Research Article

Integrability for Solutions of Anisotropic Obstacle Problems

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This paper deals with anisotropic obstacle problem for the $\mathcal{A}$-harmonic equation

$$\sum_{i=1}^{n} D_i (a_i(x, Du(x))) = 0.$$ 

An integrability result is given under suitable assumptions, which show higher integrability of the boundary datum, and the obstacle force solutions $u$ have higher integrability as well.

1. Introduction and Statement of Result

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. For $p_i > 1$, $i = 1, 2, \ldots, n$, we denote $p_m = \max_{i=1,2,\ldots,n} p_i$ and $\overline{p}$ is the harmonic mean of $p_i$, that is,

$$\frac{1}{\overline{p}} = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}. \quad (1.1)$$

The anisotropic Sobolev space $W^{1,(p_i)}(\Omega)$ is defined by

$$W^{1,(p_i)}(\Omega) = \left\{ v \in W^{1,1}(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, 2, \ldots, n \right\}. \quad (1.2)$$

Let us consider solutions $u \in W^{1,(p_i)}(\Omega)$ of the following $\mathcal{A}$-harmonic equation:

$$\sum_{i=1}^{n} D_i (a_i(x, Du(x))) = 0, \quad (1.3)$$
where $D = (D_1, D_2, \ldots, D_n)$ is the gradient operator, and the Carathéodory functions $a_i(x, \xi) : \Omega \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, n$, satisfy

$$|a_i(x, z)| \leq c_2(h(x) + |z|)^{p_i - 1}, \quad (1.4)$$

for almost every $x \in \Omega$, for every $z \in \mathbb{R}^n$, and for any $i = 1, 2, \ldots, n$, and there exists $\tilde{v} \in (0, +\infty)$ such that

$$\tilde{v} \sum_{i=1}^{n} |z_i - \tilde{z}_i|^{p_i} \leq \sum_{i=1}^{n} (a_i(x, z) - a_i(x, \tilde{z})) (z_i - \tilde{z}_i), \quad (1.5)$$

for almost every $x \in \Omega$, for any $z, \tilde{z} \in \mathbb{R}^n$. The integrability condition for $h(x) \geq 0$ in (1.4) will be given later.

Let $\varphi$ be any function in $\Omega$ with values in $R \cup \{±\infty\}$ and $\theta \in W^{1, (p_i)}(\Omega)$, and we introduce

$$\mathcal{K}^{(p_i)}_{\varphi, \theta}(\Omega) = \left\{ v \in W^{1, (p_i)}(\Omega) : v \geq \varphi, \text{ a.e. and } v - \theta \in W^{1, (p_i)}_0(\Omega) \right\}. \quad (1.6)$$

Note that

$$W^{1, (p_i)}_0(\Omega) = \left\{ v \in W^{1, 1}_0(\Omega) : D_i v \in L^{p_i}(\Omega) \text{ for every } i = 1, 2, \ldots, n \right\}. \quad (1.7)$$

The function $\varphi$ is an obstacle and $\theta$ determines the boundary values.

**Definition 1.1.** A solution to the $\mathcal{K}^{(p_i)}_{\varphi, \theta}$-obstacle problem is a function $u \in \mathcal{K}^{(p_i)}_{\varphi, \theta}(\Omega)$ such that

$$\int_{\Omega} \sum_{i=1}^{n} a_i(x, Du(x))(D_i v(x) - D_i u(x)) \, dx \geq 0, \quad (1.8)$$

whenever $v \in \mathcal{K}^{(p_i)}_{\varphi, \theta}(\Omega)$.

Higher integrability property is important among the regularity theories of nonlinear elliptic PDEs and systems, see the monograph [1] by Bensoussan and Frehse. Meyers and Elcrat [2] first considered the higher integrability for weak solutions of (1.3) in 1975. Iwaniec and Sbordone [3] obtained a regularity result for very weak solutions of the $A$-harmonic equation (1.3) by using the celebrated Gehring’s Lemma. Global integrability for anisotropic equation is contained in [4]. As far as higher integrability of $\nabla u$ is concerned, in problems with nonstandard growth a delicate interplay between the regularity with respect to $x$ and the growth with respect to $\xi$ appears: see [5]. For a global boundedness result of anisotropic variational problems, see [6]. For other related works, see [7]. We refer the readers to the classical books by Ladyženskaya and Ural’ceva [8], Morrey [9], Gilbarg and Trudinger [10] and Giaquinta [11] for some details of isotropic cases.

