Research Article

A Fixed Point Result for Boyd-Wong Cyclic Contractions in Partial Metric Spaces

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A fixed point theorem involving Boyd-Wong-type cyclic contractions in partial metric spaces is proved. We also provide examples to support the concepts and results presented herein.

1. Introduction and Preliminaries

Partial metric spaces were introduced by Matthews [1] to the study of denotational semantics of data networks. In particular, he proved a partial metric version of the Banach contraction principle [2]. Subsequently, many fixed points results in partial metric spaces appeared (see, e.g., [1, 3–19] for more details).

Throughout this paper, the letters $\mathbb{R}$ and $\mathbb{N}^*$ will denote the sets of all real numbers and positive integers, respectively. We recall some basic definitions and fixed point results of partial metric spaces.

Definition 1.1. A partial metric on a nonempty set X is a function $p : X \times X \rightarrow [0, \infty)$ such that for all $x, y, z \in X$

$$(p1) \quad x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$$

$$(p2) \quad p(x, x) \leq p(x, y),$$

$$(p3) \quad p(x, y) = p(y, x),$$

$$(p4) \quad p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X .

If p is a partial metric on X , then the function $d_p : X \times X \rightarrow [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad (1.1)$$

is a metric on X .

Example 1.2 (see, e.g., [1, 3, 11, 12]). Consider $X = [0, \infty)$ with $p(x, y) = \max\{x, y\}$. Then, (X, p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_p(x, y) = |x - y|$.

Example 1.3 (see, e.g., [1]). Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$, and define $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then, (X, p) is a partial metric space.

Example 1.4 (see, e.g., [1, 20]). Let $X := [0, 1] \cup [2, 3]$, and define $p : X \times X \rightarrow [0, \infty)$ by

$$p(x, y) = \begin{cases} \max\{x, y\} & \text{if } \{x, y\} \cap [2, 3] \neq \emptyset, \\ |x - y| & \text{if } \{x, y\} \subset [0, 1]. \end{cases} \quad (1.2)$$

Then, (X, p) is a complete partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X , which has as a base the family of open p -balls $\{B_p(x, \varepsilon), x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$.

Definition 1.5. Let (X, p) be a partial metric space and $\{x_n\}$ a sequence in X . Then,

- (i) $\{x_n\}$ converges to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow +\infty} p(x, x_n)$,
- (ii) $\{x_n\}$ is called a Cauchy sequence if $\lim_{n, m \rightarrow +\infty} p(x_n, x_m)$ exists and is finite.

Definition 1.6. A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$, such that $p(x, x) = \lim_{n, m \rightarrow +\infty} p(x_n, x_m)$.

Lemma 1.7 (see, e.g., [3, 11, 12]). Let (X, p) be a partial metric space. Then,

- (a) $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, d_p) ,
- (b) (X, p) is complete if and only if the metric space (X, d_p) is complete. Furthermore, $\lim_{n \rightarrow +\infty} d_p(x_n, x) = 0$ if and only if

$$p(x, x) = \lim_{n \rightarrow +\infty} p(x_n, x) = \lim_{n, m \rightarrow +\infty} p(x_n, x_m). \quad (1.3)$$

Lemma 1.8 (see, e.g., [3, 11, 12]). Let (X, p) be a partial metric space. Then,

- (a) if $p(x, y) = 0$, then $x = y$,
- (b) if $x \neq y$, then $p(x, y) > 0$.

Remark 1.9. If $x = y$, $p(x, y)$ may not be 0.

Lemma 1.10 (see, e.g., [3, 11, 12]). Let $x_n \rightarrow z$ as $n \rightarrow \infty$ in a partial metric space (X, p) where $p(z, z) = 0$. Then, $\lim_{n \rightarrow \infty} p(x_n, y) = p(z, y)$ for every $y \in X$.

