Research Article

Modular Identities and Explicit Evaluations of a Continued Fraction of Ramanujan

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We study a new continued fraction of Ramanujan. We prove its modular identities and give some explicit evaluations.

1. Introduction

Throughout the paper, we assume $|q| < 1$. As usual, for positive integers n and any complex number a , we write

$$(a)_n := (a; q)_n := \prod_{j=0}^{n-1} (1 - aq^j), \quad (a)_\infty := (a; q)_\infty := \prod_{n=0}^{\infty} (1 - aq^n). \quad (1.1)$$

Ramanujan's general theta-function $f(a, b)$ is defined by

$$f(a, b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad (1.2)$$

where $|ab| < 1$. After Ramanujan, we define

$$\phi(q) := f(q, q) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} = \frac{(-q; q^2)_{\infty} (q^2; q^2)_{\infty}}{(q; q^2)_{\infty} (-q^2; q^2)_{\infty}}, \quad (1.3)$$

$$\psi(q) := f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \quad (1.4)$$

$$f(-q) := f(-q, -q^2) = \sum_{k=0}^{\infty} (-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty} (-1)^k q^{k(3k+1)/2} = (q; q)_{\infty}, \quad (1.5)$$

$$\chi(q) = \frac{f(q)}{f(-q^2)} = (-q; q^2)_{\infty}. \quad (1.6)$$

Ramanujan recorded many q -continued fractions and some of their explicit values in his second notebook [1] and in his lost notebook [2]. The following beautiful continued fraction identity was recorded by Ramanujan in his second notebook and can be found in [3, p. 11, Entry 11]:

$$\frac{(-a)_{\infty} (b)_{\infty} - (a)_{\infty} (-b)_{\infty}}{(-a)_{\infty} (b)_{\infty} + (a)_{\infty} (-b)_{\infty}} = \frac{a-b}{1-q} + \frac{(a-bq)(aq-b)}{1-q^3} + \frac{q(a-bq^2)(aq^2-b)}{1-q^5} + \cdots, \quad (1.7)$$

where either q , a , and b are complex numbers with $|q| < 1$, or q , a , and b are complex numbers with $a = bq^m$ for some integer m . Several elegant q -continued fractions have representations as q -products and some of them can be expressed in terms of Ramanujan's theta-functions. An account of this can be found in Chapter 32 of Berndt's book [4] (also see [5]). The most famous one, of course, is the Rogers-Ramanujan continued fraction $R(q)$ defined by

$$R(q) := \frac{q^{1/5}}{1} + \frac{q}{1} + \frac{q^2}{1} + \frac{q^3}{1} + \cdots. \quad (1.8)$$

The continued fraction $R(q)$ has a very beautiful and extensive theory almost all of which was developed by Ramanujan. In particular, his lost notebook [2] contains several results on the Rogers-Ramanujan continued fraction. We refer to the paper by Berndt et al. [6], Kang [7, 8] for proofs of many of these theorems.

In this paper, we examine another continued fraction $T(q)$ of Ramanujan arising from (1.7) and is defined by

$$T(q) := \frac{q}{1-q^2} + \frac{q^4}{1-q^6} + \frac{q^8}{1-q^{10}} + \cdots. \quad (1.9)$$

Note that, replacing q by q^2 and then setting $a = q$ and $b = 0$ in (1.7), we obtain (1.9).

In Section 2, we record some preliminary results. Section 3 is devoted to prove some modular identities for the continued fraction $T(q)$. Finally, in Section 4, we give some explicit evaluations of $T(q)$.

We complete this introduction by defining Ramanujan's modular equation from Berndt's book [3]. The complete elliptic integral of the first kind $K(k)$ is defined by

$$K(k) := \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \frac{\pi}{2} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right), \quad (1.10)$$

where $0 < k < 1$, ${}_2F_1$ denotes the ordinary or Gaussian hypergeometric function. The number k is called the modulus of K , and $k' := \sqrt{1-k^2}$ is called the complementary modulus. Let K, K', L , and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l , and l' , respectively. Suppose that the equality

$$n \frac{K'}{K} = \frac{L'}{L} \quad (1.11)$$

holds for some positive integer n . Then, a modular equation of degree n is a relation between the moduli k and l which is implied by (1.11). If we set

$$q = \exp\left(-\pi \frac{K'}{K}\right), \quad q' = \exp\left(-\pi \frac{L'}{L}\right), \quad (1.12)$$

we see that (1.11) is equivalent to the relation $q^n = q'$. Thus, a modular equation can be viewed as an identity involving theta-functions at the arguments q and q^n . Ramanujan recorded his modular equations in terms of α and β , where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α . The multiplier m connecting α and β is defined by

$$m = \frac{K}{L}, \quad (1.13)$$

where $z_r = \phi^2(q^r)$.

