Research Article

An Extension of Generalized (ψ, φ) -Weak Contractions

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We prove a fixed-point theorem for a class of maps that satisfy generalized (ψ, φ) -weak contractions depending on a given function. An example is given to illustrate our extensions.

1. Introduction

Because fixed-point theory has a wide array of applications in many areas such as economics, computer science, and engineering, it plays evidently a crucial role in nonlinear analysis. One of the cornerstones of this theory is the Banach fixed-point theorem, also known as the Banach contraction mapping theorem [1], which can be stated as follows.

Let $T : X \rightarrow X$ be a contraction on a complete metric space $(X; d)$; that is, there is a nonnegative real number $k < 1$ such that $d(T(x), T(y)) \leq kd(x, y)$ for all $x, y \in X$. Then the map T admits one and only one point $x^* \in X$ such that $Tx^* = x^*$. Moreover, this fixed point is the limit of the iterative sequence $x_{n+1} = T(x_n)$ for $n = 0, 1, 2, \dots$, where x_0 is an arbitrary starting point in X . This theorem attracted a lot of attention because of its importance in the field. Many authors have started studying on fixed-point theory to explore some new contraction mappings to generalize the Banach contraction mapping theorem. In particular, Boyd and Wong [2] introduced the notion of Φ -contractions. In 1997 Alber and Guerre-Delabriere [3] defined the φ -weak contraction which is a generalization of Φ -contractions (see also [4–8]).

On the other hand, the notion of T -contractions introduced and studied by the authors of the interesting papers in [9–11]. Following this trend, we explore in this paper another extension of (ψ, φ) -weak contractions in the context of T -contractions.

2. Preliminaries

Let (X, d) be a metric space. Boyd and Wong [2] introduced the notion of Φ -contraction as follows. A map $T : X \rightarrow X$ is called a Φ -contraction if there exists an upper semicontinuous function $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ such that

$$d(Tx, Ty) \leq \Phi(d(x, y)) \quad (2.1)$$

for all $x, y \in X$. The concept of the φ -weak contraction was defined by Alber and Guerre-Delabriere [3] as a generalization of Φ -contraction under the setting of Hilbert spaces and obtained fixed-point results. A map $T : X \rightarrow X$ is a φ -weak contraction, if there exists a function $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ such that

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \quad (2.2)$$

for all $x, y \in X$ provided that the function φ satisfies the following condition:

$$\varphi(t) = 0 \quad \text{iff } t = 0. \quad (2.3)$$

Later Rhoades [7] proved analogs of the result in [3] in the context of metric spaces.

Theorem 2.1. *Let (X, d) be a complete metric space. Let $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ be a continuous and nondecreasing function such that $\varphi(t) = 0$ if and only if $t = 0$. If $T : X \rightarrow X$ is a φ weak contraction, then T has a unique fixed point.*

In [12], Dutta and Choudhury proved an extension of Rhoades.

Theorem 2.2. *Let (X, d) be a complete metric space, and let $T : X \rightarrow X$ be a self-mapping satisfying*

$$\varphi(d(Tx, Ty)) \leq \varphi(d(x, y)) - \varphi(d(x, y)), \quad \forall x, y \in X, \quad (2.4)$$

where $\varphi, \varphi : [0, +\infty) \rightarrow [0, +\infty)$ are continuous and nondecreasing functions with $\varphi(t) = \varphi(t) = 0$ if and only if $t = 0$. Then T has a unique fixed point.

Zhang and Song [8] improved Theorem 2.1 and gave the following result which states the existence of common fixed points of certain maps in metric spaces.

Theorem 2.3. *Let (X, d) be a complete metric space, and let $f, g : X \rightarrow X$ be self-mappings satisfying*

$$d(fx, gy) \leq M(x, y) - \varphi(M(x, y)), \quad \forall x, y \in X, \quad (2.5)$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, gy), \frac{d(x, fy) + d(y, gx)}{2} \right\} \quad (2.6)$$

and $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ are lower semicontinuous functions with $\varphi(t) = 0$ if and only if $t = 0$. Then f, g have a unique common fixed point.

