Research Article

On GCR-Lightlike Product of Indefinite Cosymplectic Manifolds

Varun Jain,1 Rakesh Kumar,2 and R. K. Nagaich3

1 Department of Mathematics, Multani Mal Modi College, Patiala 147001, India
2 Department of Basic & Applied Sciences, University College of Engineering, Punjabi University Patiala, Patiala 147002, India
3 Department of Mathematics, Punjabi University Patiala, Patiala 147002, India

Correspondence should be addressed to Rakesh Kumar, dr.rk37c@yahoo.co.in

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We define GCR-lightlike submanifolds of indefinite cosymplectic manifolds and give an example. Then, we study mixed geodesic GCR-lightlike submanifolds of indefinite cosymplectic manifolds and obtain some characterization theorems for a GCR-lightlike submanifold to be a GCR-lightlike product.

1. Introduction

To fill the gaps in the general theory of submanifolds, Duggal and Bejancu [1] introduced lightlike (degenerate) geometry of submanifolds. Since the geometry of CR-submanifolds has potential for applications in mathematical physics, particularly in general relativity, and the geometry of lightlike submanifolds has extensive uses in mathematical physics and relativity, Duggal and Bejancu [1] clubbed these two topics and introduced the theory of CR-lightlike submanifolds of indefinite Kaehler manifolds and then Duggal and Sahin [2], introduced the theory of CR-lightlike submanifolds of indefinite Sasakian manifolds, which were further studied by Kumar et al. [3]. But CR-lightlike submanifolds do not include the complex and real subcases contrary to the classical theory of CR-submanifolds [4]. Thus, later on, Duggal and Sahin [5] introduced a new class of submanifolds, generalized-Cauchy-Riemann-(GCR-) lightlike submanifolds of indefinite Kaehler manifolds and then of indefinite Sasakian manifolds in [6]. This class of submanifolds acts as an umbrella of invariant, screen real, contact CR-lightlike subcases and real hypersurfaces. Therefore, the study of GCR-lightlike submanifolds is the topic of main discussion in the present scenario. In [7], the present
authors studied totally contact umbilical $GCR$-lightlike submanifolds of indefinite Sasakian manifolds.

In present paper, after defining $GCR$-lightlike submanifolds of indefinite cosymplectic manifolds, we study mixed geodesic $GCR$-lightlike submanifolds of indefinite cosymplectic manifolds. In [8, 9], Kumar et al. obtained some necessary and sufficient conditions for a $GCR$-lightlike submanifold of indefinite Kaehler and Sasakian manifolds to be a $GCR$-lightlike product, respectively. Thus, in this paper, we obtain some characterization theorems for a $GCR$-lightlike submanifold of indefinite cosymplectic manifold to be a $GCR$-lightlike product.

2. Lightlike Submanifolds

Let $V$ be a real $m$-dimensional vector space with a symmetric bilinear mapping $g : V \times V \to \mathbb{R}$. The mapping $g$ is called degenerate on $V$ if there exists a vector $\xi \neq 0$ of $V$ such that

$$g(\xi, v) = 0, \quad \forall v \in V, \quad (2.1)$$

otherwise $g$ is called nondegenerate. It is important to note that a non-degenerate symmetric bilinear form on $V$ may induce either a non-degenerate or a degenerate symmetric bilinear form on a subspace of $V$. Let $W$ be a subspace of $V$ and $g \mid W$ degenerate; then $W$ is called a degenerate (lightlike) subspace of $V$.

Let $(\overline{M}, \overline{g})$ be a real $(m + n)$-dimensional semi-Riemannian manifold of constant index $q$ such that $m, n \geq 1, 1 \leq q \leq m + n - 1$, and let $(M, g)$ be an $m$-dimensional submanifold of $\overline{M}$ and $g$ the induced metric of $\overline{g}$ on $M$. Thus, if $\overline{g}$ is degenerate on the tangent bundle $TM$ of $M$, then $M$ is called a lightlike (degenerate) submanifold of $\overline{M}$ (for detail see [1]). For a degenerate metric $g$ on $M$, $TM^\perp$ is also a degenerate $n$-dimensional subspace of $T_x\overline{M}$. Thus, both $T_xM$ and $T_xM^\perp$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $\text{Rad } T_xM = T_xM \cap T_xM^\perp$, which is known as radical (null) subspace. If the mapping $\text{Rad } TM : x \in M \to \text{Rad } T_xM$ defines a smooth distribution on $M$ of rank $r > 0$, then the submanifold $M$ of $\overline{M}$ is called an $r$-lightlike submanifold and $\text{Rad } TM$ is called the radical distribution on $M$. Then, there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to $\text{Rad } TM$ in $TM$. Therefore,