Let Φ be the set of functions $\phi : [0, \infty) \rightarrow [0, \infty)$ such that

- (i) ϕ is upper semicontinuous (i.e., for any sequence $\{t_n\}$ in $[0, \infty)$ such that $t_n \rightarrow t$ as $n \rightarrow \infty$, we have $\limsup_{n \rightarrow \infty} \phi(t_n) \leq \phi(t)$),
- (ii) $\phi(t) < t$ for each $t > 0$.

Recently, Romaguera [21] obtained the following fixed point theorem of Boyd-Wong type [22].

Theorem 1.11. Let (X, p) be a complete partial metric space, and let $T : X \rightarrow X$ be a map such that for all $x, y \in X$

$$p(Tx, Ty) \leq \phi(M(x, y)), \quad (1.4)$$

where $\phi \in \Phi$ and

$$M(x, y) = \max \left\{ p(x, y), p(x, Tx), p(y, Ty), \frac{1}{2} [p(x, Ty) + p(y, Tx)] \right\}. \quad (1.5)$$

Then, T has a unique fixed point.

In 2003, Kirk et al. [23] introduced the following definition.

Definition 1.12 (see [23]). Let X be a nonempty set, m a positive integer, and $T : X \rightarrow X$ a mapping. $X = \bigcup_{i=1}^m A_i$ is said to be a cyclic representation of X with respect to T if

- (i) A_i , $i = 1, 2, \dots, m$ are nonempty closed sets,
- (ii) $T(A_1) \subset A_2, \dots, T(A_{m-1}) \subset A_m, T(A_m) \subset A_1$.

Recently, fixed point theorems involving a cyclic representation of X with respect to a self-mapping T have appeared in many papers (see, e.g., [24–28]).

Very recently, Abbas et al. [24] extended Theorem 1.11 to a class of cyclic mappings and proved the following result, but with $\phi \in \Phi$ being a continuous map.

Theorem 1.13. Let (X, p) be a complete partial metric space. Let m be a positive integer, $A_1, A_2, \dots, A_m$ nonempty closed subsets of (X, d_p) , and $Y = \bigcup_{i=1}^m A_i$. Let $T : Y \rightarrow Y$ be a mapping such that

- (i) $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T ,

- (ii) there exists $\phi : [0, \infty) \rightarrow [0, \infty)$ such that ϕ is continuous and $\phi(t) < t$ for each $t > 0$, satisfying

$$p(Tx, Ty) \leq \phi(M(x, y)), \quad (1.6)$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$, where $A_{m+1} = A_1$ and $M(x, y)$ is defined by (1.5).

Then, T has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

In the following example, $\phi \in \Phi$, but it is not continuous.

Example 1.14. Define $\phi : [0, \infty) \rightarrow [0, \infty)$ by $\phi(t) = t/2$ for all $t \in [0, 1)$ and $\phi(t) = n(n+1)/(n+2)$ for $t \in [n, n+1)$, $n \in \mathbb{N}^*$. Then, ϕ is upper semicontinuous on $[0, \infty)$ with $\phi(t) < t$ for all $t > 0$. However, it is not continuous at $t = n$ for all $n \in \mathbb{N}$.

Following Example 1.14, the main aim of this paper is to present the analog of Theorem 1.13 for a weaker hypothesis on ϕ , that is, with $\phi \in \Phi$. Our proof is simpler than that in [24]. Also, some examples are given.

2. Main Results

Our main result is the following.

Theorem 2.1. Let (X, p) be a complete partial metric space. Let m be a positive integer, $A_1, A_2, \dots, A_m$ nonempty closed subsets of (X, d_p) , and $Y = \bigcup_{i=1}^m A_i$. Let $T : Y \rightarrow Y$ be a mapping such that

- (1) $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to T ,
- (2) there exists $\phi \in \Phi$ such that

$$p(Tx, Ty) \leq \phi(M(x, y)), \quad (2.1)$$

for any $x \in A_i$, $y \in A_{i+1}$, $i = 1, 2, \dots, m$, where $A_{m+1} = A_1$ and $M(x, y)$ is defined by (1.5).

