Research Article

Norm for Sums of Two Basic Elementary Operators

A. Bachir,^{1,2,3} F. Lombarkia,^{1,2,3} and A. Segres^{1,2,3}

¹ Department of Mathematics, Faculty of Science, King Khalid University, P.O. Box 9004, Abha, Saudi Arabia

² Department of Mathematics, Faculty of Science, University of Batna, 05000 Batna, Algeria

³ Department of Mathematics, Mascara University, Algeria

Correspondence should be addressed to A. Bachir, bachir.ahmed@hotmail.com

Received 27 July 2012; Revised 8 November 2012; Accepted 25 November 2012

Academic Editor: Yuri Latushkin

Copyright © 2012 A. Bachir et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We give necessary and sufficient conditions under which the norm of basic elementary operators attains its optimal value in terms of the numerical range.

1. Introduction

Let E be a normed space over $\mathbb{K}(\mathbb{R} \text{ or } \mathbb{C})$, S_E its unit sphere, and E^* its dual topological space. Let D be the normalized duality mapping from E to E^* given by

$$D(x) = \left\{ \varphi \in E^* : \varphi(x) = \|x\|^2, \|\varphi\| = \|x\| \right\}, \quad \forall x \in E. \quad (1.1)$$

Let $B(E)$ be the normed space of all bounded linear operators acting on E . For any operator $A \in B(E)$ and $x \in E$,

$$\begin{aligned} W_x(A) &= \{ \varphi(Ax) : \varphi \in D(x) \}, \\ W(A) &= \cup \{ W_x(A) : x \in S_E \} \end{aligned} \quad (1.2)$$

is called the spatial numerical range of A , which may be defined as

$$W(A) = \{ \varphi(Ax) : x \in S_E; \varphi \in D(x) \}. \quad (1.3)$$

This definition was extended to arbitrary elements of a normed algebra $\mathcal{A}$ by Bonsall [1–3] who defined the numerical range of $a \in \mathcal{A}$ as

$$V(a) = W(A_a), \quad (1.4)$$

where A_a is the left regular representation of A in $B(\mathcal{A})$, that is, $A_a = ab$ for all $b \in \mathcal{A}$. $V(a)$ is known as the algebra numerical range of $a \in \mathcal{A}$, and, according to the above definitions, $V(a)$ is defined by

$$V(a) = \{\varphi(ab) : b \in S_{\mathcal{A}}; \varphi \in D(b)\}. \quad (1.5)$$

For an operator $A \in B(E)$, Bachir and Segres [4] have extended the usual definitions of numerical range from one operator to two operators in different ways as follows.

The spatial numerical range $W(A)_B$ of $A \in B(E)$ relative to B is

$$W(A)_B = \{\varphi(Ax) : x \in S_E; \varphi \in D(Bx)\}. \quad (1.6)$$

The spatial numerical range $G(A)_B$ of $A \in B(E)$ relative to B is

$$G(A)_B = \{\varphi(Ax) : x \in E; \|Bx\| = 1, \varphi \in D(Bx)\}. \quad (1.7)$$

The maximal spatial numerical range of $A \in B(E)$ relative to B is

$$M(A)_B = \{\varphi(Ax) : x \in S_E; \|Bx\| = \|B\|, \varphi \in D(Bx)\}. \quad (1.8)$$

For $A, B \in B(E)$, let $S_E(B) = \{(x_n)_n : x_n \in S_E, \|Bx_n\| \rightarrow \|B\|\}$, then the set

$$\mathcal{M}(A)_B = \{\lim \varphi_n(Ax_n) : (x_n)_n \in S_E(B), \varphi_n \in D(Bx_n)\} \quad (1.9)$$

is called the generalized maximal numerical range of A relative to B . It is known that $\mathcal{M}(A)_B$ is a nonempty closed subset of $\mathbb{K}$ and $M(A)_B \subseteq \mathcal{M}(A)_B \subseteq \overline{W(A)_B}$. The definition of $\mathcal{M}(A)_B$ can be rewritten, with respect to the semi-inner product $[\cdot, \cdot]$ as

$$\mathcal{M}(A)_B = \{\lim [Ax_n, Bx_n] : (x_n)_n \in S_E(B)\}, \quad (1.10)$$

with respect to an inner product $(\cdot, \cdot)$ as

$$\mathcal{M}(A)_B = \{\lim (Ax_n, Bx_n) : (x_n)_n \in S_E(B)\}. \quad (1.11)$$

We shall be concerned to estimate the norm of the elementary operator $M_{A_1, B_1} + M_{A_2, B_2}$, where A_1, A_2, B_1, B_2 are bounded linear operators on a normed space E and M_{A_1, B_1} is the basic elementary operator defined on $B(E)$ by

$$M_{A_1, B_1}(X) = A_1 X B_1. \quad (1.12)$$

We also give necessary and sufficient conditions on the operators A_1, A_2, B_1, B_2 under which $M_{A_1, B_1} + M_{A_2, B_2}$ attains its optimal value $\|A_1\|\|B_1\| + \|A_2\|\|B_2\|$.