$$TM = \text{Rad } TM \perp S(TM), \quad (2.2)$$

where $\perp$ denotes orthogonal direct sum. Let $S(TM^\perp)$, called screen transversal vector bundle, be a non-degenerate complementary vector subbundle to $\text{Rad } TM$ in $TM^\perp$. Let $\text{tr}(TM)$ and $\text{ltr}(TM)$ be complementary (but not orthogonal) vector bundles to $TM$ in $T\overline{M}|_M$ and to $\text{Rad } TM$ in $S(TM^\perp)$, called transversal vector bundle and lightlike transversal vector bundle of $M$, respectively. Then, we have

$$\text{tr}(TM) = \text{ltr}(TM) \perp S(TM^\perp), \quad (2.3)$$

$$T\overline{M}|_M = TM \oplus \text{tr}(TM) = (\text{Rad } TM \oplus \text{ltr}(TM)) \perp S(TM) \perp S(TM^\perp). \quad (2.4)$$
Let $u$ be a local coordinate neighborhood of $M$ and consider the local quasiorthonormal fields of frames of $\overline{M}$ along $M$ on $u$ as $\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\}$, where $\{\xi_1, \ldots, \xi_r\}$ and $\{N_1, \ldots, N_r\}$ are local lightlike bases of $\Gamma(\text{Rad } TM|_u)$ and $\Gamma(\text{ltr } TM|_u)$ and $\{W_{r+1}, \ldots, W_n\}$ and $\{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^1)|_u)$ and $\Gamma(S(TM)|_u)$, respectively. For these quasiorthonormal fields of frames, we have the following theorem.

**Theorem 2.1** (see [1]). Let $(M, g, S(TM), S(TM^1))$ be an $r$-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. Then, there exist a complementary vector bundle $\text{ltr } TM$ of $\text{Rad } TM$ in $S(TM^1)^\perp$ and a basis of $\Gamma(\text{ltr } TM|_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^1)^\perp|_u$, where $u$ is a coordinate neighborhood of $M$, such that

$$\widetilde{g}(N_i, \xi_j) = \delta_{ij}, \quad \overline{g}(N_i, N_j) = 0, \quad \text{for any } i, j \in \{1, 2, \ldots, r\},$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(\text{Rad } TM)$.

Let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then, according to decomposition (2.4), the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \overline{\nabla}_X U = -A_U X + \nabla_X^\perp U,$$  \tag{2.6}

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(\text{tr } TM)$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla_X^\perp U\}$ belong to $\Gamma(TM)$ and $\Gamma(\text{tr } TM)$, respectively. Here $\nabla$ is a torsion-free linear connection on $M$, $h$ is a symmetric bilinear form on $\Gamma(TM)$ that is called second fundamental form, and $A_U$ is a linear operator on $M$, known as shape operator.

According to (2.3), considering the projection morphisms $L$ and $S$ of $\text{tr } TM$ on $\text{ltr } TM$ and $\text{S}(TM^1)$, respectively, then (2.6) gives

$$\overline{\nabla}_X Y = \nabla_X Y + h^t(X, Y) + h^s(X, Y), \quad \overline{\nabla}_X U = -A_U X + D^t_X U + D^s_X U,$$  \tag{2.7}

where we put $h^t(X, Y) = L(h(X, Y))$, $h^s(X, Y) = S(h(X, Y))$, $D^t_X U = L(\nabla_X^\perp U)$, $D^s_X U = S(\nabla_X^\perp U)$.

As $h^t$ and $h^s$ are $\Gamma(\text{ltr } TM)$-valued and $\Gamma(S(TM^1))$-valued, respectively, they are called the lightlike second fundamental form and the screen second fundamental form on $M$. In particular,

$$\overline{\nabla}_X N = -A_N X + \nabla^i_X N + D^s(X, N), \quad \overline{\nabla}_X W = -A_W X + \nabla^i_X W + D^t(X, W),$$  \tag{2.8}

where $X \in \Gamma(TM)$, $N \in \Gamma(\text{ltr } TM)$, and $W \in \Gamma(S(TM^1))$. By using (2.3)-(2.4) and (2.7)-(2.8), we obtain

$$\overline{g}(h^t(X, Y), W) + \overline{g}(Y, D^t(X, W)) = g(A_W X, Y),$$  \tag{2.9}

