 4m\|R\|_g(4\beta_1 + 2(C_5)^2 + C_9). \end{aligned} \quad (6.69)$$

Hence

$$\begin{aligned}
\|f\|_{C^2(M)}(1+2mC_5) &\geq \frac{1}{4} \frac{\sigma_{k-1,1}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} (\beta_1)^2 \\
&+ \left(\sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \right) (-m) \|R\|_g (4\beta_1 + 2(C_5)^2 + C_9) \\
&+ \left(\sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \right) \left(-\frac{2}{3}m \right) \|R\|_g (4\beta_1 + 2(C_5)^2 + C_9) \\
&+ \left(\sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \right) \left[-\frac{7}{6}mC_5 \|\nabla R\|_g - \frac{2}{3}m^2(C_5)^2 \|R\|_g \right].
\end{aligned} \tag{6.70}$$

Then

$$\begin{aligned}
\|f\|_{C^2(M)}(1+2mC_5) &\geq \frac{1}{4} \frac{\sigma_{k-1,1}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} (\beta_1)^2 \\
&+ \left(\sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \right) \left(-\frac{5}{3}m \right) \|R\|_g (4\beta_1 + 2(C_5)^2 + C_9) \\
&+ \left(\sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \right) \left[-\frac{7}{6}mC_5 \|\nabla R\|_g - \frac{2}{3}m^2(C_5)^2 \|R\|_g \right].
\end{aligned} \tag{6.71}$$

But using the uniform ellipticity and the inequalities $e^{-2\|f\|_\infty} \binom{m}{k} \leq \sigma_k(\lambda(P)) \leq e^{2\|f\|_\infty} \binom{m}{k}$, we obtain

$$\sum_{i=1}^m \frac{\sigma_{k-1,i}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \leq \frac{me^{2\|f\|_\infty} F_0}{\binom{m}{k}}, \tag{6.72}$$

$$\frac{\sigma_{k-1,1}(\lambda(P_1))}{\sigma_k(\lambda(P_1))} \geq \frac{e^{-2\|f\|_\infty} E_0}{\binom{m}{k}}. \tag{6.73}$$

Then at P_1 in the chart φ_1 , we have

$$\begin{aligned}
0 &\geq \frac{1}{4} \frac{e^{-2\|f\|_\infty} E_0}{\binom{m}{k}} (\beta_1)^2 + \frac{me^{2\|f\|_\infty} F_0}{\binom{m}{k}} \left(-\frac{5}{3}m \right) \|R\|_g (4\beta_1 + 2(C_5)^2 + C_9) \\
&- \frac{me^{2\|f\|_\infty} F_0}{\binom{m}{k}} \left[\frac{7}{6}mC_5 \|\nabla R\|_g + \frac{2}{3}m^2(C_5)^2 \|R\|_g \right] - \|f\|_{C^2(M)}(1+2mC_5).
\end{aligned} \tag{6.74}$$

The previous inequality means that some polynomial of second order in the variable β_1 is negative:

$$\begin{aligned} 0 \geq & \frac{1}{4} \frac{e^{-2\|f\|_\infty} E_0}{\binom{m}{k}} (\beta_1)^2 + \frac{me^{2\|f\|_\infty} F_0}{\binom{m}{k}} \left(-\frac{20}{3} m \right) \|R\|_g \beta_1 \\ & - \frac{me^{2\|f\|_\infty} F_0}{\binom{m}{k}} \left[\frac{7}{6} m C_5 \|\nabla R\|_g + \frac{2}{3} m^2 (C_5)^2 \|R\|_g + \frac{5}{3} m \|R\|_g (2(C_5)^2 + C_9) \right] \\ & - \|f\|_{C^2(M)} (1 + 2mC_5). \end{aligned} \quad (6.75)$$

Set

$$\begin{aligned} I &:= \frac{80}{3} m^2 e^{4\|f\|_\infty} \frac{F_0}{E_0} \|R\|_g > 0, \\ J &:= 4m^2 e^{4\|f\|_\infty} \frac{F_0}{E_0} \left[\frac{7}{6} C_5 \|\nabla R\|_g + \frac{2}{3} m (C_5)^2 \|R\|_g + \frac{5}{3} (2(C_5)^2 + C_9) \|R\|_g \right] \\ &+ \frac{4 \binom{m}{k} e^{2\|f\|_\infty}}{E_0} \|f\|_{C^2(M)} (1 + 2mC_5) > 0. \end{aligned} \quad (6.76)$$

The previous inequality writes then:

$$(\beta_1)^2 - I\beta_1 - J \leq 0. \quad (6.77)$$

The discriminant of this polynomial of second order is equal to $\Delta = I^2 + 4J > 0$, which gives an upper bound for β_1 .

7. A $C^{2,\beta}$ A Priori Estimate

We infer from the C^2 estimate a $C^{2,\beta}$ estimate using a classical Evans-Trudinger theorem [18, Theorem 17.14 page 461], which achieves the proof of Theorem 1.2. Let us state this Evans-Trudinger theorem; we use Gilbarg and Trudinger's notations for classical norms and seminorms of Hölder spaces (cf. [18] and [9, page 137]).

