Research Article

The Multiple Gamma-Functions and the Log-Gamma Integrals

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In this paper, which is a companion paper to [W], starting from the Euler integral which appears in a generalization of Jensen’s formula, we shall give a closed form for the integral of \( \log \Gamma(1±t) \). This enables us to locate the genesis of two new functions \( A_{1/a} \) and \( C_{1/a} \) considered by Srivastava and Choi. We consider the closely related function \( A(a) \) and the Hurwitz zeta function, which render the task easier than working with the \( A_{1/a} \) functions themselves. We shall also give a direct proof of Theorem 4.1, which is a consequence of [CKK, Corollary 1.1], though.

1. Introduction

If \( f(z) \) is analytic in a domain \( D \) containing the circle \( C : |z| = r \) and has no zero on the circle, then the Gauss mean value theorem

\[
\log|f(0)| = \frac{1}{2\pi} \int_{0}^{2\pi} \log|f(re^{i\theta})| \, d\theta
\]

(1.1)

is true. In [1, page 207] the case is considered where \( f(z) \) has a zero \( re^{i\alpha} \) on the circle, and (1.1) turns out that the Euler integral

\[
\int_{0}^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2
\]

(1.2)

which is essential in proving a generalization of Jensen’s formula [1, pages 207-208].
Let $G$ denote the Catalan constant defined by the absolutely convergent series

$$G = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2} = L(2, \chi_4), \quad (1.3)$$

where $\chi_4$ is the nonprincipal Dirichlet character mod 4.

As a next step from (1.2) the relation

$$\int_{0}^{\pi/4} \log \sin t \, dt = -\frac{\pi}{4} \log 2 - \frac{1}{2} G \quad (1.4)$$

holds true. In this connection, in [2] we obtained some results on $G$ viewing it as an intrinsic value to the Barnes $G$-function. The Barnes $G$-function (which is $\Gamma_{-1}$ in the class of multiple gamma functions) is defined as the solution to the difference equation (cf. (2.3))

$$\log G(z + 1) - \log G(z) = \log \Gamma(z) \quad (1.5)$$

with the initial condition

$$\log G(1) = 0 \quad (1.6)$$

and the asymptotic formula to be satisfied

$$\log G(z + N + 2) = \frac{N + 1 + z}{2} \log 2\pi$$

$$+ \frac{1}{2} \left( N^2 + 2N + 1 + B_1 + z^2 + 2(N + 1)z \right) \log N$$

$$- \frac{3}{4} N^2 - N - Nz - \log A + \frac{1}{12} + O\left(N^{-1}\right), \quad (1.7)$$

$N \to \infty$, where $\Gamma(s)$ indicates the Euler gamma function (cf., e.g., [3]).

Invoking the reciprocity relation for the gamma function

$$\Gamma(s) \sin \pi s = \frac{\pi}{\Gamma(1-s)}, \quad (1.8)$$

it is natural to consider the integrals of $\log \Gamma(\alpha + t)$ or of multiple gamma functions $\Gamma_{r}$ (cf., e.g., [4, 5]). Barnes’ theorem [6, page 283] reads

$$\int_{0}^{a} \log \Gamma(\alpha + t) \, dt = -\log \frac{G(\alpha + a)}{G(\alpha)} - (1-a) \log \frac{\Gamma(\alpha + a)}{\Gamma(\alpha)}$$

$$+ a \log \Gamma(\alpha + a) - \frac{1}{2} a^2 - \frac{1}{2} (\log 2\pi + 1 - 2a) a$$

valid for nonintegral values of $a$. 

In this paper, motivated by the above, we proceed in another direction to developing some generalizations of the above integrals considered by Srivastava and Choi [7]. For $q$-analogues of the results, compare the recent book of the same authors [8]. Our main result is Theorem 2.1 which gives a closed form for $\int_0^a \log \Gamma(1 - t) \, dt$ and locates its genesis. A slight modification of Theorem 2.1 gives the counterpart of Barnes’ formula (1.9) which reads.