$$\overline{g}(h^t(X, Y), \xi) + \overline{g}(Y, h^t(X, \xi)) + g(Y, \nabla_X \xi) = 0,$$  \tag{2.10}

for any $\xi \in \Gamma(\text{Rad } TM)$, $W \in \Gamma(S(TM^1))$, and $N, N' \in \Gamma(\text{ltr } TM)$.
Let $P$ be the projection morphism of $TM$ on $S(TM)$. Then, using (2.2), we can induce some new geometric objects on the screen distribution $S(TM)$ on $M$ as

$$\nabla_X PY = \nabla_X^* PY + h^*(X, Y), \quad \nabla_X \xi = -A_X^* \xi + \nabla_X^t \xi,$$  \hfill (2.11)

for any $X,Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla_X^* PY, A_X^* X\}$ and $\{h^*(X, Y), \nabla_X^t \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. $\nabla^*$ and $\nabla^t$ are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$, respectively. Then, using (2.7), (2.8), and (2.11), we have

$$\bar{g}(h^*(X, PY)\xi) = g(A_X^* X, PY), \quad \bar{g}(h^*(X, PY), N) = g(A_N X, PY).$$  \hfill (2.12)

Next, an odd-dimensional semi-Riemannian manifold $\overline{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $(\phi, V, \eta, \bar{g})$, where $\phi$ is a $(1, 1)$ tensor field, $V$ is a vector field called structure vector field, $\eta$ is a 1-form, and $\bar{g}$ is the semi-Riemannian metric on $\overline{M}$ satisfying (see [10])

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y), \quad \bar{g}(X, V) = \eta(X),$$

$$\phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1,$$  \hfill (2.13)

for any $X, Y \in \Gamma(TM)$.

An indefinite almost contact metric manifold $\overline{M}$ is called an indefinite cosymplectic manifold if (see [11])

$$\nabla_X \phi = 0,$$  \hfill (2.14)

$$\nabla_X V = 0.$$  \hfill (2.15)

### 3. Generalized Cauchy-Riemann Lightlike Submanifolds

Calin [12] proved that if the characteristic vector field $V$ is tangent to $(M, g, S(TM))$, then it belongs to $S(TM)$. We assume that the characteristic vector $V$ is tangent to $M$ throughout this paper. Thus, we define the generalized Cauchy-Riemann lightlike submanifolds of an indefinite cosymplectic manifold as follows.

**Definition 3.1.** Let $(M, g, S(TM), S(TM^1))$ be a real lightlike submanifold of an indefinite cosymplectic manifold $(\overline{M}, \bar{g})$ such that the structure vector field $V$ is tangent to $M$; then $M$ is called a generalized-Cauchy-Riemann- (GCR-) lightlike submanifold if the following conditions are satisfied:

(A) there exist two subbundles $D_1$ and $D_2$ of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM),$$  \hfill (3.1)
Let such that Rad \( TM \) is one-dimensional distribution spanned by \( V \), and \( L \) and \( S \) are vector subbundles of \( \text{ltr}(TM) \) and \( S(TM)^\perp \), respectively.

The following proposition shows that the class of GCR-lightlike submanifolds is an umbrella of invariant, contact CR and contact SCR-lightlike submanifolds.

**Proposition 3.2.** A GCR-lightlike submanifold \( M \) of an indefinite cosymplectic manifold \( \overline{M} \) is contact CR-submanifold (resp., contact SCR-lightlike submanifold) if and only if \( D_1 = \{0\} \) (resp., \( D_2 = \{0\} \)).

**Proof.** Let \( M \) be a contact CR-lightlike submanifold; then \( \phi \text{Rad} TM \) is a distribution on \( M \) such that \( \text{Rad} TM \cap \phi \text{Rad} TM = \{0\} \). Therefore, \( D_2 = \text{Rad} TM \) and \( D_1 = \{0\} \). Since \( \text{ltr}(TM) \cap \phi \text{ltr}(TM) = \{0\} \), this implies that \( \phi \text{ltr}(TM) \subset S(TM) \). Conversely, suppose that \( M \) is a GCR-lightlike submanifold of an indefinite Cosymplectic manifold such that \( D_1 = \{0\} \). Then, from (3.1), we have \( D_2 = \text{Rad}(TM) \), and therefore \( \text{Rad} TM \cap \phi \text{Rad} TM = \{0\} \). Hence, \( \phi \text{Rad} TM \) is a vector subbundle of \( S(TM) \). This implies that \( M \) is a contact CR-lightlike submanifold of an indefinite cosymplectic manifold. Similarly the other assertion follows.