Theorem 7.1. *Let Ω be a bounded domain (i.e., an open connected set) of $\mathbb{R}^n$, $n \geq 2$. Let one denote by $\mathbb{R}^{n \times n}$ the set of real $n \times n$ symmetric matrices. $u \in C^4(\Omega, \mathbb{R})$ is a solution of*

$$G[u] = G(x, D^2 u) = 0 \quad \text{on } \Omega, \quad (E')$$

where $G \in C^2(\Omega \times \mathbb{R}^{n \times n}, \mathbb{R})$ is elliptic with respect to u and satisfies the following hypotheses.

- (1) G is uniformly elliptic with respect to u , that is, there exist two real numbers $\lambda, \Lambda > 0$ such that

$$\forall x \in \Omega, \forall \xi \in \mathbb{R}^n, \quad \lambda |\xi|^2 \leq G_{ij}(x, D^2 u(x)) \xi_i \xi_j \leq \Lambda |\xi|^2. \quad (7.1)$$

- (2) G is concave with respect to u in the variable r . Since G is of class C^2 , this condition of concavity is equivalent to

$$\forall x \in \Omega, \forall \zeta \in \mathbb{R}^{n \times n}, \quad G_{ij,ks}(x, D^2u(x)) \zeta_{ij} \zeta_{ks} \leq 0. \quad (7.2)$$

Then for all $\Omega' \subset\subset \Omega$, one has the following interior estimate:

$$\left[D^2u \right]_{\beta; \Omega'} \leq C, \quad (7.3)$$

where $\beta \in]0, 1]$ depends only on n, λ , and Λ and $C > 0$ depends only on $n, \lambda, \Lambda, |u|_{2; \Omega'}, \text{dist}(\Omega', \partial\Omega), G_x, G_r, G_{xx}$ et G_{rx} . The notation G_{rx} used here denotes the matrix $G_{rx} = [G_{ij, x_\ell}]_{i,j, \ell=1 \dots n}$ evaluated at $(x, D^2u(x))$. It is the same for the notations G_x, G_r , and G_{xx} [18, page 457].

7.1. The Evans-Trudinger Method

Let us suppose that there exists a constant $C_{11} > 0$ such that for all $i \in \mathbb{N}$, we have $\|\varphi_{t_i}\|_{C^2(M, \mathbb{R})} \leq C_{11}$. In the following, we remove the index i from φ_{t_i} to lighten the notations. In order to construct a $C^{2, \beta}$ estimate with $0 < \beta < 1$, we prepare the framework of application of Theorem 7.1.

Let $\mathcal{R} = (U_j, \phi_j)_{1 \leq j \leq N}$ be a finite covering of the compact manifold M by charts, and let $\rho = (\theta_j)_{1 \leq j \leq N}$ be a partition of unity of class C^∞ subordinate to this covering. The family of continuity equations writes in the chart (U_s, ϕ_s) where $1 \leq s \leq N$ is a fixed integer as follows:

$$F_k \left(\left[\delta_i^j + g^{j\bar{\ell}} \circ \phi_s^{-1}(x) \frac{\partial(\varphi_t \circ \phi_s^{-1})}{\partial z_i \partial \bar{z}_\ell}(x) \right]_{1 \leq i, j \leq m} \right) - t f \circ \phi_s^{-1}(x) - \ln(A_t) = 0 \quad (E'_{k,t})$$

$$x \in \phi_s(U_s) \subset \mathbb{R}^{2m}.$$

Besides, we have $\partial/\partial z_a \partial \bar{z}_b = (1/4)(D_{ab} + D_{(a+m)(b+m)} + iD_{a(b+m)} - iD_{(a+m)b})$ where the D_{abs} denotes real derivatives; thus our equation writes:

$$G(x, D^2(\varphi_t \circ \phi_s^{-1})) = 0 \quad x \in \phi_s(U_s) \subset \mathbb{R}^{2m} \quad \text{with,} \quad (E''_{k,t})$$

$$G(x, r) = F_k \left(\left[\delta_i^j + \frac{1}{4} g^{j\bar{\ell}}(\phi_s^{-1}(x)) (r_{i\ell} + r_{(i+m)(\ell+m)} + i r_{i(\ell+m)} - i r_{(i+m)\ell}) \right]_{1 \leq i, j \leq m} \right) - t f \circ \phi_s^{-1}(x) - \ln(A_t). \quad (7.4)$$

