Research Article

Solution of Fuzzy Matrix Equation System

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The main is to develop a method to solve an arbitrary fuzzy matrix equation system by using the embedding approach. Considering the existing solution to $n \times n$ fuzzy matrix equation system is done. To illustrate the proposed model a numerical example is given, and obtained results are discussed.

1. Introduction

The concept of fuzzy numbers and fuzzy arithmetic operations was first introduced by Zadeh [1], Dubois, and Prade [2]. We refer the reader to [3] for more information on fuzzy numbers and fuzzy arithmetic. Fuzzy systems are used to study a variety of problems including fuzzy metric spaces [4], fuzzy differential equations [5], fuzzy linear systems [6–8], and particle physics [9, 10].

One of the major applications of fuzzy number arithmetic is treating fuzzy linear systems [11–20], several problems in various areas such as economics, engineering, and physics boil down to the solution of a linear system of equations. Friedman et al. [21] introduced a general model for solving a fuzzy $n \times n$ linear system whose coefficient matrix is crisp, and the right-hand side column is an arbitrary fuzzy number vector. They used the parametric form of fuzzy numbers and replaced the original fuzzy $n \times n$ linear system by a crisp $2n \times 2n$ linear system and studied duality in fuzzy linear systems $Ax = Bx + y$ where $A$ and $B$ are real $n \times n$ matrix, the unknown vector $x$ is vector consisting of $n$ fuzzy numbers, and the constant $y$ is vector consisting of $n$ fuzzy numbers, in [22]. In [6–8, 23, 24] the authors presented conjugate gradient, LU decomposition method for solving general fuzzy linear systems, or symmetric fuzzy linear systems. Also, Abbasbandy et al. [25] investigated the existence of a minimal solution of general dual fuzzy linear equation system of the form $Ax + f = Bx + c$, where $A$ and $B$ are real $m \times n$ matrices, the unknown vector $x$ is vector consisting of $n$ fuzzy numbers, and the constants $f$ and $c$ are vectors consisting of $m$ fuzzy numbers.
In this paper, we give a new method for solving a $n \times n$ fuzzy matrix equation system whose coefficients matrix is crisp, and the right-hand side matrix is an arbitrary fuzzy number matrix by using the embedding method given in Cong-Xin and Min [26] and replace the original $n \times n$ fuzzy linear system by two $n \times n$ crisp linear systems. It is clear that, in large systems, solving $n \times n$ linear system is better than solving $2n \times 2n$ linear system. Since perturbation analysis is very important in numerical methods. Recently, Ezzati [27] presented the perturbation analysis for $n \times n$ fuzzy linear systems. Now, according to the presented method in this paper, we can investigate perturbation analysis in two crisp matrix equation systems instead of $2n \times 2n$ linear system as the authors of Ezzati [27] and Wang et al. [28].

\section{Preliminaries}

Parametric form of an arbitrary fuzzy number is given in [29] as follows. A fuzzy number $u$ in parametric form is a pair $(\underline{u}, \overline{u})$ of functions $\underline{u}(r), \overline{u}(r), 0 \leq r \leq 1$, which satisfy the following requirements:

1. $\underline{u}(r)$ is a bounded left continuous nondecreasing function over $[0, 1]$,
2. $\overline{u}(r)$ is a bounded left continuous nonincreasing function over $[0, 1]$, and
3. $\underline{u}(r) \leq \overline{u}(r), 0 \leq r \leq 1$.

The set of all these fuzzy numbers is denoted by $E$ which is a complete metric space with Hausdorff distance. A crisp number $\alpha$ is simply represented by $\underline{u}(r) = \overline{u}(r) = \alpha, 0 \leq r \leq 1$.