2. Preliminary Results

In this section, we record some results that will be used in the subsequent sections.

Lemma 2.1 (see [3, p. 124, Entry 12(i) and (ii)]). *One has*

$$f(q) = \sqrt{z_1} 2^{-1/6} \{\alpha(1-\alpha)\}^{1/24} q^{-1/24}, \quad f(-q) = \sqrt{z_1} 2^{-1/6} (1-\alpha)^{1/6} \alpha^{1/24} q^{-1/24}. \quad (2.1)$$

Lemma 2.2 (see [3, p. 214, Entry 24(iii)]). *If β has degree 2 over α , then*

$$\begin{aligned} m\sqrt{\alpha-1} + \sqrt{\beta} &= 1, \\ m^2\sqrt{\alpha-1} + \beta &= 1. \end{aligned} \quad (2.2)$$

Lemma 2.3 (see [3, p. 230, Entry 5(ii)]). *If β has degree 3 over α , then*

$$(\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4} = 1. \quad (2.3)$$

Lemma 2.4 (see [3, p. 215, (24.22)]). *If β has degree 4 over α , then*

$$\sqrt{\beta} = \left(\frac{1 - (1-\alpha)^{1/4}}{1 + (1-\alpha)^{1/4}} \right)^2. \quad (2.4)$$

Lemma 2.5 (see [3, p. 280-281, Entry 13(v) and (vi)]). *If β has degree 5 over α , then*

$$\begin{aligned} m &= \frac{1 + \left((1-\beta)^5 / (1-\alpha) \right)^{1/8}}{1 + \left\{ (1-\alpha)^3 (1-\beta) \right\}^{1/8}}, \\ \frac{5}{m} &= \frac{1 - \left((1-\alpha)^5 / (1-\beta) \right)^{1/8}}{1 - \left\{ (1-\alpha)(1-\beta)^3 \right\}^{1/8}}. \end{aligned} \quad (2.5)$$

Lemma 2.6 (see [3, p. 314, Entry 19(i)]). *If β has degree 7 over α , then*

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1. \quad (2.6)$$

3. Modular Identities for $T(q)$

In this section, we use Ramanujan's modular equations to prove certain modular identities for $T(q)$.

Theorem 3.1. *One has*

$$T(q) = \frac{f(q) - f(-q)}{f(q) + f(-q)}. \quad (3.1)$$

Proof. Replacing q by q^2 and the setting $a = q$ and $b = 0$ in (1.7) and simplifying, we obtain

$$\frac{(-q; q^2)_\infty - (q; q^2)_\infty}{(-q; q^2)_\infty + (q; q^2)_\infty} = \frac{q}{1-q^2} + \frac{q^4}{1-q^6} + \frac{q^8}{1-q^{10}} + \cdots. \quad (3.2)$$

Employing (1.6) and (1.9) in (3.2) and simplifying, we complete the proof. $\square$

Corollary 3.2. *One has*

$$\frac{1 + T(q)}{1 - T(q)} = \frac{f(q)}{f(-q)}. \quad (3.3)$$

Proof. Dividing numerator and denominator on right-hand side of the identity in Theorem 3.1 by $f(-q)$ and simplifying, we complete the proof. $\square$

Theorem 3.3. *One has*

$$(i) \alpha = 1 - \left(\frac{1 - T(q)}{1 + T(q)} \right)^8, \quad (ii) \beta = 1 - \left(\frac{1 - T(q^n)}{1 + T(q^n)} \right)^8, \quad (3.4)$$

where β has degree n over α .