This map G is concave in the variable r as the map F appearing in the C^2 estimate (cf. (6.36)), (namely, for all fixed x of $\phi_s(U_s)$, $G(x, \cdot)$ is concave on $\rho_{\phi_s^{-1}(x)}^{-1}(\lambda^{-1}(\Gamma_k)) \subset S_{2m}(\mathbb{R})$). For all $s \in \{1, \dots, N\}$, let us consider Ω_s a bounded domain of $\mathbb{R}^{2m}$ strictly included in $\phi_s(U_s)$:

$$\Omega_s \subset\subset \phi_s(U_s). \quad (7.5)$$

The notation $S' \subset\subset S$ means that S' is strictly included in S , namely, that $\overline{S'} \subset S$. We will explain later how these domains Ω_s are chosen. The map G is of class C^2 and the solution $\psi_t^s := \varphi_t \circ \phi_s^{-1} \in C^4(\Omega_s, \mathbb{R})$ since $\varphi_t \in C^{\ell, \alpha}(M)$ with $\ell \geq 5$. The equation $(E''_{k,t})$ on $\Omega_s \subset \phi_s(U_s)$ is now written in the form corresponding to the Theorem 7.1; it remains to check the hypotheses of this *theorem* on Ω_s , namely, that

- (1) G is uniformly elliptic with respect to $\psi_t^s = \varphi_t \circ \phi_s^{-1}$; that is, there exist two real numbers $\lambda_s, \Lambda_s > 0$ such that

$$\forall x \in \Omega_s, \forall \xi \in \mathbb{R}^{2m}, \quad \lambda_s |\xi|^2 \leq G_{ij} \left(x, D^2(\psi_t^s)(x) \right) \xi_i \xi_j \leq \Lambda_s |\xi|^2. \quad (7.6)$$

Moreover, we will impose ourselves to find real numbers λ_s, Λ_s independent of t .

- (2) G is concave with respect to ψ_t^s in the variable r . Since G is of class C^2 , this concavity condition is equivalent to

$$\forall x \in \Omega_s, \forall \zeta \in \mathbb{R}^{2m \times 2m}, \quad G_{ij, k\ell} \left(x, D^2(\psi_t^s)(x) \right) \zeta_{ij} \zeta_{k\ell} \leq 0. \quad (7.7)$$

This has been checked before.

- (3) The derivatives G_x, G_r, G_{xx} , and G_{rx} are controlled (these quantities are evaluated at $(x, D^2(\psi_t^s)(x))$).

Once these three points checked, and since we have a C^2 estimate of φ_t by C_{11} , Theorem 7.1 allows us to deduce that for all open set $\Omega'_s \subset\subset \Omega_s$ there exist two real numbers $\beta_s \in]0, 1]$ and $Cste_s > 0$ depending only on $m, \lambda_s, \Lambda_s, \text{dist}(\Omega'_s, \partial\Omega_s)$, on the uniform estimate of $|\psi_t^s|_{2, \Omega'_s}$, and on the uniform estimates of the quantities G_x, G_r, G_{xx} , and G_{rx} , so in particular β_s and $Cste_s$ are independent of t , such that

$$\left[D^2(\psi_t^s) \right]_{\beta_s, \Omega'_s} \leq Cste_s. \quad (7.8)$$

The Choice of Ω_s and Ω'_s

Let us denote by K_s the support of the function $\theta_s \circ \phi_s^{-1}$:

$$K_s := \text{supp}(\theta_s \circ \phi_s^{-1}) = \phi_s(\text{supp } \theta_s) \subset \phi_s(U_s). \quad (7.9)$$

The set K_s is compact, and it is included in the open set $\phi_s(U_s)$ of $\mathbb{R}^{2m}$, and $\mathbb{R}^{2m}$ is separated locally compact; then by the theorem of intercalation of relatively compact open sets, applied twice, we deduce the existence of two relatively compact open sets Ω_s and Ω'_s such that

$$K_s \subset \Omega'_s \subset\subset \Omega_s \subset\subset \phi_s(U_s). \quad (7.10)$$

The set Ω_s is required to be connected: for this, it suffices that K_s be connected even after restriction to a connected component in Ω_s of a point of K_s ; indeed, this connected component is an open set of Ω_s since Ω_s is locally connected (as an open set of $\mathbb{R}^{2m}$); moreover it is bounded since Ω_s is bounded.

Application of the Theorem

Let $\beta := \min \beta_s$; the norm $\|\cdot\|_{C^{2,\beta}}$ is submultiplicative; then

$$\begin{aligned} \|\varphi_t\|_{C^{2,\beta}(M)}^{\mathcal{R},\mathcal{D}} &= \sum_{s=1}^N \left| \left(\theta_s \circ \phi_s^{-1} \right) \times \left(\varphi_s \circ \phi_s^{-1} \right) \right|_{2,\beta;\Omega'_s} \\ &\leq \sum_{s=1}^N \left| \theta_s \circ \phi_s^{-1} \right|_{2,\beta;\Omega'_s} \times |\varphi_t^s|_{2,\beta;\Omega'_s}. \end{aligned} \quad (7.11)$$