2. Equality of Norms

Our next aim is to give necessary and sufficient conditions on the set $\{A_1, A_2, B_1, B_2\}$ of operators for which the norm of $M_{A_1, B_1} + M_{A_2, B_2}$ equals $\|A_1\|\|B_1\| + \|A_2\|\|B_2\|$.

Lemma 2.1. *For any of the operators $A, B, C \in B(E)$ and all $\alpha, \beta \in \mathbb{K}$, one has*

$$\begin{aligned}\mathcal{M}(\alpha A + \beta B)_B &= \alpha \mathcal{M}(A)_B + \beta \|B\|^2; \\ \mathcal{M}(\alpha A + \beta C)_B &\subseteq \alpha \mathcal{M}(A)_B + \beta \mathcal{M}(C)_B.\end{aligned}\tag{2.1}$$

Proof. The proof is elementary. □

Theorem 2.2. *Let A_1, A_2, B_1, B_2 be operators in $B(E)$.*

If $\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2} \cup \mathcal{M}(A_2)_{A_1}$ and $\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2} \cup \mathcal{M}(B_2)_{B_1}$, then

$$\|M_{A_1, B_1} + M_{A_2, B_2}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|.\tag{2.2}$$

Proof. The proof will be done in four steps; we choose one and the others will be proved similarly. Suppose that $\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2}$ and $\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2}$, then there exist $(x_n)_n \in S_E(A_2)$, $\varphi_n \in D(A_2 x_n)$ such that $\|A_1\|\|A_2\| = \lim_n \varphi_n(A_1 x_n)$ and there exist $(y_n)_n \in S_E(B_2)$, $\psi_n \in D(B_2 y_n)$ such that $\|B_1\|\|B_2\| = \lim_n \psi_n(B_1 y_n)$. Define the operators $X_n \in B(E)$ as follows:

$$X_n(y_n) = (\varphi_n \otimes x_n)(y_n) = \varphi_n(y_n)x_n, \quad \forall n.\tag{2.3}$$

Then $\|X_n\| \leq \|B_2\|$, for all $n \geq 1$, and

$$\begin{aligned}\|(M_{A_1+B_1} + M_{A_2+B_2})X_n(y_n)\| &= \|(A_1 X_n B_1 + A_2 X_n B_2)y_n\| \\ &= \|A_1 X_n(B_1 y_n) + A_2 X_n(B_2 y_n)\| \\ &= \frac{\|\varphi_n\|}{\|\varphi_n\|} \|A_1 \varphi_n(B_1 y_n)x_n + A_2 \psi_n(B_2 y_n)x_n\| \\ &\geq \frac{1}{\|\varphi_n\|} \|\varphi_n(\varphi_n(B_1 y_n)A_1 x_n + \psi_n(B_2 y_n)A_2 x_n)\| \\ &= \frac{1}{\|\varphi_n\|} \|\varphi_n(B_1 y_n)\varphi_n(A_1 x_n) + \|B_2 y_n\|^2 \|A_2 x_n\|^2\|.\end{aligned}\tag{2.4}$$

$$\|M_{A_1, B_1} + M_{A_2, B_2}\| \geq \frac{\|(M_{A_1, B_1} + M_{A_2, B_2})X_n(y_n)\|}{\|X_n\|}, \quad \forall n \geq 1.\tag{2.5}$$

Hence

$$\|M_{A_1, B_1} + M_{A_2, B_2}\| \geq \frac{\left\| \varphi_n(B_1 y_n) \varphi_n(A_1 x_n) + \|B_2 y_n\|^2 \|A_2 x_n\|^2 \right\|}{\|A_2\| \|B_2\|}, \quad \forall n \geq 1. \quad (2.6)$$

Letting $n \rightarrow \infty$,

$$\|M_{A_1, B_1} + M_{A_2, B_2}\| \geq \|A_1\| \|B_1\| + \|A_2\| \|B_2\|. \quad (2.7)$$

Since

$$\|M_{A_1, B_1} + M_{A_2, B_2}\| \leq \|A_1\| \|B_1\| + \|A_2\| \|B_2\|, \quad (2.8)$$

therefore

$$\|M_{A_1, B_1} + M_{A_2, B_2}\| = \|A_1\| \|B_1\| + \|A_2\| \|B_2\|. \quad (2.9)$$

□

Corollary 2.3. *Let E be a normed space and $A, B \in B(E)$. Then, the following assertions hold:*

- (1) *if $\|A\| \|B\| \in \mathcal{M}(A)_B$, then $\|A + B\| = \|A\| + \|B\|$;*
- (2) *if $\|A\| \in \mathcal{M}(I)_A$ and $\|B\| \in \mathcal{M}(I)_B$, then $\|M_{A, B} + I\| = 1 + \|A\| \|B\|$.*

Remark 2.4. In the previous corollary, if we set $B = I$, then we obtain an important equation called the Daugavet equation:

$$\|A + I\| = 1 + \|A\|. \quad (2.10)$$

It is well known that every compact operator on $C[0, 1]$ [5] or on $L_1[0, 1]$ [6] satisfies (2.10).