The following construction helps in understanding the example of GCR-lightlike submanifold. Let \( (R_{q^2}^{2m+1}, \phi_0, V, \eta, \overline{g}) \) be with its usual Cosymplectic structure and given by

\[
\eta = dz, \quad V = \delta z, \\
\overline{g} = \eta \otimes \eta - \frac{q}{2} \left( dx^i \otimes dx^i + dy^i \otimes dy^i \right) + \sum_{i=q+1}^{m} \left( dx^i \otimes dx^i + dy^i \otimes dy^i \right), \\
\phi_0(X_1, X_2, \ldots, X_{m-1}, X_m, Y_1, Y_2, \ldots, Y_{m-1}, Y_m, Z) \\
= (-X_2, X_1, \ldots, -X_m, X_{m-1}, -Y_2, Y_1, \ldots, -Y_m, Y_{m-1}, 0),
\]

where \((x^i; y^i; z)\) are the Cartesian coordinates. \(\square\)
Example 3.3. Let $\overline{M} = (R^{13}_4, \overline{g})$ be a semi-Euclidean space and $M$ a 9-dimensional submanifold of $\overline{M}$ that is given by

$$
x^4 = x^1 \cos \theta - y^1 \sin \theta, \quad y^4 = x^1 \sin \theta + y^1 \cos \theta, \quad x^2 = y^3, \quad x^5 = \sqrt{1 + (y^5)^2},
$$

(3.5)

where $\overline{g}$ is of signature $(-, -, +, +, +, -, +, +, +, +)$ with respect to the canonical basis $\{d\theta, dx_1, dx_2, dx_3, dx_4, dx_5, dx_6, dy_1, dy_2, dy_3, dy_4, dy_5, dy_6, dz\}$. Then, the local frame of $TM$ is given by

$$
\begin{align*}
\xi_1 &= \partial_x + \cos \theta \partial_y + \sin \theta \partial_z, \\
\xi_2 &= \partial_y, \\
\xi_3 &= \partial_z, \\
X_1 &= \partial_x - \partial_y, \\
X_2 &= \partial_x, \\
X_3 &= \partial_y, \\
X_4 &= y^5 \partial_x + x^5 \partial_y, \\
X_5 &= \partial_x + \partial_y, \\
X_6 &= V = \partial_z.
\end{align*}
$$

(3.6)

Hence, $M$ is a 3-lightlike as $\text{Rad} TM = \text{span}\{\xi_1, \xi_2, \xi_3\}$. Also, $\phi_0 \xi_1 = -\xi_2$ and $\phi_0 \xi_3 = X_1$; these imply that $D_1 = \text{span}\{\xi_1, \xi_2\}$ and $D_2 = \text{span}\{\xi_3\}$, respectively. Since $\phi_0 X_2 = -X_3$, $D_0 = \text{span}\{X_2, X_3\}$. By straightforward calculations, we obtain

$$
S(TM) = \text{span}\{W = x^5 \partial_x - y^5 \partial_y\},
$$

(3.7)

where $\phi_0(W) = X_4$; this implies that $S = S(TM)$. Moreover, the lightlike transversal bundle $\text{ltr}(TM)$ is spanned by

$$
\begin{align*}
N_1 &= \frac{1}{2} (\partial_x + \cos \theta \partial_y + \sin \theta \partial_z), \\
N_2 &= \frac{1}{2} (-\sin \theta \partial_x - \partial_y + \cos \theta \partial_z), \\
N_3 &= \frac{1}{2} (-\partial_x + \partial_y),
\end{align*}
$$

(3.8)

where $\phi_0(N_1) = -N_2$, $\phi_0(N_3) = X_3$. Hence, $L = \text{span}\{N_3\}$. Therefore, $\overline{D} = \text{span}\{\phi_0(N_3), \phi_0(W)\}$. Thus, $M$ is a GCR-lightlike submanifold of $R^{13}_4$.

Let $Q$, $P_1$, $P_2$ be the projection morphism on $D$, $\phi S = M_2$, $\phi L = M_1$, respectively; therefore

$$
X = QX + V + P_1 X + P_2 X,
$$

(3.9)

for $X \in \Gamma(TM)$. Applying $\phi$ to (3.9), we obtain

$$
\phi X = fX + \omega P_1 X + \omega P_2 X,
$$

(3.10)

where $fX \in \Gamma(D)$, $\omega P_1 X \in \Gamma(L)$, and $\omega P_2 X \in \Gamma(S)$, or, we can write (3.10) as

$$
\phi X = fX + \omega X,
$$

(3.11)