But, by (7.8) we have $|\varphi_t^s|_{2,\beta;\Omega'_s} = |\varphi_t^s|_{2;\Omega'_s} + [D^2(\varphi_t^s)]_{\beta_s;\Omega'_s} \leq |\varphi_t^s|_{2;\Omega'_s} + Cste_s \leq Cste'_s$ where $Cste'_s$ depends only on $m, \lambda_s, \Lambda_s, \text{dist}(\Omega'_s, \partial\Omega_s), C_{11}$ (the constant of the C^2 estimate) and the uniform estimates of the quantities G_x, G_r, G_{xx} , and G_{rx} . We obtain consequently the needed $C^{2,\beta}$ estimate:

$$\|\varphi_t\|_{C^{2,\beta}(M)}^{\mathcal{R},\mathcal{D}} \leq \sum_{s=1}^N \left| \theta_s \circ \phi_s^{-1} \right|_{2,\beta;\Omega'_s} \times Cste'_s =: C_{12}. \quad (7.12)$$

Let us now check the hypotheses 1 and 3 above.

7.2. Uniform Ellipticity of G on Ω_s

Let $x \in \Omega_s$ and $\xi \in \mathbb{R}^{2m}$:

$$\begin{aligned} \sum_{i,j=1}^{2m} G_{ij}(x,r) \xi_i \xi_j &= d(G(x,\cdot))_r(M) \quad \text{with } M = [\xi_i \xi_j]_{1 \leq i,j \leq m} \in S_{2m}(\mathbb{R}) \\ &= d\left(F_k \circ \rho_{\phi_s^{-1}(x)}\right)_r(M) \\ &= d(F_k)_{\rho_{\phi_s^{-1}(x)}(r)} \cdot d\left(\rho_{\phi_s^{-1}(x)}\right)_r(M). \end{aligned} \quad (7.13)$$

Let us recall that $\rho_P(r) = [\delta_i^j + (1/4) \sum_{\ell,o=1}^m (g^{-1/2}(P))_{i\ell} (g^{-1/2}(P))_{oj} (r_{\ell o} + r_{(\ell+m)(o+m)} + i r_{\ell(o+m)} - i r_{(\ell+m)o})]_{1 \leq i,j \leq m}$ (cf. (6.36)); we consequently obtain

$$\begin{aligned} \sum_{i,j=1}^{2m} G_{ij}(x, D^2(\varphi_t^s)(x)) \xi_i \xi_j \\ = d(F_k)_{\rho_{\phi_s^{-1}(x)}(D^2(\varphi_t^s)(x))} \cdot \left[\frac{1}{4} \sum_{\ell,o=1}^m (g^{-1/2}(\phi_s^{-1}(x)))_{i\ell} (g^{-1/2}(\phi_s^{-1}(x)))_{oj} \right. \\ \left. \times (M_{\ell o} + M_{(\ell+m)(o+m)} + i M_{\ell(o+m)} - i M_{(\ell+m)o}) \right]_{1 \leq i,j \leq m}. \end{aligned} \quad (7.14)$$

In the following, we denote $\widetilde{M} := [(1/4)(M_{\ell s} + M_{(\ell+m)(s+m)} + iM_{\ell(s+m)} - iM_{(\ell+m)s})]_{1 \leq \ell, s \leq m}$. Thus

$$\begin{aligned} \widetilde{M} &= \left[\frac{1}{4} (\xi_{\ell} \xi_s + \xi_{\ell+m} \xi_{s+m} + i \xi_{\ell} \xi_{s+m} - i \xi_{\ell+m} \xi_s) \right]_{1 \leq \ell, s \leq m} \in \mathcal{M}_m(\mathbb{C}) \\ &= \left[\frac{1}{4} (\xi_{\ell} - i \xi_{\ell+m}) \underbrace{(\xi_s + i \xi_{s+m})}_{=: \widetilde{\xi}_s} \right]_{1 \leq \ell, s \leq m} \\ &= \left[\frac{1}{4} \widetilde{\xi}_{\ell} \widetilde{\xi}_s \right]_{1 \leq \ell, s \leq m}. \end{aligned} \quad (7.15)$$

Besides, let us denote $d_i = \sigma_{k-1,i}[\lambda(g^{-1} \widetilde{g}_{\varphi_i}(\phi_s^{-1}(x)))] / \sigma_k[\lambda(g^{-1} \widetilde{g}_{\varphi_i}(\phi_s^{-1}(x)))]$ and $g^{-1/2}$ instead of $g^{-1/2}(\phi_s^{-1}(x))$ in order to lighten the formulas. We obtain by the invariance formula (2.7) that