A Banach space E is said to have the Daugavet property if every rank-one operator on E satisfies (2.10). So that from our Corollary 2.3 if $1 \in \mathcal{M}(I)_A$ or $1 \in \mathcal{M}(A)_I$ for every rank-one operator A , then E has the Daugavet property.

The reverse implication in the previous theorem is not true, in general, as shown in the following example which is a modification of that given by the authors Bachir and Segres [4, Example 3.17].

Example 2.5. Let c_0 be the classical space of sequences $(x_n)_n \subset \mathbb{C} : x_n \rightarrow 0$, equipped with the norm $\|(x_n)_n\| = \max_n |x_n|$ and let L be an infinite-dimensional Banach space. Taking the Banach space $E = L \oplus c_0$ equipped with the norm, for $x = (x_1 + x_2) \in E$, $\|x\| = \|x_1 + x_2\| = \max\{\|x_1\|, \|Tx_1\| + \|x_2\|\}$, where T is any norm-one operator from L to c_0 which does not attain its norm (by Josefson-Nissenzweig's theorem [7]), we can find a sequence $(\varphi_n)_n \subset S_{E^*}$ such that φ_n converges weakly to 0. Therefore we get the desired operator $T : L \rightarrow c_0$ defined by

$$(Tx)_n = \frac{n}{n+1} \varphi_n(x). \quad (2.11)$$

Let A_1, A_2, B_1, B_2 be operators defined on E as follows:

$$\begin{aligned} A_1(x_1 + x_2) &= 0 + Tx_1; \\ A_2x &= A_2(x_1 + x_2) = x_1 + 0; \\ B_1(x_1 + x_2) &= x_1 - x_2; \\ B_2 &= I, \quad \forall x = (x_1 + x_2) \in L \times c_0, \end{aligned} \quad (2.12)$$

where I is the identity operator on E . It easy to check that A_1, A_2, B_1 are linear bounded operators and $\|A_1\| = \|A_2\| = \|B_1\| = \|B_2\| = 1$. If we choose $X_0 = I$ and $x_0 = x_1 + 0$ such that $1 = \|Tx_1\| \geq \|x_1\|$, then $\|X_0\| = \|x_0\| = 1$ and

$$\begin{aligned} \|M_{A_1, B_1} + M_{A_2, B_2}\| &\geq \|(M_{A_1, B_1} + M_{A_2, B_2})X_0(x_0)\| \\ &= \|(A_1X_0B_1 + A_2X_0B_2)(x_0)\| \\ &= \|0 + Tx_1 + x_1 + 0\| \\ &= \max\{\|x_1\|, 2\|Tx_1\|\} \\ &= 2, \end{aligned} \quad (2.13)$$

and from

$$\|M_{A_1, B_1} + M_{A_2, B_2}\| \leq \|A_1\|\|A_2\| + \|B_1\|\|B_2\| = 2 \quad (2.14)$$

we get

$$\|M_{A_1, B_1} + M_{A_2, B_2}\| = 2 = \|A_1\|\|A_2\| + \|B_1\|\|B_2\|. \quad (2.15)$$

It is clear from the definitions of $\mathcal{M}(A_1)_{A_2}$ and $W(A_1)_{A_2}$ that

$$\mathcal{M}(A_1)_{A_2} \subseteq \overline{W(A_1)_{A_2}} \quad (2.16)$$

(for details, see [4]).

The next result shows that the reverse is true under certain conditions, before that we recall the definition of Birkhoff-James orthogonality in normed spaces.

Definition 2.6. Let E be a normed space and $x, y \in E$. We say that x is orthogonal to y in the sense of Birkhoff-James ([8, 9]), in short $x \perp_{B-J} y$, iff

$$\forall \lambda \in \mathbb{K} : \|x + \lambda y\| \geq \|x\|. \quad (2.17)$$

If F, G are linear subspaces of E , we say that F is orthogonal to G in the sense of $\perp_{B-J}$, written as $F \perp_{B-J} G$ iff $x \perp_{B-J} y$ for all $x \in F$ and all $y \in G$.

If $T \in B(E)$, we will denote by $\text{Ran}(T)$ and $T^\dagger$ the range and the dual adjoint, respectively, of the operator T .

Theorem 2.7. Let A_1, A_2, B_1, B_2 be operators in $B(E)$.