**Corollary 1.1.** Except for integral values of $a$, one has

$$
\int_0^a \log \Gamma(a - t) \, dt = \log \frac{G(a - a)}{G(a)} + (1 - a) \log \frac{\Gamma(a - a)}{\Gamma(a)}
$$

$$
+ a \log(1 - a) + \frac{1}{2} a^2 + \frac{1}{2} \left( \log 2 + 1 - 2a \right) a.
$$

Srivastava and Choi introduced two functions $\log A_{1/a}$ and $\log C_{1/a}$ by (2.9) and (2.9) with formal replacement of $1/a$ by $-1/a$, respectively. They state $C_{1/a} = A_{-1/a}$, which is rather ambiguous as to how we interpret the meaning because (2.9) is defined for $a > 0$ [7, page 347, l.11]. They use this $C_{1/a}$ function to express the integral $\int_0^a \log \Gamma(1 - t) \, dt$, without giving proof. This being the case, it may be of interest to locate the integral of $\log \Gamma(1 - t)$ [7, (13), page 349], thereby $\log C_{1/a}$ [7, page 347].

For this purpose we use a more fundamental function $A(a)$ than $A_{1/a}$ defined by

$$
\log A(a) = -\zeta'(-1, a) + \frac{1}{12},
$$

(1.11)

where $\zeta(s, a)$ is the Hurwitz zeta-function

$$
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \quad \text{Re } s = \sigma > 1
$$

(1.12)

in the first instance. For its theory, compare, for instance, [3], [9, Chapter 3].

We shall prove the following corollary which gives the right interpretation of the function $C_{1/a}$.

**Corollary 1.2.** For $0 < a < 1$, $\log C_{1/a}$

$$
\log C_{1/a} = \log A(1 - a) - \frac{1}{4} a^2,
$$

(1.13)

or

$$
\log C_{1/a} = \log A_{1-1/a} + \frac{1}{4} (1 - a)^2 + (1 - a) \log(1 - a) - \frac{1}{4} a^2.
$$

(1.14)
2. Barnes Formula

There is a generalization of (1.4) as well as (1.2) in the form [7, equation (28), page 31]:

$$\int_0^a \log \sin \pi t \, dt = a \log \frac{\sin \pi a}{2\pi} + \log \frac{G(1+a)}{G(1-a)}, \quad a \notin \mathbb{Z}. \quad (2.1)$$

Equation (2.1) is Barnes’ formula [6, page 279] which is equivalent to Kinkelin’s 1860 result [10] [7, equation (26), page 30]:

$$\int_0^a \pi t \cot \pi t \, dt = \log \frac{G(1-z)}{G(1+z)} + z \log 2\pi. \quad (2.2)$$

Since (1.5) is equivalent to

$$G(z+1) = G(z) \Gamma(z), \quad (2.3)$$

it follows that

$$\int_0^a \log \sin \pi t \, dt = a \log \frac{\sin \pi a}{2\pi} + \log \frac{G(a)}{G(1-a)} + \log \Gamma(a). \quad (2.4)$$

Putting $a = 1/2$, we obtain

$$\pi^{-1} \int_0^{\pi/2} \log \sin x \, dx = \int_0^{1/2} \log \sin \pi t \, dt = -\frac{1}{2} \log 2\pi + \log \Gamma \left( \frac{1}{2} \right) = -\frac{1}{2} \log 2, \quad (2.5)$$

which is (1.2).

The counterpart of (2.1) follows from the reciprocity relation (1.8), known as Alexeievsky’s Theorem [7, equation (42), page 32].

$$\int_0^a \log \Gamma(1+t) \, dt = \frac{1}{2} \left( \log 2\pi - 1 \right) a - \frac{a^2}{2} + a \log \Gamma(a+1) - \log G(a+1), \quad (2.6)$$

which in turn is a special case of (1.9).

Indeed, in [7, page 207], only (1.9) and the integral of $\log G(t+a)$ are in closed form and the integral of $\log \Gamma_3(t+a)$ is not. A general formula is given by Barnes [4] with constants to be worked out. We shall state a concrete form for this integral in Section 3, using the relation [7, equation (455), page 210] between $\log \Gamma_3(t+a)$ and the integral of $q$ and appealing to a closed form for the latter in [11].