Proof. We employ Lemma 2.1 in Corollary 3.2 to complete the proof. $\square$

Theorem 3.4. *Let $u = T(q)$ and $v = T(-q)$. Then,*

$$u + v = 0. \quad (3.5)$$

Proof. Replacing q by $-q$ in Corollary 3.2, we obtain

$$\frac{1 + T(-q)}{1 - T(-q)} = \frac{f(-q)}{f(q)}. \quad (3.6)$$

Now, eliminating $f(q)/f(-q)$ between (3.6) and Corollary 3.2 and simplifying, we complete the proof. $\square$

Theorem 3.5. *Let $u = T(q)$ and $v = T(q^2)$. Then,*

$$u^2 - v - 2u^2v - u^4v + 6u^2v^2 - v^3 - 2u^2v^3 - u^4v^3 + u^2v^4 = 0. \quad (3.7)$$

Proof. Eliminating m in (2.2) and then simplifying, we deduce that

$$\left(1 + \beta + (\beta - 1)\sqrt{1 - \alpha} \right)^2 - 4\beta = 0. \quad (3.8)$$

From Theorem 3.3(i), we have

$$\sqrt{1 - \alpha} = \left(\frac{1 - T(q)}{1 + T(q)} \right)^4. \quad (3.9)$$

Now, employing Theorem 3.3(ii) with $n = 2$ and (3.9) in (3.8) and factorizing using *Mathematica*, we obtain

$$\begin{aligned} & (-1 + v)^8 \left(u^2 - v - 2u^2v - u^4v + 6u^2v^2 - v^3 - 2u^2v^3 - u^4v^3 + u^2v^4 \right) \\ & \times \left(1 + 2u^2 + u^4 - 16u^2v + 6v^2 + 12u^2v^2 + 6u^4v^2 - 16u^2v^3 + v^4 + 2u^2v^4 + u^4v^4 \right) = 0. \end{aligned} \quad (3.10)$$

It can be seen that the first and the last factors in (3.10) do not vanish for $|q| \rightarrow 0$. So, by identity theorem, we have

$$u^2 - v - 2u^2v - u^4v + 6u^2v^2 - v^3 - 2u^2v^3 - u^4v^3 + u^2v^4 = 0. \quad (3.11)$$

□

Theorem 3.6. Let $u = T(q)$ and $v = T(q^3)$. Then,

$$u^3 - v - 3u^2v + 3uv^2 + 3u^3v^2 - 3u^2v^3 - u^4v^3 + uv^4 = 0. \quad (3.12)$$

Proof. From Lemma 2.3, we obtain

$$\alpha\beta - \left(1 - (1 - \alpha)^{1/4}(1 - \beta)^{1/4}\right)^4 = 0. \quad (3.13)$$

From Theorem 3.3, we deduce that

$$\begin{aligned} \alpha &= 1 - \left(\frac{1-u}{1+u}\right)^8, & \beta &= 1 - \left(\frac{1-v}{1+v}\right)^8, \\ (1-\alpha)^{1/4} &= \left(\frac{1-u}{1+u}\right)^2, & (1-\beta)^{1/4} &= \left(\frac{1-v}{1+v}\right)^2, \end{aligned} \quad (3.14)$$

where β has degree 3 over α .

Employing (3.14) in (3.13) and factorizing using *Mathematica*, we arrive at

$$\begin{aligned} &\left(-u^3 + v + 3u^2v - 3uv^2 - 3u^3v^2 + 3u^2v^3 + u^4v^3 - uv^4\right) \\ &\times \left(-u + 3u^2v + u^4v - 3uv^2 - 3u^3v^2 + v^3 + 3u^2v^3 - u^3v^4\right) = 0. \end{aligned} \quad (3.15)$$

It can be seen that the second factor of (3.15) does not vanish for $|q| \rightarrow 0$, so by identity theorem, we have

$$u^3 - v - 3u^2v + 3uv^2 + 3u^3v^2 - 3u^2v^3 - u^4v^3 + uv^4 = 0. \quad (3.16)$$

□

Theorem 3.7. Let $u = T(q)$ and $v = T(q^4)$. Then,

$$\begin{aligned} &u^4 - v - 4u^2v + 2u^4v - 4u^6v - u^8v + 28u^4v^2 - 7v^3 - 28u^2v^3 + 14u^4v^3 - 28u^6v^3 \\ &- 7u^8v^3 + 70u^4v^4 - 7v^5 - 28u^2v^5 + 14u^4v^5 - 28u^6v^5 - 7u^8v^5 + 28u^4v^6 - v^7 - 4u^2v^7 \\ &+ 2u^4v^7 - 4u^6v^7 - u^8v^7 + u^4v^8 = 0. \end{aligned} \quad (3.17)$$