If $\|M_{A_1, B_1} + M_{A_2, B_2}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|$,

$$\text{Ran} \left(A_2^\dagger \right)_{\perp_{B-J}} \text{Ran} \left(A_1^\dagger - \frac{\|A_1\|}{\|A_2\|} A_2^\dagger \right), \quad \text{Ran} (B_2)_{\perp_{B-J}} \text{Ran} \left(B_1 - \frac{\|B_1\|}{\|B_2\|} B_2 \right), \quad (2.18)$$

then

$$\|A_1\|\|A_2\| \in \mathcal{M} \left(A_1^\dagger \right)_{A_2^\dagger}, \quad \|B_1\|\|B_2\| \in \mathcal{M} (B_1)_{B_2}. \quad (2.19)$$

Moreover, if

$$\text{Ran} \left(A_1^\dagger \right)_{\perp_{B-J}} \text{Ran} \left(A_2^\dagger - \frac{\|A_2\|}{\|A_1\|} A_1^\dagger \right), \quad \text{Ran} (B_1)_{\perp_{B-J}} \text{Ran} \left(B_2 - \frac{\|B_2\|}{\|B_1\|} B_1 \right), \quad (2.20)$$

then

$$\|A_1\|\|A_2\| \in \mathcal{M} \left(A_1^\dagger \right)_{A_2^\dagger} \cap \mathcal{M} \left(A_2^\dagger \right)_{A_1^\dagger}, \quad \|B_1\|\|B_2\| \in \mathcal{M} (B_1)_{B_2} \cap \mathcal{M} (B_2)_{B_1}. \quad (2.21)$$

Proof. If $\|M_{A_1, B_1} + M_{A_2, B_2}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|$, then we can find two normalized sequences $(X_n)_n \subseteq B(E)$ and $(x_n)_n \subseteq E$ such that

$$\lim_n \|A_1 X_n B_1 x_n + A_2 X_n B_2 x_n\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|. \quad (2.22)$$

We have for all $n \geq 1$

$$\begin{aligned} \|A_1 X_n B_1 x_n\| &\leq \|A_1\|\|B_1 x_n\| \leq \|A_1\|\|B_1\| \\ \|A_2 X_n B_2 x_n\| &\leq \|A_2\|\|B_2 x_n\| \leq \|A_2\|\|B_2\|, \end{aligned} \quad (2.23)$$

so we can deduce from the above inequalities and (2.10) that $\lim_n \|B_1 x_n\| = \|B_1\|$ and $\lim_n \|B_2 x_n\| = \|B_2\|$. From the assumptions $\text{Ran}(B_2)_{\perp_{B-J}} \text{Ran}(B_1 - (\|B_1\|/\|B_2\|)B_2)$ we get

$$\overline{\text{Ran} \left(B_1 - \frac{\|B_1\|}{\|B_2\|} B_2 \right)} \cap \overline{\text{Ran}(B_2)} = \{0\}. \quad (2.24)$$

Set $\chi_n = (B_1 - (\|B_1\|/\|B_2\|)B_2)x_n$ and $y_n = B_2 x_n$ for all n and define the function ϕ_n on the closed subspace F spanned by $\{x_n, y_n\}$ for all n as

$$\phi_n(a\chi_n + by_n) = b\|y_n\|^2 = b\|B_2 x_n\|, \quad \forall a, b \in \mathbb{K}. \quad (2.25)$$

It is clear that ϕ_n is linear for all n and

$$|\phi_n(a\chi_n + by_n)| = |b|\|B_2x_n\|^2 = \|a\chi_n + by_n\|\|B_2x_n\|\frac{\|by_n\|}{\|a\chi_n + by_n\|}. \quad (2.26)$$

From the assumptions $\text{Ran}(B_2) \perp_{B-J} \text{Ran}(B_1 - (\|B_1\|/\|B_2\|)B_2)$ it follows that

$$|\phi(a\chi_n + by_n)| \leq \|B_2x_n\|\|a\chi_n + by_n\|, \quad \forall a, b \in \mathbb{K}, \forall n. \quad (2.27)$$

This means that ϕ_n is continuous for each n on the subspace F with $\|\phi_n\| = \|B_2x_n\|$ (by (2.27) and $\phi_n(y_n) = \|y_n\|\|B_2x_n\|$). Then by Hahn-Banach theorem there is $\widetilde{\phi}_n \in E^*$ with $\widetilde{\phi}_n|_F = \phi_n$ and $\|\phi_n\| = \|\widetilde{\phi}_n\|$, for each n . So

$$\widetilde{\phi}_n(\chi_n) = \widetilde{\phi}_n\left(\left(B_1 - \frac{\|B_1\|}{\|B_2\|}B_2\right)x_n\right) = 0, \quad (2.28)$$

hence

$$\begin{aligned} \lim_n \widetilde{\phi}_n(\chi_n) &= \widetilde{\phi}_n\left(\left(B_1 - \frac{\|B_1\|}{\|B_2\|}B_2\right)x_n\right) = 0, \\ \widetilde{\phi}_n(B_2x_n) &= \|B_2x_n\|^2, \quad \|\widetilde{\phi}_n\| = \|B_2x_n\|. \end{aligned} \quad (2.29)$$