$$\begin{aligned} \sum_{i,j=1}^{2m} G_{ij} \left(x, D^2(\psi_t^s(x)) \right) \xi_i \xi_j &= d(F_k)_{[g]^{-1/2} \widetilde{g}_{\varphi_i} [g]^{-1/2}} \cdot ([g]^{-1/2} \widetilde{M} [g]^{-1/2}) \\ &= d(F_k)_{\text{diag}(\lambda_1, \dots, \lambda_m)} \cdot ({}^t \overline{U} [g]^{-1/2} \widetilde{M} [g]^{-1/2} U) \\ &\quad \text{where } U \in U_m(\mathbb{C}) \text{ with} \\ &\quad {}^t \overline{U} [g]^{-1/2} \widetilde{g}_{\varphi_i} [g]^{-1/2} U = \text{diag}(\lambda_1, \dots, \lambda_m) \\ &\quad \text{we are at the point } \phi_s^{-1}(x) \\ &= \sum_{i=1}^m d_i ({}^t \overline{U} [g]^{-1/2} \widetilde{M} [g]^{-1/2} U)_{ii} \\ &= \sum_{i=1}^m d_i \left({}^t \left([g]^{-1/2} U \right) \widetilde{M} ([g]^{-1/2} U) \right)_{ii} \\ &= \sum_{i, \ell, j=1}^m d_i \left(\overline{[g]^{-1/2} U} \right)_{\ell i} \widetilde{M}_{\ell j} ([g]^{-1/2} U)_{ji} \\ &= \sum_{i, \ell, j=1}^m d_i \left(\overline{[g]^{-1/2} U} \right)_{\ell i} \frac{1}{4} \widetilde{\xi}_{\ell} \widetilde{\xi}_j ([g]^{-1/2} U)_{ji} \\ &= \frac{1}{4} \sum_{i=1}^m d_i \underbrace{\left(\sum_{j=1}^m \widetilde{\xi}_j ([g]^{-1/2} U)_{ji} \right)}_{=: \alpha_i} \underbrace{\left(\sum_{\ell=1}^m \widetilde{\xi}_{\ell} \left(\overline{[g]^{-1/2} U} \right)_{\ell i} \right)}_{=: \overline{\alpha_i}} \\ &= \frac{1}{4} \sum_{i=1}^m d_i |\alpha_i|^2. \end{aligned} \quad (7.16)$$

But by Proposition 6.10 and the inequalities $e^{-2\|f\|_\infty} \binom{m}{k} \leq \sigma_k(\lambda(g^{-1}\tilde{g}_{\varphi_t}(P))) \leq e^{2\|f\|_\infty} \binom{m}{k}$, we have for (6.72)

$$\frac{e^{-2\|f\|_\infty} E_0}{\binom{m}{k}} \leq d_i \leq \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}}. \quad (7.17)$$

Combining (7.16) and (7.17), we obtain

$$\begin{aligned} \frac{1}{4} \frac{e^{-2\|f\|_\infty} E_0}{\binom{m}{k}} \left(\sum_{i=1}^m |\alpha_i|^2 \right) &\leq \sum_{i,j=1}^{2m} G_{ij} \left(x, D^2(\varphi_t^s)(x) \right) \xi_i \xi_j \\ &\leq \frac{1}{4} \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \left(\sum_{i=1}^m |\alpha_i|^2 \right). \end{aligned} \quad (7.18)$$

But

$$\begin{aligned} \sum_{i=1}^m |\alpha_i|^2 &= \sum_{i=1}^m \left| \sum_{j=1}^m \tilde{\xi}_j ([g]^{-1/2} U)_{ji} \right|^2 \\ &= \sum_{i=1}^m \left(\sum_{j=1}^m \tilde{\xi}_j ([g]^{-1/2} U)_{ji} \right) \left(\sum_{\ell=1}^m \tilde{\xi}_\ell \overline{([g]^{-1/2} U)_{\ell i}} \right) \\ &= \sum_{j,\ell=1}^m \left\{ \sum_{i=1}^m ([g]^{-1/2} U)_{ji} \overline{([g]^{-1/2} U)_{\ell i}} \right\} \tilde{\xi}_j \tilde{\xi}_\ell \\ &= \sum_{j,\ell=1}^m \left(([g]^{-1/2} U) \times {}^t \overline{([g]^{-1/2} U)} \right)_{j\ell} \tilde{\xi}_j \tilde{\xi}_\ell. \end{aligned} \quad (7.19)$$

And $([g]^{-1/2} U) \times {}^t \overline{([g]^{-1/2} U)} = [g]^{-1/2} U {}^t \overline{U} [g]^{-1/2} = [g]^{-1/2} {}^t \overline{[g]^{-1/2}} = [g]^{-1/2} [g]^{-1/2} = [g]^{-1}$; then

$$\sum_{i=1}^m |\alpha_i|^2 = \sum_{j,\ell=1}^m ([g]^{-1})_{j\ell} \tilde{\xi}_j \tilde{\xi}_\ell = \sum_{j,\ell=1}^m g^{\ell\bar{j}} (\phi_s^{-1}(x)) \tilde{\xi}_\ell \tilde{\xi}_j. \quad (7.20)$$

Consequently, and since $|\tilde{\xi}|^2 = |\xi|^2$, the checking of the hypothesis of uniform ellipticity of the Theorem 7.1 is reduced to find two real numbers $\lambda_s^o, \Lambda_s^o > 0$ such that

$$\forall x \in \Omega_s, \forall \tilde{\xi} \in \mathbb{C}^m, \quad \lambda_s^o |\tilde{\xi}|^2 \leq \sum_{j,\ell=1}^m g^{\ell\bar{j}} (\phi_s^{-1}(x)) \tilde{\xi}_\ell \tilde{\xi}_j \leq \Lambda_s^o |\tilde{\xi}|^2. \quad (7.21)$$