For arbitrary fuzzy numbers $x = (\underline{x}(r), \overline{x}(r))$, $y = (\underline{y}(r), \overline{y}(r))$, and real number $k$, we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as [29]

1. $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,
2. $x + y = (\underline{x}(r) + \underline{y}(r), \overline{x}(r) + \overline{y}(r))$, and
3. $kx = \begin{cases} (k\underline{x}, k\overline{x}), & k \geq 0, \\ (k\overline{x}, k\underline{x}), & k < 0. \end{cases}$

\textit{Definition 2.1.} The $n \times n$ linear system is as follows:

\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= y_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= y_2, \\
&\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= y_n,
\end{align*}

(2.1)

where the given matrix of coefficients $A = (a_{ij}), 1 \leq i, j \leq n$ is a real $n \times n$ matrix, the given $y_i \in E, 1 \leq i \leq n$, with the unknowns $x_j \in E, 1 \leq j \leq n$ is called a fuzzy linear system (FLS). The operations in (2.1) is described in next section.

Here, a numerical method for finding solution [21] of a fuzzy $n \times n$ linear system is given.
Definition 2.2 (see [21]). A fuzzy number vector \((x_1, x_2, \ldots, x_n)^t\) given by

\[ x_j = \left( \underline{x}_j(r), \bar{x}_j(r) \right); \quad 1 \leq j \leq n, \quad 0 \leq r \leq 1 \]  

is called a solution of the fuzzy linear system (2.1) if

\[
\sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} a_{ij}\underline{x}_j = y_i, \\
\sum_{j=1}^{n} a_{ij}x_j = \sum_{j=1}^{n} a_{ij}\bar{x}_j = \bar{y}_i.
\]  

(2.3)

If, for a particular \(i\), \(a_{ij} > 0\), for all \(j\), we simply get

\[
\sum_{j=1}^{n} a_{ij}x_j = y_i, \quad \sum_{j=1}^{n} a_{ij}\underline{x}_j = \underline{y}_i.
\]  

(2.4)

Finally, we conclude this section by a reviewing on the proposed method for solving fuzzy linear system [21].

The authors [21] wrote the linear system of (2.1) as follows:

\[ SX = Y, \]  

(2.5)

where \(s_{ij}\) are determined as follows:

\[
a_{ij} \geq 0 \Rightarrow s_{ij} = a_{ij}, \quad s_{i+n,j+n} = a_{ij}, \\
a_{ij} < 0 \Rightarrow s_{i,j+n} = -a_{ij}, s_{i+n,j} = -a_{ij},
\]  

(2.6)

and any \(s_{ij}\) which is not determined by (2.1) is zero and

\[
X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ -\underline{x}_1 \\ \vdots \\ -\underline{x}_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ -\bar{y}_1 \\ -\bar{y}_n \end{bmatrix}.
\]  

(2.7)

The structure of \(S\) implies that \(s_{ij} \geq 0, 1 \leq i, j \leq 2n\) and that

\[ S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}, \]  

(2.8)
where \( B \) contains the positive entries of \( A \), and \( C \) contains the absolute values of the negative entries of \( A \), that is, \( A = B - C \).

**Theorem 2.3** (see [21]). The inverse of nonnegative matrix

\[
S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}
\]

is

\[
S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix},
\]

where

\[
D = \frac{1}{2} \left[ (B + C)^{-1} + (B - C)^{-1} \right], \quad E = \frac{1}{2} \left[ (B + C)^{-1} - (B - C)^{-1} \right].
\]

**Corollary 2.4** (see [30]). The solution of (2.5) is obtained by

\[
X = S^{-1}Y.
\]

### 3. Fuzzy Matrix Equation System

A matrix system such as

\[
\begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\begin{pmatrix}
  x_{11} & x_{12} & \cdots & x_{1n} \\
  x_{21} & x_{22} & \cdots & x_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_{n1} & x_{n2} & \cdots & x_{nn}
\end{pmatrix}
= \begin{pmatrix}
  y_{11} & y_{12} & \cdots & y_{1n} \\
  y_{21} & y_{22} & \cdots & y_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  y_{n1} & y_{n2} & \cdots & y_{nn}
\end{pmatrix},
\]

where \( a_{ij}, 1 \leq i, j \leq n \), are real numbers, the elements \( y_{ij} \) in the right-hand matrix are fuzzy numbers, and the unknown elements \( x_{ij} \) are ones, is called a fuzzy matrix equation system (FMES).