Then, T has a unique fixed point $z \in \bigcap_{i=1}^m A_i$.

Proof. Let $x_0 \in Y = \bigcup_{i=1}^m A_i$. Consider the Picard iteration $\{x_n\}$ given by $Tx_n = x_{n+1}$ for $n = 0, 1, 2, \dots$. If there exists n_0 such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0+1} = Tx_{n_0} = x_{n_0}$ and the existence of the fixed point is proved.

Assume that $x_n \neq x_{n+1}$, for each $n \geq 0$. Having in mind that $Y = \bigcup_{i=1}^m A_i$, so for each $n \geq 0$, there exists $i_n \in \{1, 2, \dots, m\}$ such that $x_n \in A_{i_n}$ and $x_{n+1} = Tx_n \in T(A_{i_n}) \subseteq A_{i_n+1}$. Then, by (2.1)

$$p(x_{n+1}, x_{n+2}) = p(Tx_n, Tx_{n+1}) \leq \phi(M(x_n, x_{n+1})), \quad (2.2)$$

where

$$\begin{aligned}
 M(x_n, x_{n+1}) &= \max \left\{ p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \right. \\
 &\quad \left. \frac{p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)}{2} \right\} \\
 &= \max \left\{ p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1})}{2} \right\} \quad (2.3) \\
 &= \max \left\{ p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})}{2} \right\} \\
 &= \max \{ p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}) \}.
 \end{aligned}$$

Therefore,

$$M(x_n, x_{n+1}) = \max \{ p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}) \} \quad \forall n \geq 0. \quad (2.4)$$

If for some $k \in \mathbb{N}$, we have $M(x_k, x_{k+1}) = p(x_{k+1}, x_{k+2})$, so by (2.2)

$$0 < p(x_{k+1}, x_{k+2}) \leq \phi(p(x_{k+1}, x_{k+2})) < p(x_{k+1}, x_{k+2}), \quad (2.5)$$

which is a contradiction. It follows that

$$M(x_n, x_{n+1}) = p(x_n, x_{n+1}) \quad \forall n \geq 0. \quad (2.6)$$

Thus, from (2.2), we get that

$$0 < p(x_{n+1}, x_{n+2}) \leq \phi(p(x_{n+1}, x_{n+2})) < p(x_{n+1}, x_{n+2}). \quad (2.7)$$

Hence, $\{p(x_n, x_{n+1})\}$ is a decreasing sequence of positive real numbers. Consequently, there exists $\gamma \geq 0$ such that $\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = \gamma$. Assume that $\gamma > 0$. Letting $n \rightarrow \infty$ in the above inequality, we get using the upper semicontinuity of ϕ

$$0 < \gamma \leq \limsup_{n \rightarrow \infty} \phi(p(x_{n+1}, x_{n+2})) \leq \phi(\gamma) < \gamma, \quad (2.8)$$

which is a contradiction, so that $\gamma = 0$, that is,

$$\lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0. \quad (2.9)$$

By (1.1), we have $d_p(x, y) \leq 2p(x, y)$ for all $x, y \in X$, and then from (2.9)

$$\lim_{n \rightarrow \infty} d_p(x_n, x_{n+1}) = 0. \quad (2.10)$$

Also, by (p2),

$$\lim_{n \rightarrow \infty} p(x_n, x_n) = 0. \quad (2.11)$$

In the sequel, we will prove that $\{x_n\}$ is a Cauchy sequence in the partial metric space $(Y = \bigcup_{i=1}^m A_i, p)$. By Lemma 1.7, it suffices to prove that $\{x_n\}$ is Cauchy sequence in the metric space (Y, d_p) . We argue by contradiction. Assume that $\{x_n\}$ is not a Cauchy sequence in (Y, d_p) . Then, there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$ such that

$$d_p(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \quad (2.12)$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.12). Then,

$$d_p(x_{n(k)-1}, x_{m(k)}) < \varepsilon. \quad (2.13)$$

We use (2.13) and the triangular inequality

$$\begin{aligned} \varepsilon &\leq d_p(x_{n(k)}, x_{m(k)}) \leq d_p(x_{n(k)}, x_{n(k)-1}) + d_p(x_{n(k)-1}, x_{m(k)}) \\ &< \varepsilon + d_p(x_{n(k)}, x_{n(k)-1}). \end{aligned} \quad (2.14)$$