Formula (2.6) is stated in the following form [7, equation (12), page 349]:

$$\int_0^a \log \Gamma(1+t) \, dt = \frac{1}{2} \left( \log 2\pi - 1 \right) a - \frac{3}{4} a^2 + \log A - \log A_{1/\alpha}, \quad (2.7)$$
where \( \log A \) is the Glaisher-Kinkelin constant defined by [7, equation (2), page 25]

\[
\log A = \lim_{N \to \infty} \left( \sum_{n=1}^{N} n \log n - \frac{1}{2} \left( N^2 + N + B_2 \right) \log N + \frac{1}{4} N^2 \right),
\]

(2.8)

and \( \log A_{1/a} \) is defined by [7, equation (9), page 347]

\[
\log A_{1/a} = \lim_{N \to \infty} \left( \sum_{n=1}^{N} (n + a) \log(n + a) - \frac{1}{2} \left( N^2 + (2a + 1)N + a^2 + a + B_2 \right) \log(N + a) + \frac{1}{4} N^2 + \frac{a}{2} N \right),
\]

(2.9)

for \( a > 0 \).

Comparing (2.6) and (2.7), we immediately obtain

\[
\log A_{1/a} = \log G(a + 1) - a \log \Gamma(a + 1) + \log A - \frac{a^2}{4}
\]

\[
= \log G(a) + (1-a) \log \Gamma(a) + \log a - \frac{a^2}{4} - a \log a,
\]

(2.10)

on using the difference relation \( \Gamma(a + 1) = a \Gamma(a) \).

Thus, in a sense we have located the genesis of the function \( \log A_{1/a} \), although they prove (2.7) by an elementary method [7, page 348].

Indeed, \( A_{1/a} \) and \( A(a) \) are almost the same:

\[
\log A_{1/a} = \log A(a) - \frac{1}{4} a^2 - a \log a,
\]

(2.11)

a proof being given below. However, \( \log A(a) \) is more directly connected with \( \zeta(-1, a) \) for which we have rich resources of information as given in [9, Chapter 3].

We prove the following theorem which gives a closed form for \( \int_{0}^{a} \log \Gamma(1-t) \, dt \), thereby giving the genesis of the constant \( C_{1/a} \).

**Theorem 2.1.** For \( a \not\in \mathbb{Z} \), one has

\[
\int_{0}^{a} \log \Gamma(1-t) \, dt = \log G(1-a) + a \log \Gamma(1-a) + \frac{1}{2} a^2 + \frac{1}{2} (\log 2\pi - 1)a.
\]

(2.12)

If \( 0 < a < 1 \), then

\[
\int_{0}^{a} \log \Gamma(1-t) \, dt = \log A(1-a) - \log A + \frac{1}{2} a^2 + \frac{1}{2} (\log 2\pi - 1)a.
\]

(2.13)
Proof. We evaluate the integral

\[ I = \int_0^a \log \Gamma(1 + t) \sin \pi t \, dt \]  

(2.14)

in two ways. First,

\[ I = a \log \pi + a \log a - a - \int_0^a \log \Gamma(1 - t) \, dt. \]  

(2.15)

On the other hand, noting that \( I \) is the sum of (2.1) and (2.7), we deduce that

\[ I = a \log \frac{\sin \pi a}{2\pi} + \log G(a + 1) + \log A - \log G(1 - a) \]

\[ + \frac{1}{2} (\log 2\pi - 1)a - \frac{3}{4} a^2 - \log A_{1/a}. \]  

(2.16)

Substituting (1.5), we obtain

\[ I = a \log \frac{\sin \pi a}{2\pi} + a \log \Gamma(a) + \log A(a) - \log A_{1/a} \]

\[ - \log G(1 - a) + \frac{1}{2} (\log 2\pi - 1)a - \frac{3}{4} a^2. \]  

(2.17)

The first two terms on the right of (2.17) become

\[ a \log \frac{\Gamma(a) \sin \pi a}{2\pi} = a \log \frac{1}{2} \Gamma(1 - a) = -a(\log 2 + \log \Gamma(1 - a)), \]  

(2.18)

while the 3rd and the 4th terms give, in view of (2.11), \((1/4)a^2 + a \log a\).

Hence, altogether

\[ I = -a \log 2 - a \log \Gamma(1 - a) - \log G(1 - a) + a \log a - \frac{1}{2} a^2 + \frac{1}{2} (\log 2\pi - 1)a. \]  

(2.19)

Comparing (2.15) and (2.19) proves (2.12), completing the proof.