Thus, $0 \in \mathcal{M}(B_1 - (\|B_1\|/\|B_2\|)B_2)_{B_2}$ and by Lemma 2.1

$$0 \in \left(\mathcal{M}(B_1)_{B_2} - \frac{\|B_1\|}{\|B_2\|}\|B_2\|^2\right) = \mathcal{M}(B_1)_{B_2} - \|B_1\|\|B_2\|. \quad (2.30)$$

Therefore,

$$\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2}. \quad (2.31)$$

□

From $\|M_{A_1, B_1} + M_{A_2, B_2}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|$ we can find a normalized sequences $(X_n)_n \subseteq B(E)$ such that

$$\lim_n \|A_1X_nB_1 + A_2X_nB_2\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|. \quad (2.32)$$

Since $\|A_1X_nB_1 + A_2X_nB_2\| = \|B_1^\dagger X_n^\dagger A_1^\dagger + B_2^\dagger X_n^\dagger A_2^\dagger\|$, for each n , then we can find a normalized $\phi_{n_k} \in E^\dagger$ such that

$$\lim_{k,n} \|B_1^\dagger X_n^\dagger A_1^\dagger \phi_{n_k} + B_2^\dagger X_n^\dagger A_2^\dagger \phi_{n_k}\| = \|A_1\|\|B_1\| + \|A_2\|\|B_2\|. \quad (2.33)$$

We argue similarly and get

$$\lim_{k,n} \|A_1^\dagger \phi_{n_k}\| = \|A_1^\dagger\|, \quad \lim_{k,n} \|A_2^\dagger \phi_{n_k}\| = \|A_2^\dagger\|. \quad (2.34)$$

Following the same steps as in the previous case we obtain $\|A_1\| \|A_2\| \in \mathcal{M}(A_1^\dagger)_{A_2^\dagger}$.

Moreover, if we have $\text{Ran}(A_1^\dagger) \perp_{B-J} \text{Ran}(A_2^\dagger - (\|A_2\|/\|A_1\|)A_1^\dagger)$ and $\text{Ran}(B_1) \perp_{B-J} \text{Ran}(B_2 - (\|B_2\|/\|B_1\|)B_1)$, it suffices to reverse, in the proof of the previous case, the role of $A_1^\dagger$ into $A_2^\dagger$ and B_1 into B_2 .

For the completeness of the previous theorem we need to prove the following result which is very interesting.

We recall that Phelps [10] has proved that, for a Banach space E , $\cup\{D(x) : x \in E\}$ is dense in E^* ; this property is called subreflexivity of the space E . Using this fact, Bonsall and Duncan [2] has proved that for any operator $T \in B(E)$ we have $\overline{W(T)} = \overline{W(T^\dagger)}$. The following result generalizes the Bollobas result in the case $\mathcal{M}(A)_B$, where $A, B \in B(E)$.

Proposition 2.8. *Let E be a Banach space with smooth dual and let $A, B \in B(E)$ such that B is a surjective operator. Then $\mathcal{M}(A^\dagger)_{B^\dagger} \subseteq \mathcal{M}(A)_B$.*

Proof. Let $a \in \mathcal{M}(A^\dagger)_{B^\dagger}$, then there are $\varphi_n \in D(B^\dagger \varphi_n)$, $(\varphi_n)_n \in S_{E^*}(B^\dagger)$ such that $a = \lim_n \varphi_n(A^\dagger \varphi_n)$.

By the subreflexivity of E there exist sequences $(\varphi_{n_k})_{n_k} \subseteq E^*$ and $(x_{n_k}) \subseteq E$ such that $\varphi_{n_k} \in D(x_{n_k})$ and $\|\varphi_{n_k} - \|Bx_{n_k}\|\varphi_n\|$ to 0. It follows that the sequence $(\hat{x}_{n_k}) \subseteq E^{**}$ has an E^{**} -weak convergent subsequence $(\hat{x}_{n_m})_{n_m}$, that is,

$$\hat{x}_{n_m}(f) \longrightarrow \Psi(f), \quad \forall f \in E^*, \Psi \in E^{**}. \quad (2.35)$$

On the one hand, we have

$$\|Bx_{n_m}\|^2 = \left[B^\dagger(\varphi_{n_m} - \|Bx_{n_m}\|\varphi_n) \right](x_{n_m}) + \|Bx_{n_m}\| \left(B^\dagger \varphi_n \right)(x_{n_m}). \quad (2.36)$$