Letting $k \rightarrow \infty$ in (2.14) and using (2.10), we find

$$\lim_{k \rightarrow \infty} d_p(x_{n(k)}, x_{m(k)}) = \varepsilon. \quad (2.15)$$

On the other hand

$$\begin{aligned} d_p(x_{n(k)}, x_{m(k)}) &\leq d_p(x_{n(k)}, x_{n(k)+1}) + d_p(x_{n(k)+1}, x_{m(k)+1}) + d_p(x_{m(k)+1}, x_{m(k)}), \\ d_p(x_{n(k)+1}, x_{m(k)+1}) &\leq d_p(x_{n(k)+1}, x_{n(k)}) + d_p(x_{n(k)}, x_{m(k)}) + d_p(x_{m(k)}, x_{m(k)+1}). \end{aligned} \quad (2.16)$$

Letting $k \rightarrow +\infty$ in the two above inequalities and using (2.10) and (2.15),

$$\lim_{k \rightarrow \infty} d_p(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \quad (2.17)$$

Similarly, we have

$$\lim_{k \rightarrow \infty} d_p(x_{n(k)}, x_{m(k)+1}) = \lim_{k \rightarrow +\infty} d_p(x_{m(k)}, x_{n(k)+1}) = \varepsilon. \quad (2.18)$$

Also, by (1.1), (2.11), and (2.15)–(2.18), we may find

$$\begin{aligned}\lim_{k \rightarrow \infty} p(x_{n(k)}, x_{m(k)}) &= \lim_{k \rightarrow \infty} p(x_{n(k)}, x_{m(k)+1}) = \frac{\epsilon}{2}, \\ \lim_{k \rightarrow \infty} p(x_{n(k)+1}, x_{m(k)+1}) &= \lim_{k \rightarrow \infty} p(x_{m(k)}, x_{n(k)+1}) = \frac{\epsilon}{2}.\end{aligned}\tag{2.19}$$

On the other hand, for all k , there exists $j(k)$, $0 \leq j(k) \leq p$, such that $n(k) - m(k) + j(k) \equiv 1(p)$. Then, $x_{m(k)-j(k)}$ (for k large enough, $m(k) > j(k)$) and $x_{n(k)}$ lie in different adjacently labeled sets A_i and A_{i+1} for certain $i = 1, \dots, p$. Using the contractive condition (2.1), we get

$$\begin{aligned}p(x_{n(k)+1}, x_{m(k)-j(k)+1}) &= p(Tx_{n(k)}, Tx_{m(k)-j(k)}) \\ &\leq \phi(M(x_{n(k)}, x_{m(k)-j(k)})),\end{aligned}\tag{2.20}$$

where

$$\begin{aligned}M(x_{n(k)}, x_{m(k)-j(k)}) &= \max \left\{ p(x_{n(k)}, x_{m(k)-j(k)}), p(x_{n(k)}, Tx_{n(k)}), p(x_{m(k)-j(k)}, Tx_{m(k)-j(k)}), \right. \\ &\quad \left. \frac{p(x_{n(k)}, Tx_{m(k)-j(k)}) + p(x_{m(k)-j(k)}, Tx_{n(k)})}{2} \right\} \\ &= \max \left\{ p(x_{n(k)}, x_{m(k)-j(k)}), p(x_{n(k)}, x_{n(k)+1}), p(x_{m(k)-j(k)}, x_{m(k)-j(k)+1}), \right. \\ &\quad \left. \frac{p(x_{n(k)}, x_{m(k)-j(k)+1}) + p(x_{m(k)-j(k)}, x_{n(k)+1})}{2} \right\}.\end{aligned}\tag{2.21}$$