Comparing (2.13) and [7, equation (13), page 349]

\[ \int_0^a \log \Gamma(1 - t) \, dt = \log A(1 - a) - \log A + \frac{3}{4} a^2 + \frac{1}{2} (\log 2\pi - 1)a, \]  

(2.20)

we prove Corollary 1.2.

Hence the relation between \( C_{1/a} \) and \( A_{1/a} \) is (1.14), that is, one between \( C_{1/a} \) and \( A_{1-1/a} \) rather than \( C_{1/a} = A_{-1/a} \) as Srivastava and Choi state.
At this point we shall dwell on the underlying integral representation for (the derivative of) the Hurwitz zeta-function, which makes the argument rather simple and lucid as in [12] and gives some consequences.

**Proof of (2.11).** Consider that

\[
\zeta'(s, a) - \frac{1}{12} = -\frac{1}{2} a^2 \log a - \frac{1}{4} a^2 - \frac{1}{2} a \log a \\
- \frac{B_2}{2} \log a - \frac{1}{3!} \int_0^\infty \overline{B}_3(t)(t + a)^{-2}dt
\]  

[9, (3.15), page 59], where the last integral may be also expressed as

\[
-\frac{1}{2!} \int_0^\infty \overline{B}_2(t)(t + a)^{-1}dt,
\]

and where \(\overline{B}_k(t)\) is the \(k\)th periodic Bernoulli polynomial. Then

\[
-\zeta'(-1, a) = \sum_{0 \leq N \leq x} (n + a) \log(n + a) - \frac{1}{2} (x + a)^2 \log(x + a) \\
+ \frac{1}{4} (x + a)^2 + \overline{B}_1(x)(x + a) - \frac{1}{2} \overline{B}_2(x)(x + a) + O\left(x^{-1} \log x\right);
\]

whence in particular, we have the generic formula for \(\zeta'(-1, a)\) and consequently for \(\log A(a)\) through (1.11):

\[
\log A(a) = \lim_{N \to \infty} \left( \sum_{n=0}^\infty (n + a) \log(n + a) - \frac{1}{2} \log(N + a) \\
\times \left( (N + a)^2 + N + a + B_2 \right) + \frac{1}{4} (N + a)^2 \right).
\]

This may be slightly modified in the form

\[
\log A(a) = \lim_{N \to \infty} \left( \sum_{n=0}^N (n + a) \log(n + a) \\
- \frac{1}{2} \left( N^2 + (2a + 1)N + a^2 + a + B_2 \right) \log(N + a) + \frac{1}{4} N^2 + \frac{1}{2} aN \right) \\
+ \frac{1}{4} a^2 + a \log a.
\]

Comparing (2.9) and (2.25), we verify (2.11). \(\square\)
The merit of using \( A(a) \) is that by way of \( \zeta'(-1, a) \), we have a closed form for it:

\[
\log A(a) = \frac{1}{2}a^2 \log a - \frac{1}{4}a^2 + \frac{1}{2}a \log a + \frac{B_2}{2} \log a + \frac{1}{2!} \int_0^\infty \frac{B_2(t)(t + a)^{-1}}{1} dt.
\] (2.26)

In the same way, via another important relation [7, equation (23), page 94],

\[
\log G(a) = - \left( \zeta'(-1, a) - \frac{1}{12} \right) - \log A - (1 - a) \log \Gamma(a).
\] (2.27)

Equation (2.21) gives a closed form for \( \log G(a) \), too. We also have from (1.11) and (2.27)

\[
\log A(a) = \log G(a) + (1 - a) \log \Gamma(a) + \log A = \log G(a + 1) - a \log \Gamma(a) + \log A.
\] (2.28)

There are some known expressions not so handy as given by (2.27). For example, [7, page 25] and [7, equation (440), page 206], one of which reads

\[
\frac{G'}{G}(1 + z) = \sum_{n=1}^\infty \left( \frac{n}{z+n} - 1 + \frac{z}{n} \right) + \frac{1}{2} \log 2\pi - 1 - (1 + \gamma)z,
\] (2.29)

with \( \gamma \) designating the Euler constant. Equation (2.29) is a basis of (2.2) (cf. proof of [2, Lemma 1]).