Combining the theorems above with the results of Dutta and Choudhury [12], Ćorović [13] obtained the following theorem.

Theorem 2.4. Let (X, d) be a complete metric space, and let $T, S : X \rightarrow X$ be self-mappings satisfying

$$\varphi(d(fx, gy)) \leq \varphi(M(x, y)) - \varphi(M(x, y)), \quad \forall x, y \in X, \quad (2.7)$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, gy), \frac{d(x, gy) + d(y, fx)}{2} \right\}, \quad (2.8)$$

$\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a continuous and nondecreasing function with $\varphi(t) = 0$ if and only if $t = 0$, and $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a lower semicontinuous function with $\varphi(t) = 0$ if and only if $t = 0$. Then f, g have a unique common fixed point.

The notion of the T -contraction is defined in ([10, 11]) as follows.

Definition 2.5. Let T and S be two self-mappings on a metric space (X, d) . The mapping S is said to be a T -contraction if there exists $k \in (0, 1)$ such that

$$d(TSx, TSy) \leq kd(Tx, Ty), \quad \forall x, y \in X. \quad (2.9)$$

It can be easily seen that if T is the identity map, then the T -contraction coincides with the usual contraction.

Example 2.6. Let $X = (0, \infty)$ with the usual metric $d(x, y) = |x - y|$ induced by $(\mathbb{R}, d)$. Consider the following self-mappings $T(x) = 1/x$ and $Sx = 3x$ on X . It is clear that S is not a contraction. On the contrary,

$$d(TSx, TSy) = \left| \frac{1}{3x} - \frac{1}{3y} \right| = \frac{1}{3} \left| \frac{1}{y} - \frac{1}{x} \right| \leq \frac{1}{3} d(Tx, Ty), \quad \forall x, y \in X. \quad (2.10)$$

Definition 2.7 (see, e.g., [9, 11]). Let (X, d) be a metric space. If $\{y_n\}$ is a convergent sequence whenever $\{Ty_n\}$ is convergent, then $T : X \rightarrow X$ is called *sequentially convergent*.

The aim of this work is to give a proper extension of Ćoricorić's result of using the concept of T -contraction, that is, the contraction depending on a given function. We will show the existence of a common fixed point for a class of certain maps.

3. Main Results

We start this section by recalling the following two classes of functions.

Let Ψ denote the set of all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ which satisfy

- (i) ψ is continuous and nondecreasing,
- (ii) $\psi(t) = 0$ if and only if $t = 0$.

Similarly Φ denotes the set of all functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ which satisfy

- (i) φ is lower semi continuous,
- (ii) $\varphi(t) = 0$ if and only if $t = 0$.

It is easy to see that $\psi_1(t) = t$, $\psi_2(t) = t/(t+1)$, $\psi_3(t) = t^2$ belong to Ψ and $\varphi_1(t) = \min\{t, 1\}$, $\varphi_2(t) = \ln(1+t)$ belong to Φ .

We are ready to state our main theorem that is a proper extension of Theorem 2.4.

Theorem 3.1. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ an injective, continuous, and sequentially convergent mapping. Let $f, g : X \rightarrow X$ be self-mappings. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d(Tfx, Tgy)) \leq \psi(M(Tx, Ty)) - \varphi(M(Tx, Ty)), \quad (3.1)$$

for all $x, y \in X$, where

$$M(Tx, Ty) = \max \left\{ d(Tx, Ty), d(Tx, Tfx), d(Ty, Tgy), \frac{d(Tx, Tgy) + d(Ty, Tfx)}{2} \right\}, \quad (3.2)$$

then f, g have a unique common fixed point.