where $fX$ and $\omega X$ are the tangential and transversal components of $\phi X$, respectively.
Similarly, 
\[ \phi U = BU + CU, \quad U \in \Gamma(\text{tr}(TM)), \quad (3.12) \]

where \( BU \) and \( CU \) are the sections of \( TM \) and \( \text{tr}(TM) \), respectively. Differentiating (3.10) and using (2.8)–(2.10) and (3.12), we have
\[ \begin{align*}
D^s(X, \omega P_2 Y) &= -\nabla^s_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^s(X, fY) + Ch^s(X, Y), \\
D^l(X, \omega P_1 Y) &= -\nabla^l_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^l(X, fY) + Ch^l(X, Y),
\end{align*} \quad (3.13) \]

for all \( X, Y \in \Gamma(TM) \). By using, cosymplectic property of \( \nabla \) with (2.7), we have the following lemmas.

**Lemma 3.4.** Let \( M \) be a GCR-lightlike submanifold of an indefinite cosymplectic manifold \( \overline{M} \); then one has
\[ (\nabla_X f) Y = A_{\omega Y} X + Bh(X, Y), \quad (\nabla_X \omega) Y = Ch(X, Y) - h(X, fY), \quad (3.14) \]

where \( X, Y \in \Gamma(TM) \) and
\[ (\nabla_X f) Y = \nabla_X f Y - f \nabla_X Y, \quad (\nabla_X \omega) Y = \nabla_X \omega Y - \omega \nabla_X Y. \quad (3.15) \]

**Lemma 3.5.** Let \( M \) be a GCR-lightlike submanifold of an indefinite cosymplectic manifold \( \overline{M} \); then one has
\[ (\nabla_X B) U = A_{CU} X - f A_{U} X, \quad (\nabla_X C) U = -\omega A_{U} X - h(X, BU), \quad (3.16) \]

where \( X \in \Gamma(TM) \) and \( U \in \Gamma(\text{tr}(TM)) \) and
\[ (\nabla_X B) U = \nabla_X BU - B \nabla_X U, \quad (\nabla_X C) U = \nabla_X CU - C \nabla_X U. \quad (3.17) \]

4. **Mixed Geodesic GCR-Lightlike Submanifolds**

**Definition 4.1.** A GCR-lightlike submanifold of an indefinite cosymplectic manifold is called mixed geodesic GCR-lightlike submanifold if its second fundamental form \( h \) satisfies \( h(X, Y) = 0 \), for any \( X \in \Gamma(D \oplus V) \) and \( Y \in \Gamma(D) \).

**Definition 4.2.** A GCR-lightlike submanifold of an indefinite cosymplectic manifold is called GCR geodesic GCR-lightlike submanifold if its second fundamental form \( h \) satisfies \( h(X, Y) = 0 \), for any \( X, Y \in \Gamma(D) \).

**Theorem 4.3.** Let \( M \) be a GCR-lightlike submanifold of an indefinite cosymplectic manifold \( \overline{M} \). Then, \( M \) is mixed geodesic if and only if \( A^s_X \) and \( A^l_X \not\in \Gamma(M \perp \phi D) \), for any \( X \in \Gamma(D \oplus V), W \in \Gamma(S(TM^\perp)) \) and \( \xi \in \Gamma(\text{Rad}(TM)) \).
Proof. Using, definition of GCR-lightlike submanifolds, $M$ is mixed geodesic if and only if
g(h(X,Y),W) = g(h(X,Y),ξ) = 0, for $X ∈ \Gamma(D ⊕ V), Y ∈ \Gamma(\overline{D}), W ∈ \Gamma(S(TM^⊥))$, and $ξ ∈ \Gamma(\text{Rad}(TM))$. Using (2.8) and (2.11), we get

$$\overline{g}(h(X,Y),W) = \overline{g}(-\nabla_XY, W) = -g(Y, \nabla_XW) = g(Y, A_WX),$$

$$\overline{g}(h(X,Y),ξ) = \overline{g}(-\nabla_XY, ξ) = -g(Y, \nabla_Xξ) = g(Y, A^*_ξX).$$

Therefore, from (4.1), the proof is complete. □

Theorem 4.4. Let $M$ be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, $M$ is $\overline{D}$ geodesic if and only if $A^*_ξX$ and $A_WX ∉ \Gamma(M_D \perp ωD_2)$, for any $X ∈ \Gamma(\overline{D}), ξ ∈ \Gamma(\text{Rad}(TM))$, and $W ∈ \Gamma(S(TM^⊥))$.