In the present paper, we consider integrability for solutions of anisotropic obstacle problems of the $A$-harmonic equation (1.3), which show higher integrability of the boundary
datum, and the obstacle force solutions \( u \), have higher integrability as well. The idea of this paper comes from [4], and the result can be considered as a generalization of [4, Theorem 2.1].

**Theorem 1.2.** Let \( u \in \mathcal{K}_{q,\theta}^{(p)}(\Omega) \) be a solution to the \( \mathcal{K}_{q,\theta}^{(p)} \) obstacle problem and \( \theta \in W^{1,q_i}(\Omega) \), \( q_i \in (p_i, +\infty), i = 1, 2, \ldots, n \), \( 0 \leq h \in L^{q_i}(\Omega) \) with \( q_m = \max_{i=1,\ldots,n} q_i \), \( \psi \in [\psi, +\infty] \) is such that \( \theta^* = \max\{\psi, \theta\} \in \theta + W_0^{1,q_i}(\Omega) \). Moreover, \( \bar{p} < n \). Then

\[
\begin{align*}
\theta^* \in \theta \ast L_{\text{weak}}^t(\Omega),
\end{align*}
\]

where

\[
\begin{align*}
t = \frac{\bar{p}^t}{1 - (bp^{t/\bar{p}})(p_m/p_m - 1)} > \bar{p}^t,
\end{align*}
\]

and \( b \) is any number verifying

\[
0 < b \leq \min_{j=1,\ldots,n} \left( 1 - \frac{p_j}{q_j} \left( 1 - \frac{1}{p_j} \right) \right),
\]

\[
1 - \frac{p_m - 1}{p_m} \frac{\bar{p}}{p},
\]

\[
(1.11)
\]

**Remark 1.3.** Take the obstacle function \( \psi \) to be minus infinity in Theorem 1.2, and the condition (1.4) replaced by

\[
|a_i(x, z)| \leq c_2(1 + |z_i|)^{p_i-1}
\]

for almost every \( x \in \Omega \), for every \( z \in \mathbb{R}^n \), and for any \( i = 1, 2, \ldots, n \), then we arrive at Theorem 2.1 in [4].

### 2. Proof of the Main Theorem

**Proof of Theorem 1.2.** Let \( u \in \mathcal{K}_{q,\theta}^{(p)}(\Omega) \) be a solution to the \( \mathcal{K}_{q,\theta}^{(p)} \)-obstacle problem. Take \( \theta^* = \max\{\psi, \theta\} \in \theta + W_0^{1,q_i}(\Omega) \). Let us consider \( L \in (0, +\infty) \) and

\[
\begin{align*}
\nu = \begin{cases} 
\theta^* - L, & \text{for } u - \theta^* < -L, \\
u, & \text{for } -L \leq u - \theta^* \leq L, \\
\theta^* + L, & \text{for } u - \theta^* > L.
\end{cases}
\end{align*}
\]

Then \( \nu \in \mathcal{K}_{q,\theta}^{(p)}(\Omega) \). Indeed, for the second and the third cases of the above definition for \( \nu \), we obviously have \( \nu \geq \nu \), and for the first case, \( u - \theta^* < -L \), we have \( \theta^* > u + L \geq \psi + L \); this
implies \( v = \theta_* - L \geq \varphi \). Since \( u = \theta_* = \theta \) on \( \partial \Omega \), then \( v = u \) on \( \partial \Omega \), this implies \( v = \theta \) on \( \partial \Omega \). By Definition 1.1, one has

\[
0 \leq \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} a_i(x,Du(x))(D_i\theta(x) - D_iu(x))dx \\
= \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} a_i(x,Du(x))(D_i\theta_*(x) - D_iu(x))dx.
\]

(2.2)