By the min-max principle applied on $\mathbb{C}^m$ to the Hermitian form $\langle X, Y \rangle_{g(\phi_s^{-1}(x))} = g^{a\bar{b}}(\phi_s^{-1}(x)) X_a \bar{Y}_b$ relatively to the canonical one, we have

$$\begin{aligned} \lambda_{\min} \left[g^{a\bar{b}}(\phi_s^{-1}(x)) \right]_{1 \leq a, b \leq m} |\tilde{\xi}|^2 &\leq \sum_{a, b=1}^m g^{a\bar{b}}(\phi_s^{-1}(x)) \tilde{\xi}_a \bar{\tilde{\xi}}_b \\ &\leq \lambda_{\max} \left[g^{a\bar{b}}(\phi_s^{-1}(x)) \right]_{1 \leq a, b \leq m} |\tilde{\xi}|^2. \end{aligned} \quad (7.22)$$

But the functions $P \mapsto \lambda_{\min}[g^{a\bar{b}}(P)]_{1 \leq a, b \leq m}$ and $P \mapsto \lambda_{\max}[g^{a\bar{b}}(P)]_{1 \leq a, b \leq m}$ are continuous on $\overline{\phi_s^{-1}(\Omega_s)} \subset U_s$ which is compact since it is a closed set of the compact manifold M (cf. (7.5) for the choice of the domains Ω_s), so these functions are bounded and reach their bounds; thus

$$\begin{aligned} \underbrace{\left(\min_{P \in \phi_s^{-1}(\Omega_s)} \lambda_{\min} [g^{a\bar{b}}(P)]_{1 \leq a, b \leq m} \right)}_{=: \lambda_s^o} \times |\tilde{\xi}|^2 &\leq \sum_{a, b=1}^m g^{a\bar{b}}(\phi_s^{-1}(x)) \tilde{\xi}_a \bar{\tilde{\xi}}_b \\ &\leq \underbrace{\left(\max_{P \in \phi_s^{-1}(\Omega_s)} \lambda_{\max} [g^{a\bar{b}}(P)]_{1 \leq a, b \leq m} \right)}_{=: \Lambda_s^o} \times |\tilde{\xi}|^2. \end{aligned} \quad (7.23)$$

By the inequalities (7.18) and (7.23), we deduce that

$$\begin{aligned} \lambda_s |\tilde{\xi}|^2 &\leq \sum_{i, j=1}^{2m} G_{ij} \left(x, D^2(\psi_t^s)(x) \right) \tilde{\xi}_i \bar{\tilde{\xi}}_j \leq \Lambda_s |\tilde{\xi}|^2 \\ \text{with } \lambda_s &:= \frac{1}{4} \frac{e^{-2\|f\|_\infty} E_0}{\binom{m}{k}} \lambda_s^o, \\ \Lambda_s &:= \frac{1}{4} \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \Lambda_s^o. \end{aligned} \quad (7.24)$$

The real numbers λ_s and Λ_s depend on $k, m, \|f\|_\infty, E_0, F_0, g, (U_s, \phi_s)$, and Ω_s and are independent of t, x and $\tilde{\xi}$, which achieves the proof of the global uniform ellipticity.

7.3. Uniform Estimate of $G_x, G_r, G_{xx},$ and G_{rx}

In this subsection, we estimate uniformly the quantities $G_x, G_r, G_{xx},$ and G_{rx} (recall that these quantities are evaluated at $(x, D^2(\psi_t^s)(x))$) by using the same technique as in the previous subsection for the proof of uniform ellipticity (7.24).

We have

$$|G_x|^2 = |[G_{x_i}]_{1 \leq i \leq 2m}|^2 = \sum_{i=1}^{2m} |G_{x_i}|^2 \quad \text{where } G_{x_i} = \frac{\partial G}{\partial x_i} \left(x, D^2(\psi_t^s)(x) \right). \quad (7.25)$$

For (7.14), we obtain

$$G_{x_i} = d(F_k)_{[g^{-1}\tilde{g}_{\varphi_t}(\phi_s^{-1}(x))]} \cdot \left(\underbrace{\left[\sum_{\ell=1}^m \frac{\partial(g^{q\bar{\ell}} \circ \phi_s^{-1})}{\partial x_i}(x) \partial_{o\bar{\ell}} \varphi_t(\phi_s^{-1}(x)) \right]}_{=: M^o} \right)_{1 \leq o, q \leq m} - t \frac{\partial(f \circ \phi_s^{-1})}{\partial x_i}(x) \quad (7.26)$$

and for (7.16), we infer then by the invariance formula (2.7) that

$$G_{x_i} = \sum_{j=1}^m d_j \left({}^t \bar{U} M^o U \right)_{jj} - t \frac{\partial f}{\partial x^i}(\phi_s^{-1}(x)), \quad (7.27)$$