Proof. The proof is similar to the proof of Theorem 4.3. □

Lemma 4.5. Let $M$ be a mixed geodesic GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then $A^*_ξX ∈ \Gamma(ωD_2)$, for any $X ∈ \Gamma(\overline{D}), ξ ∈ \Gamma(D_2)$.

Proof. For $X ∈ \Gamma(\overline{D})$ and $ξ ∈ \Gamma(D_2)$, using (2.7) we have

$$h(φξ, X) = 0, \quad A^*_ξX = \Gamma(D_0 ⊕ \{V\} \perp ωD_2).$$

Since $M$ is mixed geodesic, we obtain $φνXξ = νXφξ$. Here, using (2.11), we get $φ(-A^*_ξX + ν^*_Xφξ) = ν^*_Xφξ + h^*(X, φξ)$, and then, by virtue of (3.11), we obtain $-f A^*_ξX - ωA^*_ξX + φ(ν^*_Xφξ) = ν^*_Xφξ + h^*(X, φξ)$. Comparing the transversal components, we get $ωA^*_ξX = 0$; this implies that

$$A^*_ξX ∈ \Gamma(D_0 ⊕ \{V\} \perp ωD_2).$$

If $A^*_ξX ∈ D_0$, then the nondegeneracy of $D_0$ implies that there must exist a $Z_0 ∈ D_0$ such that $\overline{g}(A^*_ξX, Z_0) ≠ 0$. But using the hypothesis that $M$ is a mixed geodesic with (2.7) and (2.11), we get

$$\overline{g}(A^*_ξX, Z_0) = -\overline{g}(νXξ, Z_0) = \overline{g}(ξ, νXZ_0) = \overline{g}(ξ, νXZ_0 + h(X, Z_0)) = 0. \quad (4.4)$$

Therefore,

$$A^*_ξX ∉ \Gamma(D_0). \quad (4.5)$$

Also using (2.13), and (2.15), we get

$$\overline{g}(A^*_ξX, V) = -\overline{g}(νXξ, V) = \overline{g}(ξ, νXV) = 0. \quad (4.6)$$
Therefore,

\[ A^*_X \notin \{V\} \]  

Hence, from (4.3), (4.5), and (4.7), the result follows.

**Corollary 4.6.** Let \( M \) be a mixed geodesic GCR-lightlike submanifold of an indefinite cosymplectic manifold \( \overline{M} \). Then, \( \overline{g}(h'(X, Y), \xi) = 0 \), for any \( X \in \Gamma(\overline{D}), Y \in \Gamma(M_2) \) and \( \xi \in \Gamma(D_2) \).

**Proof.** The result follows from (2.12) and Lemma 4.5.

**Theorem 4.7.** Let \( M \) be a mixed geodesic GCR-lightlike submanifold of an indefinite cosymplectic manifold \( \overline{M} \). Then, \( A_X \in \Gamma(D \oplus \{V\}) \) and \( \nabla^t_X U \in \Gamma(L \perp S) \), for any \( X \in \Gamma(D \oplus \{V\}) \) and \( U \in \Gamma(L \perp S) \).

**Proof.** Since \( M \) is mixed geodesic GCR-lightlike submanifold \( h(X, Y) = 0 \) for any \( X \in \Gamma(D \oplus \{V\}), Y \in \Gamma(\overline{D}) \), and thus (2.6) implies that

\[ 0 = \nabla_X Y - A_X Y. \]  

(4.8)

Since \( \overline{D} \) is an anti-invariant distribution there exists a vector field \( U \in \Gamma(L \perp S) \) such that \( \phi U = Y \). Thus, from (2.8), (2.14), (3.11), and (3.12), we get

\[ 0 = \nabla_X U - \nabla_X Y = \phi(-A_X + \nabla^t_X U) - \nabla_X Y \]

\[ = -f A_X + \omega A_X + B \nabla^t_X U + C \nabla^t_X U - \nabla_X Y. \]  

(4.9)

Comparing the transversal components, we get \( \omega A_X = C \nabla^t_X U \). Since \( \omega A_X \in \Gamma(L \perp S) \) and \( C \nabla^t_X U \in \Gamma(L \perp S) \perp \), this implies that \( \omega A_X = 0 \) and \( C \nabla^t_X U = 0 \). Hence, \( A_X \in \Gamma(D \oplus \{V\}) \) and \( \nabla^t_X U \in \Gamma(L \perp S) \).

5. **GCR-Lightlike Product**

**Definition 5.1.** GCR-lightlike submanifold \( M \) of an indefinite cosymplectic manifold \( \overline{M} \) is called GCR-lightlike product if both the distributions \( D \oplus \{V\} \) and \( \overline{D} \) define totally geodesic foliation in \( M \).