Monotonicity (1.5) allows us to write

\[
\tilde{\nu} \sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} |D_iu(x) - D_i\theta_*(x)|^p dx \\
\leq \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} (a_i(x,Du(x)) - a_i(x,D\theta_*(x)))(D_iu(x) - D_i\theta_*(x))dx,
\]

which together with (2.2) implies

\[
\tilde{\nu} \sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} |D_iu(x) - D_i\theta_*(x)|^p dx \\
\leq - \int_{\{|u-\theta_*|>L\}} \sum_{i=1}^{n} a_i(x,D\theta_*)(D_iu(x) - D_i\theta_*(x))dx.
\]

(2.4)

We now use anisotropic growth (1.4) and the Hölder inequality in (2.4), obtaining that

\[
\tilde{\nu} \sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} |D_iu(x) - D_i\theta_*(x)|^p dx \\
\leq \sum_{i=1}^{n} \int_{\{|u-\theta_*|>L\}} a_i(x,D\theta_*)(D_iu(x) - D_i\theta_*(x))dx \\
\leq c_2 \sum_{i=1}^{n} (h + |D_i\theta_*|)^{p-1}|D_iu(x) - D_i\theta_*(x)|dx \\
\leq c_2 \sum_{i=1}^{n} \left( \int_{\{|u-\theta_*|>L\}} (h + |D_i\theta_*|)^p dx \right)^{(p-1)/p_i} \left( \int_{\{|u-\theta_*|>L\}} |D_iu(x) - D_i\theta_*(x)|^p dx \right)^{1/p_i}.
\]

(2.5)

Let \( t_i \) be such that

\[
p_i < t_i \leq q_i,
\]

(2.6)
for every \(i = 1, \ldots, n\); \(t_i\) will be chosen later. We use the Hölder inequality as follows:

\[
\left( \int_{\{|u-\theta_*| > L\}} (h + |D_i \theta_*|)^{p_i} dx \right)^{(p_i-1)/p_i} \\
\leq \left( \int_{\{|u-\theta_*| > L\}} (h + |D_i \theta_*|)^{t_i} dx \right)^{(p_i-1)/t_i} (||u - \theta_*| > L|)^{(t_i-p_i)(p_i-1)/hp_i}.
\]  \hspace{1cm} (2.7)

The following proof is similar to that of [4, Theorem 2.1]; we only list the necessary changes: instead of [4, (3.14)] by

\[
\left( \int_{\{|u-\theta_*| > L\}} (h + |D_i \theta_*|)^{p_i} dx \right)^{(p_i-1)/p_i} \\
\leq \left( \int_{\{|u-\theta_*| > L\}} (h + |D_i \theta_*|)^{t_i} dx \right)^{(p_i-1)/t_i} (||u - \theta_*| > L|)^b \\
\leq M(||u - \theta_*| > L|)^b,
\]  \hspace{1cm} (2.8)

where

\[
M = \max_{j=1,...,n} \left( \int_{\Omega} (h + |D_j \theta_*|)^{t_j} dx \right)^{(p_i-1)/t_j} < \infty,
\]  \hspace{1cm} (2.9)

and instead of [4, (3.19)] we use anisotropic Sobolev Embedding Theorem for \(v - u\),

\[
\left( \int_{\Omega} |v - u|^p dx \right)^{1/p'} \\
\leq c_* \left\{ \prod_{i=1}^n \left( \int_{\Omega} |D_i (v - u)|^{p_i} dx \right)^{1/p_i} \right\}^{1/n} \\
\leq c_* \left\{ \prod_{i=1}^n \left( \int_{\{|u-\theta_*| > L\}} |D_i u - D_i \theta_*|^{p_i} dx \right)^{1/p_i} \right\}^{1/n}.
\]  \hspace{1cm} (2.10)

By \(|v - u| = (|u - \theta_*| - L)1_{\{|u-\theta_*| > L\}}\), we obtain

\[
\left( \int_{\{|u-\theta_*| > L\}} (|u - \theta_*| - L)^p dx \right)^{1/p'} = \left( \int_{\Omega} |v - u|^p dx \right)^{1/p'}.
\]  \hspace{1cm} (2.11)

Following the idea of the proof of Theorem 2.1 in [4], we complete the proof of Theorem 1.2. \(\Box\)
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References

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