Proof. We will follow the lines in the proof of the main result in [13]. By injection of T , we easily check that $M(Tx, Ty) = 0$ if and only if $x = y$ is a common fixed point of f and g . Let $x_0 \in X$. We define two iterative sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$x_{2n+2} = fx_{2n+1}, \quad x_{2n+1} = gx_{2n}, \quad y_n = Tx_n, \quad \forall n = 0, 1, 2, \dots \quad (3.3)$$

We prove $\{y_n\}$ is a Cauchy sequence. For this purpose, we first claim that $\lim_{n \rightarrow \infty} d(y_{n+1}, y_n) = 0$. It follows from property of φ that if n is odd

$$\begin{aligned} \psi(d(y_{n+1}, y_n)) &= \psi(d(Tx_{n+1}, Tx_n)) = \psi(d(Tfx_n, Tgx_{n-1})) \\ &\leq \psi(M(Tx_n, Tx_{n-1})) - \varphi(M(Tx_n, Tx_{n-1})) \\ &\leq \psi(M(Tx_n, Tx_{n-1})), \end{aligned} \quad (3.4)$$

where

$$\begin{aligned}
 M(Tx_n, Tx_{n-1}) &= \max \left\{ d(Tx_n, Tx_{n-1}), d(Tfx_n, Tx_n), d(Tgx_{n-1}, Tx_{n-1}), \right. \\
 &\quad \left. \frac{d(Tgx_{n-1}, Tx_n) + d(Tfx_n, Tx_{n-1})}{2} \right\} \\
 &= \max \left\{ d(y_n, y_{n-1}), d(y_{n+1}, y_n), d(y_n, y_{n-1}), \frac{d(y_{n-1}, y_{n+1})}{2} \right\} \\
 &\leq \max \left\{ d(y_n, y_{n-1}), d(y_{n+1}, y_n), \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{2} \right\}.
 \end{aligned} \tag{3.5}$$

Hence, we have

$$\psi(d(y_{n+1}, y_n)) \leq \psi \left(\max \left\{ d(y_n, y_{n-1}), d(y_{n+1}, y_n), \frac{d(y_{n-1}, y_n) + d(y_n, y_{n+1})}{2} \right\} \right). \tag{3.6}$$

If $d(y_n, y_{n+1}) > d(y_{n-1}, y_n) \geq 0$ then $M(Tx_n, Tx_{n-1}) = d(y_n, y_{n+1})$, hence

$$\psi(d(y_n, y_{n+1})) \leq \psi(d(y_n, y_{n+1})) - \psi(d(y_n, y_{n+1})) \tag{3.7}$$

and which contradicts with $d(y_n, y_{n+1}) > 0$ and the property of φ . Thus, it follows from (3.5) that

$$d(y_{n+1}, y_n) \leq M(Tx_n, Tx_{n-1}) = d(y_n, y_{n-1}). \tag{3.8}$$

If n is even then by the same argument above, we obtain

$$d(y_{n+1}, y_n) \leq M(Tx_{n-1}, Tx_n) = d(y_n, y_{n-1}). \tag{3.9}$$

Therefore,

$$d(y_{n+1}, y_n) \leq M(Tx_n, Tx_{n-1}) = d(y_n, y_{n-1}) \tag{3.10}$$

for all n and $\{d(y_n, y_{n+1})\}$ is a nonincreasing sequence of nonnegative real numbers. Hence, there exists $r \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = \lim_{n \rightarrow \infty} M(Tx_n, Tx_{n-1}) = r. \tag{3.11}$$

By the lower semicontinuity of φ , we have

$$\varphi(r) \leq \liminf_{n \rightarrow \infty} \varphi(M(Tx_n, Tx_{n-1})). \tag{3.12}$$

Taking the upper limits as $n \rightarrow \infty$ on either side of

$$\varphi(d(y_n, y_{n+1})) \leq \varphi(M(Tx_n, Tx_{n-1})) - \varphi(M(Tx_n, Tx_{n-1})), \quad (3.13)$$

we get

$$\varphi(r) \leq \varphi(r) - \liminf_{n \rightarrow \infty} \varphi(M(Tx_n, Tx_{n-1})) \leq \varphi(r) - \varphi(r), \quad (3.14)$$

that is, $\varphi(r) \leq 0$. By the property of φ , this implies that $\varphi(r) = 0$. It follows that $r = 0$ and