**Theorem 5.2.** Let \( M \) be a GCR-lightlike submanifold of an indefinite cosymplectic manifold \( \overline{M} \). Then, the distribution \( D \oplus \{V\} \) define a totally geodesic foliation in \( M \) if and only if \( Bh(X, \phi Y) = 0 \), for any \( X, Y \in D \oplus \{V\} \).
Proof. Since $\overline{D} = \phi(L \perp S)$, $D \oplus \{ V \}$ defines a totally geodesic foliation in $M$ if and only if $g(\nabla_X Y, \phi \xi) = g(\nabla_X Y, \phi W) = 0$, for any $X, Y \in \Gamma(D \oplus \{ V \})$, $\xi \in \Gamma(D_2)$, and $W \in \Gamma(S)$. Using (2.7) and (2.14), we have

\[
g(\nabla_X Y, \phi \xi) = -\overline{g}(\overline{\nabla}_X \phi Y, \xi) = -\overline{g}(h'(X, fY), \xi), \tag{5.1}
\]

\[
g(\nabla_X Y, \phi W) = -\overline{g}(\overline{\nabla}_X \phi Y, W) = -\overline{g}(h^s(X, fY), W). \tag{5.2}
\]

Hence, from (5.1) and (5.2), the assertion follows.

Theorem 5.3. Let $M$ be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, the distribution $\overline{D}$ defines a totally geodesic foliation in $M$ if and only if $A_N X$ has no component in $\phi S \perp \phi D_2$ and $A_{\omega Y} X$ has no component in $D_2 \perp D_0$, for any $X, Y \in \Gamma(\overline{D})$ and $N \in \Gamma(\text{ltr}(TM))$.

Proof. From the definition of a GCR-lightlike submanifold, we know that $\overline{D}$ defines a totally geodesic foliation in $M$ if and only if

\[
g(\nabla_X Y, N) = g(\nabla_X Y, \phi N_1) = g(\nabla_X Y, V) = g(\nabla_X Y, \phi Z) = 0, \tag{5.3}
\]

for $X, Y \in \Gamma(\overline{D})$, $N \in \Gamma(\text{ltr}(TM))$, $Z \in \Gamma(D_0)$ and $N_1 \in \Gamma(L)$. Using (2.7) and (2.8), we have

\[
g(\nabla_X Y, N) = \overline{g}(\overline{\nabla}_X Y, N) = -\overline{g}(\overline{\nabla}_X Y, \overline{\nabla}_X N) = g(Y, A_N X). \tag{5.4}
\]

Using (2.7), (2.15), and (2.14), we obtain

\[
g(\nabla_X Y, \phi N_1) = -g(\phi \overline{\nabla}_X Y, N_1) = -g(\overline{\nabla}_X \omega Y, N_1) = g(A_{\omega Y} X, N_1), \tag{5.5}
\]

\[
g(\nabla_X Y, \phi Z) = -g(\phi \overline{\nabla}_X Y, Z) = -g(\overline{\nabla}_X \omega Y, Z) = g(A_{\omega Y} X, Z), \tag{5.6}
\]

\[
g(\nabla_X Y, V) = g(\overline{\nabla}_X Y, V) = -g(Y, \overline{\nabla}_X V) = 0. \tag{5.7}
\]

Thus, from (5.4)–(5.7), the result follows.

Theorem 5.4. Let $M$ be a GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. If $(\nabla_X f) Y = 0$, then $M$ is a GCR lightlike product.

Proof. Let $X, Y \in \Gamma(\overline{D})$; therefore $f Y = 0$. Then using (3.15) with the hypothesis, we get $f \nabla_X Y = 0$. Therefore the distribution $\overline{D}$ defines a totally geodesic foliation. Next, let $X, Y \in D \oplus \{ V \}$; therefore $\omega Y = 0$. Then using (3.14), we get $B h(X, Y) = 0$. Therefore, $D \oplus \{ V \}$ defines a totally geodesic foliation in $M$. Hence, $M$ is a GCR lightlike product.

Definition 5.5. A lightlike submanifold $M$ of a semi-Riemannian manifold is said to be an irrotational submanifold if $\overline{\nabla}_X \xi \in \Gamma(TM)$, for any $X \in \Gamma(TM)$ and $\xi \in \Gamma(\text{Rad}(TM))$. Thus, $M$ is an irrotational lightlike submanifold if and only if $h'(X, \xi) = 0$ and $h^s(X, \xi) = 0$. 
Theorem 5.6. Let $M$ be an irrotational GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, $M$ is a GCR lightlike product if the following conditions are satisfied:

(A) $\nabla_X U \in \Gamma(S(TM^1))$, for all $X \in \Gamma(TM)$, and $U \in \Gamma(\text{tr}(TM))$,

(B) $A_0^1 Y \in \Gamma(\phi(S))$, for all $Y \in \Gamma(D)$.