Proof. Squaring the modular equation in Lemma 2.4 and simplifying, we obtain

$$\beta - \left(\frac{1 - (1 - \alpha)^{1/4}}{1 + (1 - \alpha)^{1/4}} \right)^4 = 0. \quad (3.18)$$

From Theorem 3.3(i), we have

$$(1 - \alpha)^{1/4} = \left(\frac{1 - T(q)}{1 + T(q)} \right)^2. \quad (3.19)$$

Now, employing Theorem 3.3(ii) with $n = 4$ and (3.19) in (3.18) and simplifying, we complete the proof. $\square$

Theorem 3.8. Let $u = T(q)$ and $v = T(q^5)$. Then,

$$u^5 - v - 5u^2v + 10u^3v^2 + 5u^5v^2 - 10u^2v^3 - 10u^4v^3 + 5uv^4 + 10u^3v^4 - 5u^4v^5 - u^6v^5 + uv^6 = 0. \quad (3.20)$$

Proof. From Theorem 3.3, we obtain

$$c := (1 - \alpha)^{1/8} = \left(\frac{1 - u}{1 + u} \right), \quad d := (1 - \beta)^{1/8} = \left(\frac{1 - v}{1 + v} \right), \quad (3.21)$$

where β has degree 5 over α .

Employing (3.21) in (2.5), we find that

$$m = \frac{c + d^5}{c(1 + c^3d)}, \quad (3.22)$$

$$\frac{5}{m} = \frac{d - c^5}{d(1 - cd^3)}, \quad (3.23)$$

respectively.

Eliminating m between (3.22) and (3.23) and simplifying, we deduce that

$$5cd(1 + c^3d)(1 - cd^3) - (c + d^5)(d - c^5) = 0. \quad (3.24)$$

Substituting for c and d from (3.21) in (3.24) and simplifying, we arrive at

$$u^5 - v - 5u^2v + 10u^3v^2 + 5u^5v^2 - 10u^2v^3 - 10u^4v^3 + 5uv^4 + 10u^3v^4 - 5u^4v^5 - u^6v^5 + uv^6 = 0. \quad (3.25)$$

$\square$

Theorem 3.9. Let $u = T(q)$ and $v = T(q^7)$. Then,

$$\begin{aligned} u^8 - uv - 7u^3v - 7u^5v + 7u^7v + 28u^6v^2 - 7uv^3 - 49u^3v^3 + 7u^5v^3 - 7u^7v^3 + 70u^4v^4 \\ - 7uv^5 + 7u^3v^5 - 49u^5v^5 - 7u^7v^5 + 28u^2v^6 + 7uv^7 - 7u^3v^7 - 7u^5v^7 - u^7v^7 + v^8 = 0. \end{aligned} \quad (3.26)$$

Proof. From Lemma 2.6, we obtain

$$\alpha\beta - \left(1 - (1 - \alpha)^{1/4}(1 - \beta)^{1/4}\right)^8 = 0. \quad (3.27)$$

Again, from Theorem 3.3, we deduce that

$$\begin{aligned} \alpha &= 1 - \left(\frac{1-u}{1+u}\right)^8, & \beta &= 1 - \left(\frac{1-v}{1+v}\right)^8, \\ (1-\alpha)^{1/8} &= \left(\frac{1-u}{1+u}\right), & (1-\beta)^{1/8} &= \left(\frac{1-v}{1+v}\right), \end{aligned} \quad (3.28)$$

where β has degree 7 over α .

Employing (3.28) in (3.27) and simplifying using *Mathematica*, we arrive at

$$\begin{aligned} u^8 - uv - 7u^3v - 7u^5v + 7u^7v + 28u^6v^2 - 7uv^3 - 49u^3v^3 + 7u^5v^3 - 7u^7v^3 + 70u^4v^4 \\ - 7uv^5 + 7u^3v^5 - 49u^5v^5 - 7u^7v^5 + 28u^2v^6 + 7uv^7 - 7u^3v^7 - 7u^5v^7 - u^7v^7 + v^8 = 0. \end{aligned} \quad (3.29)$$

□

4. Explicit Evaluations of $T(q)$

In this section, we establish some general theorems for the explicit evaluations of the continued fraction $T(q)$ and give examples.