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \quad (3.15)$$

It is implied from (3.10) that

$$\lim_{n \rightarrow \infty} M(Tx_n, Tx_{n-1}) = 0. \quad (3.16)$$

Now, we claim that $\{y_n\}$ is a Cauchy sequence. Since $\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0$, it is sufficient to prove that $\{y_{2n}\}$ is a Cauchy sequence. Suppose on the contrary that $\{y_{2n}\}$ is not a Cauchy sequence. Then, there exist $\varepsilon > 0$ and subsequences $\{y_{2n(k)}\}$ and $\{y_{2m(k)}\}$ of $\{y_{2n}\}$ such that $n(k)$ is the smallest index for which

$$n(k) > m(k) > k, \quad d(y_{2m(k)}, y_{2n(k)}) > \varepsilon. \quad (3.17)$$

This means that

$$d(y_{2m(k)}, y_{2n(k)-2}) < \varepsilon. \quad (3.18)$$

From (3.18) and the triangle inequality, we get

$$\begin{aligned} \varepsilon &\leq d(y_{2m(k)}, y_{2n(k)}) \\ &\leq d(y_{2m(k)}, y_{2n(k)-2}) + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}) \\ &< \varepsilon + d(y_{2n(k)-2}, y_{2n(k)-1}) + d(y_{2n(k)-1}, y_{2n(k)}). \end{aligned} \quad (3.19)$$

Letting $k \rightarrow \infty$ and using (3.15), we get

$$\lim_{k \rightarrow \infty} d(y_{2m(k)}, y_{2n(k)}) = \varepsilon. \quad (3.20)$$

By the fact

$$\begin{aligned} |d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2m(k)}, y_{2n(k)})| &\leq d(y_{2n(k)}, y_{2n(k)+1}) \\ |d(y_{2m(k)-1}, y_{2n(k)}) - d(y_{2m(k)}, y_{2n(k)})| &\leq d(y_{2m(k)-1}, y_{2m(k)}) \end{aligned} \quad (3.21)$$

and using (3.15) and (3.20), we obtain

$$\lim_{k \rightarrow \infty} d(y_{2m(k)-1}, y_{2n(k)}) = \lim_{k \rightarrow \infty} d(y_{2m(k)}, y_{2n(k)+1}) = \varepsilon. \quad (3.22)$$

Moreover, from

$$|d(y_{2m(k)-1}, y_{2n(k)+1}) - d(y_{2m(k)-1}, y_{2n(k)})| \leq d(y_{2n(k)}, y_{2n(k)+1}) \quad (3.23)$$

and combining with (3.15) and (3.22), we conclude that

$$\lim_{k \rightarrow \infty} d(y_{2m(k)-1}, y_{2n(k)+1}) = \varepsilon. \quad (3.24)$$

Now, by the definition of $M(Tx, Ty)$ and from (3.10), (3.15), and (3.20)–(3.24), we can deduce that

$$\lim_{k \rightarrow \infty} M(Tx_{2m(k)-1}, Tx_{2n(k)}) = \varepsilon. \quad (3.25)$$

Due to (3.1), we have

$$\begin{aligned} \psi(d(y_{2m(k)}, y_{2n(k)+1})) &= \psi(d(Tx_{2m(k)}, Tx_{2n(k)+1})) = \psi(d(Tfx_{2m(k)-1}, Tgx_{2n(k)})) \\ &\leq \psi(M(Tx_{2m(k)-1}, Tx_{2n(k)})) - \varphi(M(Tx_{2m(k)-1}, Tx_{2n(k)})). \end{aligned} \quad (3.26)$$

Letting $k \rightarrow \infty$ and using (3.22) and (3.25), we have

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon). \quad (3.27)$$

It is a contradiction to $\varphi(t) > 0$ for every $t > 0$. This proves that $\{y_n\}$ is a Cauchy sequence.