where $U \in U_m(\mathbb{C})$ such that $({}^t \bar{U} [g^{-1}\tilde{g}_{\varphi_t}(\phi_s^{-1}(x))] U = \text{diag}(\lambda_1, \dots, \lambda_m)$ and $d_i = \sigma_{k-1,i}[\lambda(g^{-1}\tilde{g}_{\varphi_t}(\phi_s^{-1}(x)))] / \sigma_k[\lambda(g^{-1}\tilde{g}_{\varphi_t}(\phi_s^{-1}(x)))]$. We can then write:

$$\begin{aligned} G_{x_i} &= \sum_{j,p,q=1}^m d_j \bar{U}_{pj} U_{qj} M_{pq}^o - t \frac{\partial f}{\partial x^i}(\phi_s^{-1}(x)) \\ &= \sum_{j,p,q=1}^m d_j \bar{U}_{pj} U_{qj} \left(\sum_{\ell=1}^m \frac{\partial g^{q\bar{\ell}}}{\partial x^i}(\phi_s^{-1}(x)) \partial_{p\bar{\ell}} \varphi_t(\phi_s^{-1}(x)) \right) - t \frac{\partial f}{\partial x^i}(\phi_s^{-1}(x)). \end{aligned}$$

Thus $|G_{x_i}| \leq \sum_{j,p,q,\ell=1}^m \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} |\bar{U}_{pj}| |U_{qj}| \quad (7.28)$

$$\times \left(\underbrace{\max_{1 \leq a, b \leq m, 1 \leq i \leq 2m} \max_{P \in \phi_s^{-1}(\Omega_s)} \left| \frac{\partial g^{a\bar{b}}}{\partial x^i}(P) \right|}_{=: \Lambda_s^1} \right) \|\varphi_t\|_{C^2(M, \mathbb{R})} + \|f\|_{C^1(M, \mathbb{R})}.$$

But $U \in U_m(\mathbb{C})$; then $|U_{qj}| \leq 1$ for all $1 \leq q, j \leq m$, consequently

$$|G_{x_i}| \leq m^4 \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \Lambda_s^1 \underbrace{\|\varphi_t\|_{C^2(M, \mathbb{R})}}_{\leq C_{11} (C^2 \text{ estimate})} + \|f\|_{C^1(M, \mathbb{R})}, \quad (7.29)$$

which gives the needed uniform estimate for G_x :

$$|G_x| \leq \sqrt{2m} \left(m^4 \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \Lambda_s^1 C_{11} + \|f\|_{C^1(M, \mathbb{R})} \right). \quad (7.30)$$

Similarly

$$|G_r|^2 = \left| [G_{pq}]_{1 \leq p, q \leq 2m} \right|^2 = \sum_{p, q=1}^{2m} |G_{pq}|^2, \quad (7.31)$$

where $G_{pq} = \frac{\partial G}{\partial r_{pq}}(x, D^2(\psi_t^s)(x))$.

And we have

$$G_{pq} = d(F_k)_{[g^{-1}\tilde{g}_{\varphi_t}(\phi_s^{-1}(x))]} \cdot \underbrace{\left[\sum_{\ell=1}^m g^{j\bar{\ell}}(\phi_s^{-1}(x)) (\widetilde{E_{pq}})_{i\bar{\ell}} \right]_{1 \leq i, j \leq m}}_{=: M^1}, \quad (7.32)$$

where E_{pq} is the $m \times m$ matrix whose all coefficients are equal to zero except the coefficient pq which is equal to 1, and the matrix $(\widetilde{E_{pq}})$ is obtained from E_{pq} by the formula $\widetilde{M} := [(1/4)(M_{\ell s} + M_{(\ell+m)(s+m)} + iM_{\ell(s+m)} - iM_{(\ell+m)s})]_{1 \leq \ell, s \leq m}$, thus

$$G_{pq} = \sum_{j=1}^m d_j ({}^t \bar{U} M^1 U)_{jj}, \quad (7.33)$$

where U and d_i are as before for G_x .

Since $|(\widetilde{E_{pq}})_{i\bar{\ell}}| \leq 1$ for all $1 \leq i, \ell \leq m$, we obtain for G_x that

$$|G_{pq}| \leq m^4 \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \Lambda_s^2, \quad (7.34)$$

where $\Lambda_s^2 = \max_{1 \leq a, b \leq m} \max_{P \in \phi_s^{-1}(\Omega_s)} |g^{a\bar{b}}(P)|$, which gives the needed uniform estimate for G_r :

$$|G_r| \leq 2m^5 \frac{e^{2\|f\|_\infty} F_0}{\binom{m}{k}} \Lambda_s^2. \quad (7.35)$$

Concerning G_{xx} , we have

$$|G_{xx}|^2 = \left| [G_{x_p x_q}]_{1 \leq p, q \leq 2m} \right|^2 = \sum_{p, q=1}^{2m} |G_{x_p x_q}|^2, \quad (7.36)$$

where $G_{x_p x_q} = \frac{\partial^2 G}{\partial x_p \partial x_q}(x, D^2(\psi_t^s)(x))$.