**Remark 2.2.** The Glaisher-Kinkelin constant \( A \) is connected with \( A(1) \) and \( A_1 \) as follows:

\[
\log A = \log A(1) = \log A_1 + \frac{1}{4}.
\] (2.30)

This can also be seen from Vardi’s formula [7, (31), page 97]:

\[
\log A = -\zeta'(-1) + \frac{1}{12},
\] (2.31)

which is (1.11) with \( a = 1 \).

We may also give another direct proof of Corollary 1.2.

**Proof of Corollary 1.2 (another proof).** \( \log C_{1/a} \) is the limit of the expression

\[
S_N = \sum_{k=1}^N (k - 1 + a) \log(k - 1 + a) - \left( \frac{1}{2}N^2 + \left( a - \frac{1}{2} \right)N + \frac{1}{2}B_2(a) \right)
\] \times \log(N - 1 + a) + \frac{1}{4}N^2 + \frac{N}{2}(a - 1),
\] (2.32)
where $\alpha = 1 - a$. Let $N = M + 1$. Then

$$S_N = \sum_{k=0}^{M} (k + \alpha) \log(k + \alpha) - \left( \frac{1}{2} (M + 1)^2 + \left( \alpha - \frac{1}{2} \right) (M + 1) + \frac{1}{2} B_2(\alpha) \right) \times \log(M + a) + \frac{1}{4} (M + 1)^2 + \frac{M + 1}{2} \alpha - \frac{M + 1}{2}. \quad (2.33)$$

Hence, simplifying, we find that

$$S_N = \sum_{k=1}^{M} (k + \alpha) \log(k + \alpha) - \left( \frac{1}{2} M^2 + \left( \alpha + \frac{1}{2} \right) M + \frac{1}{2} \left( \alpha^2 + \alpha + B_2 \right) \right) \times \log(M + a) + \frac{1}{4} M^2 + \frac{1}{2} \alpha M + \alpha \log \alpha - \frac{(\alpha - 1)^2}{4} + \frac{1}{4} \alpha^2. \quad (2.34)$$

Hence

$$\log C_{1/a} = \log A_a + \alpha \log \alpha - \frac{(\alpha - 1)^2}{4} + \frac{1}{4} \alpha^2, \quad (2.35)$$

which is (1.14). This completes the proof. \hfill \square

As an immediate consequence of Corollary 1.2, we prove (2.36) as can be found in [7, pages 350–351].

$$A_{1/a} = \left( \frac{\pi a}{\sin \pi a} \right)^{-a} \frac{G(1 + a)}{G(1 - a)} C_{1/a}, \quad 0 < a < 1. \quad (2.36)$$

Proof of (2.36). From (2.28), (1.5), and (1.8), we obtain

$$\log A(a) - \log A(1 - a) = \log \frac{G(1 + a)}{G(1 - a)} - a \log \frac{\pi}{\sin \pi a}. \quad (2.37)$$

On the other hand, by (2.11) and (1.13), we see that the left-hand side of (2.37) is

$$\log \frac{A_{1/a}}{C_{1/a}} + a \log a, \quad (2.38)$$

whence we conclude that

$$\log \frac{A_{1/a}}{C_{1/a}} = \log \frac{G(1 + a)}{G(1 - a)} - a \log \frac{\pi a}{\sin \pi a}. \quad (2.39)$$

On exponentiating, (2.37) leads to (2.36). \hfill \square
3. Polygamma Function of Negative Order

In this section we introduce the function $\tilde{A}_k(q)$ [13]:

$$\tilde{A}_k(q) = k\zeta'(1-k,q),$$

which is closely related to the polygamma function of negative order and states some simple applications. We recall some properties of $\tilde{A}_k(q)$:

$$\tilde{A}_2(q+1) = \tilde{A}_2(q) + 2q \log q,$$
$$\tilde{A}_2\left(\frac{1}{2}\right) = -\zeta'(-1) - \frac{1}{12} \log 2,$$
$$\tilde{A}_2\left(\frac{1}{4}\right) = -\frac{1}{4} \zeta'(-1) + \frac{G}{2\pi},$$
$$\tilde{A}_2\left(\frac{3}{4}\right) = -\frac{1}{2} \zeta'(-1) - \tilde{A}_2\left(\frac{1}{4}\right).$$