Since X is a complete metric space, there exists $u \in X$ such that $\lim_{n \rightarrow \infty} y_n = u$. Since T is sequentially convergent, we can deduce that $\{x_n\}$ converges to $v \in X$. By the continuity of T , we infer that

$$u = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} Tx_n = Tv. \quad (3.28)$$

We will show that $v = fv = gv$. Indeed, suppose that $v \neq fv$, since T is injective, we have $u = Tv \neq Tfv$. Hence, $d(Tv, Tfv) > 0$. Since

$$\begin{aligned} \lim_{n \rightarrow \infty} y_{2n+1} &= \lim_{n \rightarrow \infty} y_{2n} = u, \\ \lim_{n \rightarrow \infty} d(y_{2n}, y_{2n+1}) &= 0, \end{aligned} \quad (3.29)$$

we can seek $N_0 \in \mathbb{N}$ such that for any $n \geq N_0$

$$d(y_{2n+1}, u) < \frac{d(Tv, Tfv)}{4}, \quad d(y_{2n}, u) < \frac{d(Tv, Tfv)}{4}, \quad d(y_{2n}, y_{2n+1}) < \frac{d(Tv, Tfv)}{4}. \quad (3.30)$$

Then, we have

$$\begin{aligned} d(Tv, Tfv) &\leq M(Tv, Tx_{2n}) = \max \left\{ d(Tv, Tx_{2n}), d(Tv, Tfv), d(Tx_{2n}, Tgx_{2n}), \right. \\ &\quad \left. \frac{d(Tv, Tgx_{2n}) + d(Tx_{2n}, Tfv)}{2} \right\} \\ &= \max \left\{ d(u, y_{2n}), d(Tv, Tfv), d(y_{2n}, y_{2n+1}), \right. \\ &\quad \left. \frac{d(u, y_{2n+1}) + d(y_{2n}, Tfv)}{2} \right\} \\ &\leq \max \left\{ d(u, y_{2n}), d(Tv, Tfv), d(y_{2n}, y_{2n+1}), \right. \\ &\quad \left. \frac{d(u, y_{2n+1}) + d(y_{2n}, Tv) + d(Tv, Tfv)}{2} \right\} \\ &\leq \max \left\{ \frac{d(Tv, Tfv)}{4}, d(Tv, Tfv), \frac{d(Tv, Tfv)}{4}, \right. \\ &\quad \left. \frac{d(Tv, Tfv)/4 + d(Tv, Tfv)/4 + d(Tv, Tfv)}{2} \right\} \\ &\leq \max \left\{ d(Tv, Tfv), \frac{3}{4} d(Tv, Tfv) \right\} = d(Tv, Tfv). \end{aligned} \quad (3.31)$$

Therefore, $M(Tv, Tx_{2n}) = d(Tv, Tfv)$ for every $n \geq N_0$. Since

$$\begin{aligned} \varphi(d(Tfv, y_{2n+1})) &= \varphi(d(Tfv, Tx_{2n+1})) = \varphi(d(Tfv, Tgx_{2n})) \\ &\leq \varphi(M(Tv, Tx_{2n})) - \varphi(M(Tv, Tx_{2n})) \\ &= \varphi(d(Tv, Tfv)) - \varphi(d(Tv, Tfv)) \end{aligned} \quad (3.32)$$

and letting $n \rightarrow \infty$, we arrive at

$$\varphi(d(Tfv, Tv)) \leq \varphi(d(Tv, Tfv)) - \varphi(d(Tv, Tfv)). \quad (3.33)$$

We get a contradiction. Hence, $v = fv$. By the same argument, we get $v = gv$.

Let $w \in X$ such that $w = fw = gw$. Then, we have

$$\begin{aligned} M(Tv, Tw) &= \max \left\{ d(Tv, Tw), d(Tv, Tfv), d(Tw, Tgw), \frac{d(Tv, Tgw) + d(Tfv, Tw)}{2} \right\} \\ &= \max \left\{ d(Tv, Tw), \frac{d(Tv, Tw) + d(Tv, Tw)}{2} \right\} = d(Tw, Tv). \end{aligned} \quad (3.34)$$

Thus

$$\begin{aligned} \varphi(d(Tv, Tw)) &= \varphi(d(Tfv, Tgw)) \leq \varphi(M(Tv, Tw)) - \varphi(M(Tv, Tw)) \\ &= \varphi(d(Tv, Tw)) - \varphi(d(Tv, Tw)). \end{aligned} \quad (3.35)$$

This implies that $d(Tv, Tw) = 0$, or $Tv = Tw$. Since T is injective, we have $w = v$. The theorem is proved. $\square$

Remark 3.2. (1) In Theorem 3.1, if we choose $Tx = x$ for all $x \in X$, then we get Theorem 2.4.