Proof. Let (A) hold; then, using (2.8), we get $A_N X = 0$, $A_W X = 0$, $D^X(X,W) = 0$, and $\nabla_X N = 0$ for $X \in \Gamma(TM)$. These equations imply that the distribution $\overline{D}$ defines a totally geodesic foliation in $M$, and, with (2.9), we get $\overline{g}(h^*(X,Y),W) = 0$. Hence, the nondegeneracy of $S(TM^1)$ implies that $h^*(X,Y) = 0$. Therefore, $h^*(X,Y)$ has no component in $S$. Finally, from (2.10) and the hypothesis that $M$ is irrotational, we have $\overline{g}(h^*(X,Y),\xi) = \overline{g}(Y,A_0^1 X)$, for $X \in \Gamma(TM)$ and $Y \in \Gamma(D)$. Assume that (B) holds; then $h^*(X,Y) = 0$. Therefore, $h^*(X,Y)$ has no component in $L$. Thus, the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in $M$. Hence, $M$ is a GCR lightlike product.

Definition 5.7 (see [13]). If the second fundamental form $h$ of a submanifold, tangent to characteristic vector field $V$, of a Sasakian manifold $\overline{M}$ is of the form

$$h(X,Y) = \{g(X,Y) - \eta(X)\eta(Y)\}a + \eta(X)h(Y,V) + \eta(Y)h(X,V), \quad (5.8)$$

for any $X,Y \in \Gamma(TM)$, where $a$ is a vector field transversal to $M$, then $M$ is called a totally contact umbilical submanifold of a Sasakian manifold.

Theorem 5.8. Let $M$ be a totally contact umbilical GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Then, $M$ is a GCR-lightlike product if $Bh(X,Y) = 0$, for any $X,Y \in \Gamma(TM)$.

Proof. Let $X,Y \in \Gamma(D \oplus \{V\})$; then the hypothesis that $Bh(X,Y) = 0$ implies that the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in $M$.

If we assume that $X,Y \in \Gamma(D)$, then, using (3.14), we have $-f \nabla_X Y = A_{\omega Y} X + Bh(X,Y)$, and taking inner product with $Z \in \Gamma(D_0)$ and using (2.6) and (2.14), we obtain

$$-g(f \nabla_X Y,Z) = g(A_{\omega Y} X + Bh(X,Y),Z) = g\left(\nabla_X Y,\phi Z\right) = -g(Y,\nabla_X Z'), \quad (5.9)$$

where $\phi Z = Z' \in \Gamma(D_0)$. For any $X \in \Gamma(\overline{D})$ from (3.14), we have $\omega P \nabla_X Z = h(X,f Z) - Ch(X,Z)$. Therefore, using the hypothesis with (5.8), we get $\omega P \nabla_X Z = 0$; this implies that $\nabla_X Z \in \Gamma(D)$, and thus (5.9) becomes $g(f \nabla_X Y,Z) = 0$. Then, the nondegeneracy of the distribution $D_0$ implies that the distribution $\overline{D}$ defines a totally geodesic foliation in $M$. Hence, the assertion follows.

Theorem 5.9. Let $M$ be a totally geodesic GCR-lightlike submanifold of an indefinite cosymplectic manifold $\overline{M}$. Suppose that there exists a transversal vector bundle of $M$ which is parallel along $\overline{D}$ with respect to Levi-Civita connection on $M$, that is, $\overline{\nabla}_X U \in \Gamma(\text{tr}(TM))$, for any $U \in \Gamma(\text{tr}(TM))$, $X \in \Gamma(\overline{D})$, Then, $M$ is a GCR-lightlike product.

Proof. Since $M$ is a totally geodesic GCR-lightlike $Bh(X,Y) = 0$, for $X,Y \in \Gamma(D \oplus \{V\})$; this implies $D \oplus \{V\}$ defines a totally geodesic foliation in $M$. 

Next $\nabla_X U \in \Gamma(\text{tr}(TM))$ implies $A_X X = 0$, and hence, by Theorem 5.3, the distribution $\mathcal{D}$ defines a totally geodesic foliation in $M$. Hence, the result follows.

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**References**


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