Then

$$\|Bx_{n_m}\|^2 \leq \left\| B^\dagger(\varphi_{n_m} - \|Bx_{n_m}\|\varphi_n) \right\| + \|Bx_{n_m}\| \left\| B^\dagger \varphi_n \right\|. \quad (2.37)$$

Thus

$$\|Bx_{n_m}\| \left| \|Bx_{n_m}\| - \left\| B^\dagger \varphi_n \right\| \right| \leq \left\| B^\dagger \right\| \left\| \varphi_{n_m} - \|Bx_{n_m}\|\varphi_n \right\|. \quad (2.38)$$

On the other hand,

$$\begin{aligned}
 \left| \hat{x}_{n_m}(B^\dagger \varphi_n) - \|B^\dagger \varphi_n\| \right| &\leq \left| \hat{x}_{n_m}(B^\dagger \varphi_n) - \hat{x}_{n_m}\left(\frac{B^\dagger \varphi_{n_m}}{\|Bx_{n_m}\|}\right) \right| \\
 &\quad + \left| \frac{1}{\|Bx_{n_m}\|} \hat{x}_{n_m}(B^\dagger \varphi_{n_m}) - \|B^\dagger \varphi_n\| \right| \\
 &= \left| \hat{x}\left(B^\dagger \varphi_n - \frac{1}{\|Bx_{n_m}\|} B^\dagger \varphi_{n_m}\right) \right| + \left| \|Bx_{n_m} - \|B^\dagger\| \varphi_n\| \right| \\
 &\longrightarrow 0 \quad \text{as } m \longrightarrow \infty.
 \end{aligned} \tag{2.39}$$

So $\lim_m \hat{x}_{n_m}(B^\dagger \varphi_n) = \|B^\dagger \varphi_n\|$ and $\|B^\dagger \varphi_n\| \Psi_n \in D(B^\dagger \varphi_n)$. Then by smoothness of the space E^* we get $\|B^\dagger \varphi_n\| \Psi_n = \Psi_n$, for all n . Next,

$$\begin{aligned}
 \left| \hat{x}_{n_m}(A^\dagger \varphi_{n_m}) - \frac{\|Bx_{n_m}\|}{\|B^\dagger \varphi_n\|} \varphi_n(A^\dagger \varphi_n) \right| &\leq \left| \hat{x}_{n_m}(A^\dagger \varphi_{n_m}) - \hat{x}_{n_m}(\|Bx_{n_m}\| A^\dagger \varphi_n) \right| \\
 &\quad + \|Bx_{n_m}\| \left| \hat{x}_{n_m}(A^\dagger \varphi_n) - \frac{1}{\|B^\dagger \varphi_n\|} \varphi_n(A^\dagger \varphi_n) \right| \\
 &= \left| \hat{x}_{n_m}(A^\dagger \varphi_{n_m} - A^\dagger \varphi_n) \right| \\
 &\quad + \|Bx_{n_m}\| \left| \hat{x}_{n_m}(A^\dagger \varphi_n) - \varphi_n(A^\dagger \varphi_n) \right| \\
 &\longrightarrow 0 \quad \text{as } m \longrightarrow \infty.
 \end{aligned} \tag{2.40}$$

Then $\lim_m \hat{x}_{n_m}(A^\dagger \varphi_{n_m}) = \varphi_n(A^\dagger \varphi_n)$ or $\lim_m \varphi_{n_m}(Ax_{n_m}) = \varphi(A^\dagger \varphi_n)$ and therefore

$$\lim_n \left[\lim_m \varphi_{n_m}(Ax_{n_m}) \right] = \lim_n \varphi_n(A^\dagger \varphi_n) = a \tag{2.41}$$

which means that $a \in \mathcal{M}(A)_B$. □

Corollary 2.9. Let E be a Banach space with smooth dual and $A_1, A_2, B_1, B_2 \in B(E)$.

If $\|M_{A_1, B_1} + M_{A_2, B_2}\| = \|A_1\| \|B_1\| + \|A_2\| \|B_2\|$ and $\text{Ran}(A_2^\dagger) \perp_{B-J} \text{Ran}(A_1^\dagger - (\|A_1\|/\|A_2\|)A_2^\dagger)$ with A_2 being surjective, and $\text{Ran}(B_2) \perp_{B-J} \text{Ran}(B_1 - (\|B_1\|/\|B_2\|)B_2)$, then

$$\|A_1\| \|A_2\| \in \mathcal{M}(A_1)_{A_2}, \quad \|B_1\| \|B_2\| \in \mathcal{M}(B_1)_{B_2}. \tag{2.42}$$