(2) In Theorem 3.1, if we fix $\varphi(t) = t$ for all t , then we obtain another extension of Theorem 2.3.

(3) In Theorem 3.1, if we choose $f = g$, then we get the uniqueness and existence of fixed point of generalized φ -weak T -contractions.

The following example shows that Theorem 3.1 is a proper extension of Theorem 2.4.

Example 3.3. Let $X = [1, +\infty)$ and d be the usual metric in X . Consider the maps $f(x) = g(x) = 4\sqrt{x}$. It is easy to see that 16 is the unique fixed point of f and g . We claim that f and g are not generalized φ -weak contraction. Indeed, if there exist lower semicontinuous functions $\varphi, \varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(t) > 0$, $\varphi(t) > 0$ for $t \in (0, \infty)$ and $\varphi(0) = \varphi(0) = 0$, such that

$$\varphi(d(fx, gy)) \leq \varphi(M(x, y)) - \varphi(M(x, y)), \quad \forall x, y \in X, \quad (3.36)$$

then

$$\varphi(4|\sqrt{x} - \sqrt{y}|) \leq \varphi(M(x, y)) - \varphi(M(x, y)), \quad \forall x, y \in X, \quad (3.37)$$

where $M(x, y) = \max\{d(x, y), d(fx, x), d(gy, y), (1/2)[d(gx, y) + d(fy, x)]\}$. For $x = 4$ and $y = 1$, we obtain

$$M(x, y) = \max \left\{ 3, 4, 3, \frac{7}{2} \right\} = 4. \quad (3.38)$$

It follows from (3.37) that

$$\varphi(4) \leq \varphi(4) - \varphi(4). \quad (3.39)$$

Hence, $\varphi(4) \leq 0$. We arrive at a contradiction with $\varphi(t) > 0$ for $t \in (0, \infty)$.

Consider the map $Tx = \ln x + 1$, for all $x \in X$. It is easy to see that T is injective, continuous, and sequentially convergent. Let $\psi(t) = t$ and $\varphi(t) = t/3$, for all $t \in [0, +\infty)$. Now, we show that f and g are generalized φ -weak T -contractions. It reduces to check the following inequality:

$$|\ln 4\sqrt{x} - \ln 4\sqrt{y}| \leq \frac{2}{3}M(Tx, Ty), \quad \forall x, y \in [1, +\infty). \quad (3.40)$$

We have

$$|\ln 4\sqrt{x} - \ln 4\sqrt{y}| = \frac{1}{2} \left| \ln \frac{x}{y} \right| \quad (3.41)$$

$$\begin{aligned} M(Tx, Ty) &= \max \left\{ |\ln x - \ln y|, |\ln 4\sqrt{x} - \ln x|, |\ln 4\sqrt{y} - \ln y|, \right. \\ &\quad \left. \frac{|\ln 4\sqrt{y} - \ln x| + |\ln 4\sqrt{x} - \ln y|}{2} \right\} \\ &\geq |\ln x - \ln y| = \left| \ln \frac{x}{y} \right|. \end{aligned} \quad (3.42)$$

It follows from (3.41) and (3.42) that

$$|\ln 4\sqrt{x} - \ln 4\sqrt{y}| \leq \frac{1}{2}M(Tx, Ty) \quad (3.43)$$

for every $x, y \in X$. This proves that (3.40) is true.

By the same method used in the proof of Theorem 3.1, we get the following theorem.

Theorem 3.4. *Let (X, d) be a complete metric space and $T : X \rightarrow X$ an injective, continuous, and sequentially convergent mapping. Let $f, g : X \rightarrow X$ be self-mappings. If there exist $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(d(fTx, gTy)) \leq \psi(d(Tx, Ty)) - \varphi(d(Tx, Ty)) \quad (3.44)$$

for all $x, y \in X$, then f, g have a unique common fixed point.

Proof. It follows from the proof of Theorem 3.1 with necessary modifications. $\square$

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