Equation (3.3) is [2, equation (2.31)], which is used in proving [2, Theorem 2] and can be read off from the distribution property [9, equation (3.72), page 76] as follows:

$$\sum_{a=1}^{4} \zeta\left(s, \frac{a}{4}\right) = 4^s \zeta(s).$$

Differentiation gives

$$\sum_{n=1}^{4} \zeta'(s, \frac{a}{4}) = 4^s ((\log 4) \zeta(s) + \zeta'(s)).$$

Putting $s = -1$, we obtain

$$\zeta'(-1) + \zeta'\left(-1, \frac{1}{2}\right) + \zeta'\left(-1, \frac{1}{4}\right) + \zeta'\left(-1, \frac{3}{4}\right) = 4^{-1} ((\log 4) \zeta(-1) + \zeta'(-1)),$$

which we solve in $\zeta'(-1, 3/4)$:

$$\zeta'\left(-1, \frac{3}{4}\right) = \frac{1}{4} ((2 \log 2) \zeta(-1) + \zeta'(-1))$$
$$- \zeta'(-1) - \frac{1}{2} \tilde{A}_2\left(\frac{1}{2}\right) - \zeta'\left(-1, \frac{1}{4}\right).$$
Substituting (3.2) and \( \zeta(-1) = -B_2/2 = -1/12 \) and simplifying, we conclude that
\[
\zeta'(-1, \frac{3}{4}) = \frac{1}{4} \zeta'(-1) - \zeta'(-1, \frac{1}{4}) \tag{3.8}
\]
and that
\[
\tilde{A}_2\left(\frac{3}{4}\right) = 2\zeta'(-1, \frac{3}{4}) = -\frac{1}{2} \zeta'(-1) - 2\zeta'(-1, \frac{1}{4}), \tag{3.9}
\]
whence (3.3).

Using these, we deduce from (2.37) the following.

**Example 3.1.**

\[
\log A_{1/4} = \frac{5}{64} + \frac{1}{2} \log 2 - \frac{1}{8} \log A - \frac{G}{2\pi}. \tag{3.10}
\]

**Proof.** By (1.11) and (3.1), for \( q > 0 \),

\[
\log A(q) = -\frac{1}{2} \tilde{A}_2(q) + \frac{1}{12}. \tag{3.11}
\]

Since \( \log A(1/4) - \log A(3/4) = -1/2(\tilde{A}_2(1/4) - \tilde{A}_2(3/4)) \), it follows from (3.3) that the left-hand side of (2.37) is

\[
-\tilde{A}_2\left(\frac{1}{4}\right) - \frac{1}{4} \zeta'(-1), \tag{3.12}
\]

which is

\[
2 \log A\left(\frac{1}{4}\right) - \frac{1}{6} + \frac{1}{4} \left(\log A - \frac{1}{12}\right) \tag{3.13}
\]

where we used (2.31).

The right-hand side of (2.37), \( \log(G(5/4)/G(3/4)) - 1/4 \log(\pi/\sin(\pi/4)) \), becomes \(-G/2\pi\), in view of known values of \( G \) [7, page 30].

Hence, altogether, (2.37) with \( a = 1/4 \) reads

\[
-\frac{G}{2\pi} = 2 \log A\left(\frac{1}{4}\right) + \frac{1}{4} \log A - \frac{3}{16}. \tag{3.14}
\]

Invoking (2.11), this becomes (3.10).

We note that (3.14) gives a proof of the third equality in (3.2). Both (2.36) and (3.10) are contained in [14, 1999a] and are given as exercises in [7].
4. The Triple Gamma Function

For general material, we refer to [7, page 42]. As can been seen on [7, page 207], the important integral \( \int_0^z \log \Gamma_3(t + a) \, dt \) is not in closed form. Recently, Chakraborty-Kanemitsu-Kuzumaki [5, Corollary 1.1] have given a general expressions for all the integrals in \( \log \Gamma_3 \), by appealing to Barnes’ original results.