A calculation shows that

$$\begin{aligned}
 G_{x_p x_q} = & -t \frac{\partial^2 f}{\partial x^p \partial x^q} (\phi_s^{-1}(x)) \\
 & + \sum_{i,j,\ell=1}^m \frac{\partial F_k}{\partial B_i^j} \left([g^{-1} \tilde{g}_{\varphi_t}(\phi_s^{-1}(x))] \right) \frac{\partial^2 g^{j\bar{\ell}}}{\partial x^p \partial x^q} (\phi_s^{-1}(x)) \partial_{i\bar{\ell}} \varphi_t (\phi_s^{-1}(x)) \\
 & + \underbrace{\sum_{i,j,\ell,\mu,o,v=1}^m \frac{\partial^2 F_k}{\partial B_\mu^o \partial B_i^j} \left([g^{-1} \tilde{g}_{\varphi_t}(\phi_s^{-1}(x))] \right)}_{=: \mathcal{E}} \\
 & \times \frac{\partial g^{o\bar{v}}}{\partial x^p} (\phi_s^{-1}(x)) \frac{\partial g^{j\bar{\ell}}}{\partial x^q} (\phi_s^{-1}(x)) \partial_{\mu\bar{v}} \varphi_t (\phi_s^{-1}(x)) \partial_{i\bar{\ell}} \varphi_t (\phi_s^{-1}(x)).
 \end{aligned} \tag{7.37}$$

All the terms are uniformly bounded; it remains to justify that the term in second derivative $\mathcal{E}$ is also uniformly bounded:

$$\begin{aligned}
 \mathcal{E} &= d^2(F_k)_{[g^{-1} \tilde{g}_{\varphi_t}(\phi_s^{-1}(x))]} \cdot (E_{\mu o}, E_{ij}) \quad \text{then by the invariance formula (2.7)} \\
 &= \sum_{a,b,c,d=1}^m \frac{\partial^2 F_k}{\partial B_a^b \partial B_c^d} [\text{diag}(\lambda_1, \dots, \lambda_m)] ({}^t \bar{U} E_{\mu o} U)_{ab} ({}^t \bar{U} E_{ij} U)_{cd},
 \end{aligned} \tag{7.38}$$

where $U \in U_m(\mathbb{C})$ is like before.

But we know the second derivatives of F_k at a diagonal matrix by (2.5). Besides, we have $0 < \sigma_{k-1,i}(\lambda)/\sigma_k(\lambda) = d_i \leq e^{2\|f\|_\infty} F_0 / \binom{m}{k}$ by (7.17), and since $e^{-2\|f\|_\infty} \binom{m}{k} \leq \sigma_k(\lambda)$, it remains only to control the quantities $|\sigma_{k-2,ij}(\lambda)|$ with $i \neq j$ to prove that $\mathcal{E}$ is uniformly bounded. But since $\lambda \in \Gamma_k$, we have $\sigma_{k-2,ij}(\lambda) > 0$ [11]. Moreover, by the pinching of the eigenvalues, we deduce automatically that

$$\sigma_{k-2,ij}(\lambda) \leq \binom{m-2}{k-2} (C'_2)^{k-1} =: F_1, \tag{7.39}$$

which achieves the checking of the fact that G_{xx} is uniformly bounded.

Similarly, we establish a uniform estimate of G_{xr} using this calculation:

$$\begin{aligned}
 G_{x_o, p q} &= \frac{\partial^2 G}{\partial x_o \partial r_{pq}} (x, D^2(\varphi_t^s)(x)) \\
 &= \sum_{i,j,\ell=1}^m \frac{\partial F_k}{\partial B_i^j} \left([g^{-1} \tilde{g}_{\varphi_t}(\phi_s^{-1}(x))] \right) \frac{\partial g^{j\bar{\ell}}}{\partial x^o} (\phi_s^{-1}(x)) (\widetilde{E_{pq}})_{i\bar{\ell}}
 \end{aligned}$$

$$\begin{aligned}
& + \sum_{i,j,\ell,v,\mu,\gamma=1}^m \frac{\partial^2 F_k}{\partial B_v^\mu \partial B_i^\gamma} \left(\left[g^{-1} \tilde{g}_{\varphi_t} \left(\phi_s^{-1}(x) \right) \right] \right) \\
& \times \frac{\partial g^{\mu\bar{\gamma}}}{\partial x^o} \left(\phi_s^{-1}(x) \right) \partial_{v\bar{\gamma}} \varphi_t \left(\phi_s^{-1}(x) \right) g^{j\bar{\ell}} \left(\phi_s^{-1}(x) \right) \left(\widetilde{E_{pq}} \right)_{i\bar{\ell}},
\end{aligned} \tag{7.40}$$

which achieves the proof of the $C^{2,\beta}$ estimate.

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