As (2.19), using (2.9), we may get

$$\lim_{k \rightarrow \infty} p(x_{n(k)}, x_{m(k)-j(k)}) = \lim_{k \rightarrow \infty} p(x_{n(k)+1}, x_{m(k)-j(k)+1}) = \frac{\epsilon}{2},\tag{2.22}$$

$$\lim_{k \rightarrow \infty} p(x_{n(k)}, x_{m(k)-j(k)+1}) = \lim_{k \rightarrow \infty} p(x_{n(k)+1}, x_{m(k)-j(k)}) = \frac{\epsilon}{2}.\tag{2.23}$$

By (2.22) and (2.23), we get that

$$\lim_{k \rightarrow \infty} M(x_{n(k)}, x_{m(k)-j(k)}) = \frac{\epsilon}{2}.\tag{2.24}$$

Letting $n \rightarrow \infty$ in (2.20), we get using (2.22), (2.24), and the upper semicontinuity of ϕ

$$0 < \frac{\epsilon}{2} \leq \limsup_{k \rightarrow \infty} \phi(M(x_{n(k)}, x_{m(k)-j(k)})) \leq \phi\left(\frac{\epsilon}{2}\right) < \frac{\epsilon}{2}, \quad (2.25)$$

which is a contradiction.

This shows that $\{x_n\}$ is a Cauchy sequence in the complete subspace $Y = \bigcup_{i=1}^m A_i$ equipped with the metric d_p . Thus, there exists $u = \lim_{n \rightarrow \infty} x_n \in (Y, d_p)$. Notice that the sequence $\{x_n\}_{n \in \mathbb{N}}$ has an infinite number of terms in each A_i , $i = 1, \dots, m$, so since (Y, d_p) is complete, from each A_i , $i = 1, \dots, m$ one can extract a subsequence of $\{x_n\}$ that converges to u . Because A_i , $i = 1, \dots, m$ are closed in (Y, d_p) , it follows that

$$u \in \bigcap_{i=1}^m A_i. \quad (2.26)$$

Thus, $\bigcap_{i=1}^m A_i \neq \emptyset$.

For simplicity, set $A = \bigcap_{i=1}^m A_i$. Clearly, A is also closed in (Y, d_p) , so it is a complete subspace of (Y, d_p) and then (A, p) is a complete partial metric space. Consider the restriction of T on A , that is, $T|_A : A \rightarrow A$. Then, $T|_A$ satisfies the assumptions of Theorem 1.11, and thus $T|_A$ has a unique fixed point in Z . $\square$

3. Examples

We give some examples illustrating our results.

Example 3.1. Let $X = \mathbb{R}$ and $p(x, y) = \max\{|x|, |y|\}$. It is obvious that (X, p) is a complete partial metric space.

Set $A_1 = [-8, 0]$, $A_2 = [0, 8]$, and $Y = A_1 \cup A_2 = [-8, 8]$. Define $T : Y \rightarrow Y$ by

$$Tx = \begin{cases} \frac{-x}{4} & \text{if } x \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases} \quad (3.1)$$

Notice that $T([-8, -1]) = 0$ and $T([-1, 0]) = [0, 1/4]$, and hence $T(A_1) \subseteq A_2$. Analogously, $T((1, 8]) = 0$ and $T([0, 1]) = [-1/4, 0]$, and hence $T(A_2) \subseteq A_1$.

Take

$$\phi(t) = \begin{cases} \frac{t}{3} & \text{if } t \in [0, 1), \\ \frac{n^2}{n^2 + 1} & \text{if } t \in [n, n + 1), \quad n \in \mathbb{N}^*. \end{cases} \quad (3.2)$$

Clearly, T satisfies condition (2.1). Indeed, we have the following cases.

Case 1. ($x \in [-8, -1)$ and $y \in (1, 8]$). Inequality (2.1) turns into

$$p(Tx, Ty) = \max\{0, 0\} = 0 \leq \phi(M(x, y)), \quad (3.3)$$

which is necessarily true.