Moreover, if $\text{Ran}(A_1^\dagger) \perp_{B-J} \text{Ran}(A_2^\dagger - (\|A_2\|/\|A_1\|)A_1^\dagger)$, A_2 is surjective, and $\text{Ran}(B_1) \perp_{B-J} \text{Ran}(B_2 - (\|B_2\|/\|B_1\|)B_1)$, then

$$\|A_1\| \|A_2\| \in \mathcal{M}(A_1)_{A_2} \cap \mathcal{M}(A_2)_{A_1}, \quad \|B_1\| \|B_2\| \in \mathcal{M}(B_1)_{B_2} \cap \mathcal{M}(B_2)_{B_1}. \tag{2.43}$$

Corollary 2.10. Let E be a Banach space with smooth dual and $A_1, A_2, B_1, B_2 \in B(E)$ such that A_1, A_2 are surjective operators. If $\text{Ran}(A_i^\dagger) \perp_{B-J} \text{Ran}(A_j^\dagger - (\|A_j\|/\|A_i\|)A_i^\dagger)$ and $\text{Ran}(B_i) \perp_{B-J} \text{Ran}(B_j - (\|B_j\|/\|B_i\|)B_i)$, ($i, j = 1, 2$ such that $i \neq j$) then the following assertions are equivalent:

- (1) $\|M_{A_1, B_1} + M_{A_2, B_2}\| = \|A_1\| \|B_1\| + \|A_2\| \|B_2\|$;
- (2) $\|A_1\| \|A_2\| \in \mathcal{M}(A_1)_{A_2} \cap \mathcal{M}(A_2)_{A_1}$ and $\|B_1\| \|B_2\| \in \mathcal{M}(B_1)_{B_2} \cap \mathcal{M}(B_2)_{B_1}$.

As a particular case, we obtain the following.

Corollary 2.11. Let E be a Banach space with smooth dual and A, B are surjective operators in $B(H)$. If

$$\text{Ran}(B^\dagger) \perp_{B-J} \text{Ran}\left(A^\dagger - \frac{\|A\|}{\|B\|} B^\dagger\right), \quad \text{Ran}(A) \perp_{B-J} \text{Ran}\left(B - \frac{\|B\|}{\|A\|} A\right), \quad (2.44)$$

then the following assertions are equivalent:

- (1) $\|A\| \|B\| \in \mathcal{M}(A)_B \cap \mathcal{M}(B)_A$;
- (2) $\|M_{A, B} + M_{B, A}\| = 2\|A\| \|B\|$.

3. Hilbert Space Case

Let $E = \mathcal{H}$ be a complex Hilbert space and $A \in B(\mathcal{H})$. The maximal numerical range of A [11] denoted by $W_0(A)$ is defined by

$$\{\lambda \in \mathbb{C} : \exists (x_n), \|x_n\| = 1, \text{ such that } \lim \langle Ax_n, x_n \rangle = \lambda \text{ and } \lim \|Ax_n\| = \|A\|\}, \quad (3.1)$$

and its normalized maximal range, denoted by $W_N(A)$, is given by

$$W_N(A) = \begin{cases} W_0\left(\frac{A}{\|A\|}\right) & \text{if } A \neq 0 \\ 0 & \text{if } A = 0. \end{cases} \quad (3.2)$$

The set $W_0(A)$ is nonempty, closed, convex, and contained in the closure of the numerical range of A .

In this section we prove that if $E = \mathcal{H}$, the conditions

$$\|A_1\| \|A_2\| \in \mathcal{M}(A_1)_{A_2} \cap \mathcal{M}(A_2)_{A_1}, \quad \|B_1\| \|B_2\| \in \mathcal{M}(B_1)_{B_2} \cap \mathcal{M}(B_2)_{B_1} \quad (3.3)$$

would imply that

$$\begin{aligned} \|A_2^* A_1\| &= \|A_1\| \|A_2\|, & \|B_2 B_1^*\| &= \|B_1\| \|B_2\|, \\ W_N(A_2^* A_1) \cap W_N(B_2 B_1^*) &\neq \emptyset. \end{aligned} \quad (3.4)$$

Proposition 3.1. Let $\mathcal{H}$ be a complex Hilbert space, $A_1, A_2, B_1, B_2 \in B(\mathcal{H})$.

If $\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2} \cap \mathcal{M}(A_2)_{A_1}$ and $\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2} \cap \mathcal{M}(B_2)_{B_1}$, then $\|A_2^*A_1\| = \|A_1\|\|A_2\|$ and $\|B_2B_1^*\| = \|B_1\|\|B_2\|$ and $W_N(A_2^*A_1) \cap W_N(B_2B_1^*) \neq \emptyset$.

Proof. If $A_1 = 0$ or $A_2 = 0$ and $B_1 = 0$ or $B_2 = 0$, the result is obvious.