Using matrix notation, we have

\[
AX = Y.
\]

A fuzzy number matrix

\[
X = (x_1, \ldots, x_i, \ldots, x_n)
\]

is called a solution of the fuzzy matrix system (2.1) if

\[
Ax_j = y_j, \quad 1 \leq j \leq n.
\]
In this section, we propose a new method for solving FMES.

**Theorem 3.1.** Suppose that the inverse of matrix \( A \) exists and \( x_j = (x_{j1}, x_{j2}, \ldots, x_{jn})^T \) is a solution of this equation. Then \( x_j + \overline{x_j} = (x_{j1} + \overline{x_{j1}}, x_{j2} + \overline{x_{j2}}, \ldots, x_{jn} + \overline{x_{jn}})^T \) is the solution of the following systems:

\[
A \left( x_j + \overline{x_j} \right) = y_j + \overline{y_j}, \quad j = 1, 2, \ldots, n, \tag{3.5}
\]

where \( y_j + \overline{y_j} = (y_{j1} + \overline{y_{j1}}, y_{j2} + \overline{y_{j2}}, \ldots, y_{jn} + \overline{y_{jn}})^T, \quad j = 1, 2, \ldots, n. \)

**Proof.** It is the same as the proof of Theorem 3 in [27].

For solving (3.2), we first solve the following system:

\[
a_{11} (x_{j1} + \overline{x_{j1}}) + \cdots + a_{1n} (x_{jn} + \overline{x_{jn}}) = (y_{j1} + \overline{y_{j1}}),
\]

\[
a_{21} (x_{j1} + \overline{x_{j1}}) + \cdots + a_{2n} (x_{jn} + \overline{x_{jn}}) = (y_{j2} + \overline{y_{j2}}),
\]

\[
\vdots
\]

\[
a_{n1} (x_{j1} + \overline{x_{j1}}) + \cdots + a_{nn} (x_{jn} + \overline{x_{jn}}) = (y_{jn} + \overline{y_{jn}}),
\]

\[
j = 1, 2, \ldots, n. \tag{3.6}
\]

Using matrix notation, we have

\[
A \left( X + \overline{X} \right) = \left( Y + \overline{Y} \right). \tag{3.7}
\]

Suppose that the solution of (3.7) is as

\[
d_j = \begin{bmatrix} d_{j1} \\ d_{j2} \\ \vdots \\ d_{jn} \end{bmatrix} = x_j + \overline{x_j} = \begin{bmatrix} x_{j1} + \overline{x_{j1}} \\ x_{j2} + \overline{x_{j2}} \\ \vdots \\ x_{jn} + \overline{x_{jn}} \end{bmatrix}, \quad j = 1, 2, \ldots, n. \tag{3.8}
\]

Let matrices \( B \) and \( C \) have defined as Section 2. Now using matrix notation for (3.7), we get in parametric form \( (B - C) (X(r) + \overline{X}(r)) = (Y(r) + \overline{Y}(r)) \). We can write this system as follows:

\[
B X(r) - C \overline{X}(r) = Y(r), \tag{3.9}
\]

\[
B \overline{X}(r) - C X(r) = \overline{Y}(r).
\]
By substituting \( \overline{X}(r) = D - \overline{X}(r) \) and \( \overline{X}(r) = D - \overline{X}(r) \) in the first and second equation of above system, respectively, we have

\[
\begin{align*}
(B + C)\overline{X}(r) &= \overline{Y}(r) + CD, \quad (3.10) \\
(B + C)\overline{X}(r) &= \overline{Y}(r) + CD, \quad (3.11)
\end{align*}
\]

therefore, we have

\[
\begin{align*}
\overline{X}(r) &= (B + C)^{-1}(\overline{Y}(r) + CD), \\
\overline{X}(r) &= (B + C)^{-1}(\overline{Y}(r) + CD).
\end{align*}
\]

Therefore, we can solve fuzzy matrix equation system (3.2) by solving (3.7)–(3.10).