Case 2. ($x \in [-8, -1)$ and $y \in [0, 1]$). Inequality (2.1) becomes

$$\begin{aligned} p(Tx, Ty) &= \max\left\{0, \frac{|y|}{4}\right\} = \frac{y}{4} \leq \phi(M(x, y)) \\ &= \phi\left(\max\left\{p(x, y), p(x, Tx), p(Ty, y), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\right\}\right) \\ &= \phi\left(\max\left\{|x|, |x|, |y|, \frac{1}{2}[|x| + |y|]\right\}\right) \\ &= \phi(|x|). \end{aligned} \quad (3.4)$$

It is clear that $1/2 \leq \phi(t) < 1$ for all $t > 1$. Hence, (3.4) holds.

Case 3. ($x \in [-1, 0]$ and $y \in (1, 8]$). Inequality (2.1) turns into

$$\begin{aligned} p(Tx, Ty) &= \max\left\{\frac{|x|}{4}, 0\right\} = \frac{|x|}{4} \leq \phi(M(x, y)) \\ &= \phi\left(\max\left\{p(x, y), p(x, Tx), p(Ty, y), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\right\}\right) \\ &= \phi\left(\max\left\{|y|, |x|, |y|, \frac{1}{2}[|x| + |y|]\right\}\right) \\ &= \phi(|y|) = \phi(y), \end{aligned} \quad (3.5)$$

which is true again by the fact that $1/2 \leq \phi(t) < 1$ for all $t > 1$.

Case 4. ($x \in [-1, 0]$ and $y \in [0, 1]$). Inequality (2.1) becomes

$$\begin{aligned} p(Tx, Ty) &= \max\left\{\frac{|x|}{4}, \frac{|y|}{4}\right\} \leq \phi(M(x, y)) \\ &= \phi\left(\max\left\{p(x, y), p(x, Tx), p(Ty, y), \frac{1}{2}[p(x, Ty) + p(Tx, y)]\right\}\right) \\ &= \phi\left(\max\left\{\max\{|x|, |y|\}, |x|, |y|, \frac{1}{2}\left[\max\left\{\frac{|x|}{4}, |y|\right\} + \max\left\{|x|, \frac{|y|}{4}\right\}\right]\right\}\right). \end{aligned} \quad (3.6)$$

Let us examine all possibilities:

$$p(Tx, Ty) = \begin{cases} \frac{|x|}{4} & \text{if } |x| \geq |y|, \\ \frac{|y|}{4} & \text{if } \frac{|y|}{4} \leq |x| \leq |y|, \\ \frac{|y|}{4} & \text{if } |x| \leq \frac{|y|}{4}, \end{cases} \quad (3.7)$$

$$M(x, y) \leq \begin{cases} |x| & \text{if } |x| \geq |y|, \\ |y| & \text{if } \frac{|y|}{4} \leq |x| \leq |y|, \\ |y| & \text{if } |x| \leq \frac{|y|}{4}. \end{cases}$$

Thus, (2.1) holds for $\phi(t) = t/3$.

The rest of the assumptions of Theorem 2.1 are also satisfied. The function T has 0 as a unique fixed point.

However, since ϕ is not a continuous function, we could not apply Theorem 1.13.

Example 3.2. Let $X = [0, 1]$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then, (X, p) is a complete partial metric space. Take $A_1 = \dots = A_p = X$. Define $T : X \rightarrow X$ by $Tx = x/2$. Consider $\phi : [0, \infty) \rightarrow [0, \infty)$ given by Example 1.14.

For all $x, y \in X$, we have

$$p(Tx, Ty) = \max\left\{\frac{x}{2}, \frac{y}{2}\right\} = \phi(p(x, y) \leq \phi(M(x, y))). \quad (3.8)$$

Then all the assumptions of Theorem 2.1 are satisfied. The function T has 0 as a unique fixed point.

Similarly, Theorem 1.13 is not applicable.

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