The proof will be done in four steps, we choose one and the others will be proved similarly. Suppose that $A_1 \neq 0$ and $A_2 \neq 0$, if $\|A_1\|\|A_2\| \in \mathcal{M}(A_1)_{A_2}$, then there exists a sequence $(x_n)_n \in S_{\mathcal{H}}(A_2)$ such that

$$\|A_1\|\|A_2\| = \lim \langle A_1x_n, A_2x_n \rangle. \quad (3.5)$$

We have $|\langle A_2^*A_1x_n, x_n \rangle| \leq \|A_2^*A_1\| \leq \|A_1\|\|A_2\|$; this yields

$$\lim \|A_2^*A_1x_n\| = \|A_2^*A_1\| = \|A_1\|\|A_2\|. \quad (3.6)$$

From (3.5) and (3.6) we get

$$\|A_2^*A_1\| = \|A_1\|\|A_2\|, \quad 1 \in W_0\left(\frac{A_2^*A_1}{\|A_2^*A_1\|}\right). \quad (3.7)$$

Suppose now that $B_1 \neq 0$ and $B_2 \neq 0$, if $\|B_1\|\|B_2\| \in \mathcal{M}(B_1)_{B_2}$, then there exists a sequence $(y_n)_n \in S_{\mathcal{H}}(B_2)$ such that

$$\|B_1\|\|B_2\| = \lim \langle B_1y_n, B_2y_n \rangle. \quad (3.8)$$

Since $\lim_n \|B_1y_n\| = \|B_1\|$, then $\lim_n (B_1^*B_1y_n - \|B_1\|^2y_n) = 0$.

Suppose that $w_n = B_1y_n/\|B_1\|$, then $y_n = B_1^*w_n/\|B_1\| + z_n$ such that $\lim_n z_n = 0$.

Hence

$$\begin{aligned} \langle B_2y_n, B_1y_n \rangle &= \left\langle B_2 \left(\frac{B_1^*w_n}{\|B_1\|} \right), \|B_1\|w_n \right\rangle \\ &= \langle B_2B_1^*w_n, w_n \rangle + \langle B_2z_n, \|B_1\|w_n \rangle. \end{aligned} \quad (3.9)$$

From this, we derive that

$$\lim \|B_2B_1^*w_n\| = \|B_2B_1^*\| = \|B_1\|\|B_2\|. \quad (3.10)$$

From (3.8) and (3.10) we have

$$\|B_2B_1^*\| = \|B_1\|\|B_2\|, \quad 1 \in W_0\left(\frac{B_2B_1^*}{\|B_2B_1^*\|}\right). \quad (3.11)$$

From (3.7) and (3.11) we get $\|A_2^*A_1\| = \|A_1\|\|A_2\|$ and $\|B_2B_1^*\| = \|B_1\|\|B_2\|$ and $W_N(A_2^*A_1) \cap W_N(B_2B_1^*) \neq \emptyset$. $\square$

Remark 3.2. We remark that in the case $E = \mathcal{H}$ we obtain an implication given by Boumazgour [12].

Acknowledgments

The authors would like to sincerely thank the anonymous referees for their valuable comments which improved the paper. This research was supported by a grant from King Khalid University (no. KKU_S130_33).

References

- [1] F. F. Bonsall and J. Duncan, *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras*, vol. 2 of *London Mathematical Society Lecture Note Series*, Cambridge University Press, London, UK, 1971.
- [2] F. F. Bonsall and J. Duncan, *Numerical Ranges. II*, vol. 2, Cambridge University Press, New York, NY, USA, 1973.
- [3] F. F. Bonsall, "The numerical range of an element of a normed algebra," *Glasgow Mathematical Journal*, vol. 10, pp. 68–72, 1969.
- [4] A. Bachir and A. Segres, "Numerical range and orthogonality in normed spaces," *Filomat*, vol. 23, no. 1, pp. 21–41, 2009.
- [5] I. K. Daugavet, "A property of completely continuous operators in the space C ," *Uspekhi Matematicheskikh Nauk*, vol. 18, no. 5, pp. 157–158, 1963 (Russian).
- [6] G. Ja. Lozanovskii, "On almost integral operators in KB-spaces," *Vestnik Leningrad University. Mathematics*, vol. 21, no. 7, pp. 35–44, 1966.
- [7] J. Diestel, *Sequences and Series in Banach Spaces*, vol. 92 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1984.
- [8] G. Birkhoff, "Orthogonality in linear metric spaces," *Duke Mathematical Journal*, vol. 1, no. 2, pp. 169–172, 1935.
- [9] R. C. James, "Orthogonality and linear functionals in normed linear spaces," *Transactions of the American Mathematical Society*, vol. 61, pp. 265–292, 1947.
- [10] R. R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, vol. 1364 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2nd edition, 1993.
- [11] J. G. Stampfli, "The norm of a derivation," *Pacific Journal of Mathematics*, vol. 33, pp. 737–747, 1970.
- [12] M. Boumazgour, "Norm inequalities for sums of two basic elementary operators," *Journal of Mathematical Analysis and Applications*, vol. 342, no. 1, pp. 386–393, 2008.