In this section, we shall give a direct derivation of a closed form by combining [7, (455), page 210] and [11, Corollary 3] (with \( \lambda = 3 \)). The first reads

\[
2 \int_0^z \log \Gamma_3(t + a) \, dt = - \int_0^z t^3 \psi(t + a) \, dt + 2z \log \Gamma_3(z + a)
\]

\[
-2(2a - 3) \frac{\log \Gamma_3(z + a)}{\log \Gamma_3(a)} + \left(3a^2 - 9a + 7\right) \frac{\log \Gamma_3(z + a)}{\log \Gamma_3(a)}
\]

\[
- (a - 1)^3 \frac{\log \Gamma_3(z + a)}{\log \Gamma_3(a)} + \frac{3}{8} z^4 + \frac{1}{3} (1 - \log 2\pi) z^3
\]

\[
+ \left(- \frac{3}{4} a^2 + \frac{7}{4} a - \frac{9}{8} + \frac{1}{4} (2a - 3) \log 2\pi + \log A\right) z^2,
\]

\[
+ \left(a^2 - \frac{3}{2} a + \frac{1}{4} + \frac{1}{2} (a - 2 - 3a + 2) \log 2\pi + 2(3 - 2a) \log A\right) z,
\]

while the second reads (cf. also [15])

\[
\int_0^z t^3 \log \psi(t + a) \, dt = - \sum_{r=0}^3 C_3(r, a) \log \frac{\Gamma_{r+1}(a + z)}{\Gamma_{r+1}(a)}
\]

\[
- \sum_{l=1}^3 (-1)^l \left(\begin{array}{c} 3 \\ l \end{array}\right) \zeta'(l - 3) + \frac{B_{l+1}(a)}{l(4 - l)}\right) z^l + \frac{11}{24} z^4,
\]

where \( C_3(r, a) \) are defined by

\[
C_3(r, a) = (-1)^r r! \sum_{m=r}^3 \binom{3}{m} (-1)^m S(m, n) (a - 1)^{3-m}
\]

and where \( S(m, n) \) are the Stirling numbers of the second kind [7, page 58]. To express the values of \( \zeta'(l - 3) \), we appeal to [7]

(i) \( \zeta'(0) = -(1/2) \log 2\pi \) [7, (20), page 92],

(ii) \( \zeta'(-2) = \log B = \zeta(3)/4\pi^2 \) [7, pages 99-100].
and (2.31). After some elementary but long calculations, we arrive at

\[
\int_0^z t^3 \log q(t + a) dt = -3! \log \frac{\Gamma_4(a + z)}{\Gamma_4(a)} - 6(a - 2) \log \frac{\Gamma_3(a + z)}{\Gamma_3(a)}
\]

\[
- \left(3a^2 - 9a + 7\right) \log \frac{\Gamma_2(a + z)}{\Gamma_2(a)} - (a - 1)^3 \log \frac{\Gamma(a + z)}{\Gamma(a)} \frac{11}{24} z^4
\]

\[
+ \left(- \frac{1}{2} \log 2\pi + \frac{1}{3} B_1(a)\right) z^3 - \left(\frac{1}{4} - 3 \log A + \frac{1}{4} B_2(a)\right) z^2
\]

\[
+ 3 \left(\log B + \frac{1}{3} B_3(a)\right) z.
\]

Combining we have the following.

**Theorem 4.1** (see [5, Example 2.3]). Except for the singularities of the multiple gamma function, one has

\[
\int_0^z \log \Gamma_3(t + a) dt = 3! \log \frac{\Gamma_4(a + z)}{\Gamma_4(a)} + z \log \Gamma_3(z + a)
\]

\[
+ (a - 3) \log \frac{\Gamma_3(a + z)}{\Gamma_3(a)} - \frac{1}{24} z^4 - \frac{1}{6} \left(a - \frac{3}{2} - \frac{1}{2} \log 2\pi\right) z^3
\]

\[
+ \frac{1}{8} \left(-2a^2 + 6a - \frac{10}{3} + (2a - 3) \log 2\pi - 8 \log A\right) z^2
\]

\[
- \frac{1}{2} \left(a^3 - \frac{5}{2} a^2 + 2a - \frac{1}{4} + \frac{1}{2} (a^2 - 3a + 2) \log 2\pi\right)
\]

\[
+ 2(2a - 3) \log A + 3 \log B\right) z.
\]

This theorem enables us to put many formulas in [7] in closed form including, for instance, [7, (698), page 245]. Compare [5].

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**References**