For $q := e^{-\pi\sqrt{n}}$, Ramanujan's two class invariants G_n and g_n are defined by

$$G_n = 2^{-1/4}q^{-1/24}\chi(q), \quad g_n = 2^{-1/4}q^{-1/24}\chi(-q). \quad (4.1)$$

The class invariants G_n and g_n are connected by the relation [4, p. 187, Entry 2.1]:

$$g_{4n} = 2^{1/4}g_nG_n. \quad (4.2)$$

The singular modulus α_n is defined by $\alpha_n := \alpha(e^{-\pi\sqrt{n}})$, where n is a positive integer and unique positive number between 0 and 1 satisfying

$$\sqrt{n} = \frac{{}_2F_1(1/2, 1/2; 1; 1 - \alpha_n)}{{}_2F_1(1/2, 1/2; 1; \alpha_n)}. \quad (4.3)$$

Class invariants and singular moduli are intimately related by the equalities [4, p. 185, (1.6)]:

$$G_n = (4\alpha_n(1 - \alpha_n))^{-1/24}, \quad g_n = \left(4\alpha_n(1 - \alpha_n)^{-2}\right)^{-1/24}. \quad (4.4)$$

An account of Ramanujan's class invariants and singular moduli can be found in Chapter 34 of Berndt's book [4].

Theorem 4.1. *One has*

$$T\left(e^{-\pi\sqrt{n}}\right) = \frac{1 - (1 - \alpha_n)^{1/8}}{1 + (1 - \alpha_n)^{1/8}}. \quad (4.5)$$

Proof. We set $q := e^{-\pi\sqrt{n}}$ in Theorem 3.3(i) and use the definition of singular moduli α_n and simplifying, we complete the proof. $\square$

In the scattered places of his first notebook [1], Ramanujan calculated over 30 singular moduli α_n . See Chapter 34 of Berndt's book [4] for details. Thus, one can use Theorem 4.1 to find the values of $T(e^{-\pi\sqrt{n}})$ if the corresponding values of α_n are known. For example, from [4, p. 281, Theorem 9.2], we note that

$$\alpha_2 = \left(\sqrt{2} - 1\right)^2. \quad (4.6)$$

Employing (4.6) in Theorem 4.1, we calculate

$$T\left(e^{-\pi\sqrt{2}}\right) = \frac{1 - \left(-2 + 2\sqrt{2}\right)^{1/8}}{1 + \left(-2 + 2\sqrt{2}\right)^{1/8}}. \quad (4.7)$$

Many other values of $T(e^{-\pi\sqrt{n}})$ can be computed by using the known values of α_n .

Theorem 4.2. *One has*

$$T\left(e^{-\pi\sqrt{n}}\right) = \left(\frac{g_{4n} - 2^{1/4}g_n^2}{g_{4n} - 2^{1/4}g_n^2}\right). \quad (4.8)$$

Proof. Dividing numerator and denominator of right-hand side of Theorem 3.1 and employing (1.6), we obtain

$$T(q) = \frac{\chi(q) - \chi(-q)}{\chi(q) + \chi(-q)}. \quad (4.9)$$

Setting $q := e^{-\pi\sqrt{n}}$, employing the definitions of G_n and g_n from (4.1) in (4.9) and simplifying, we obtain

$$T(e^{-\pi\sqrt{n}}) = \frac{G_n - g_n}{G_n + g_n}. \quad (4.10)$$

Substituting for G_n from (4.2) in (4.10) and simplifying, we complete the proof. $\square$

Theorem 4.2 implies that if we know the values of g_n and g_{4n} for any positive number n , then corresponding values of $T(e^{-\pi\sqrt{n}})$ can easily be calculated. Saikia [9] evaluated several values of g_n and g_{4n} for positive number n . For example, noting from [9, Theorem 3.5], we have

$$g_3 = 2^{-1/6} (2 + \sqrt{3})^{1/8}, \quad g_{12} = 2^{1/6} (2 + \sqrt{3})^{1/8}. \quad (4.11)$$

Employing (4.11) in Theorem 4.2, we obtain

$$T(e^{-\pi\sqrt{3}}) = \frac{2 - 2^{3/4} (2 + \sqrt{3})^{1/8}}{2 + 2^{3/4} (2 + \sqrt{3})^{1/8}}. \quad (4.12)$$

Many other values of $T(e^{-\pi\sqrt{n}})$ can be determined by using the values of g_n and g_{4n} evaluated in [9].

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