**Theorem 3.2.** Let in (3.3) \( j = 1 \), also \( g \) and \( G \) are the number of multiplication operations that are required to calculate

\[
X = (x_1, x_2, \ldots, x_n, -\overline{x}_1, -\overline{x}_2, \ldots, -\overline{x}_n)^T = S^{-1}Y,
\]

(\textit{the proposed method in Friedman et al. [21]}) and

\[
x_j = (x_{j1}, x_{j2}, \ldots, x_{jn}, \overline{x}_{j1}, \overline{x}_{j2}, \ldots, \overline{x}_{jn})^T,
\]

from (3.7)–(3.10), respectively. Then \( G \leq g \) and \( g - G = n^2 \).

**Proof.** According to Section 2, we have

\[
S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix},
\]

where

\[
D = \frac{1}{2}(B + C)^{-1} + (B - C)^{-1}, \quad E = \frac{1}{2}(B + C)^{-1} - (B - C)^{-1}.
\]

Therefore, for determining \( S^{-1} \), we need to compute \( (B + C)^{-1} \) and \( (B - C)^{-1} \). Now, assume that \( M \) is \( n \times n \) matrix and denote by \( h(M) \) the number of multiplication operations that are required to calculate \( M^{-1} \). It is clear that

\[
h(S) = h(B + C) + h(B - C) = 2h(A),
\]

and hence

\[
g = 2h(A) + 4n^2.
\]
For computing $x_j + \overline{x}_j = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)$ from (3.7) and $x_j = (x_{j1}, x_{j2}, \ldots, x_{jn})^T$ from (3.10) the number of multiplication operations is $h(A) + n^2$ and $h(B + C) + 2n^2$, respectively. Clearly $h(B + C) = h(A)$, so

$$G = 2h(A) + 3n^2,$$

(3.19)

and hence $g - G = n^2$. This proves theorem.

\begin{remark}
In (3.3) if $j = 1$, then this paper is similar to [27].
\end{remark}

\begin{example}
Consider the $2 \times 2$ fuzzy matrix equation system as follows:

$$\begin{pmatrix}
2 & -1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{pmatrix}
= \begin{pmatrix}
(3r - 3, 3 - 3r) & (4r - 4, 6 - 6r) \\
(2r + 1, 5 - 2r) & (3r, 7 - 4r)
\end{pmatrix}.
$$

(3.20)

By using (3.7) and (3.10), we have

$$\begin{pmatrix}
x_{11}(r) + \overline{x}_{11}(r) & x_{12}(r) + \overline{x}_{12}(r) \\
x_{21}(r) + \overline{x}_{21}(r) & x_{22}(r) + \overline{x}_{22}(r)
\end{pmatrix}
= \begin{pmatrix}
2 & 3 - r \\
4 & 4
\end{pmatrix},$$

(3.21)

$$\begin{pmatrix}
x_{11}(r) & x_{12}(r) \\
x_{21}(r) & x_{22}(r)
\end{pmatrix}
= \begin{pmatrix}
r & r \\
1 + r & 2r
\end{pmatrix},$$

and hence

$$\begin{pmatrix}
\overline{x}_{11}(r) & \overline{x}_{12}(r) \\
\overline{x}_{21}(r) & \overline{x}_{22}(r)
\end{pmatrix}
= \begin{pmatrix}
2 - r & 3 - 2r \\
3 - r & 4 - 2r
\end{pmatrix}.$$

(3.22)

Obviously, $x_{11}, x_{12}, x_{21}$ and $x_{22}$, are fuzzy numbers.

\section{4. Conclusions}

In this paper, we propose a general model for solving fuzzy matrix equation system. The original system with matrix coefficient $A$ is replaced by two $n \times n$ crisp matrix equation systems.

\section{References}


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