Research Article

Generalizations of the Simpson-Like Type Inequalities for Co-Ordinated s -Convex Mappings in the Second Sense

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A generalized identity for some partial differentiable mappings on a bidimensional interval is obtained, and, by using this result, the author establishes generalizations of Simpson-like type inequalities for coordinated s -convex mappings in the second sense.

1. Introduction

In recent years, a number of authors have considered error estimate inequalities for some known and some new quadrature formulas. Sometimes they have considered generalizations of the Simpson-like type inequality which gives an error bound for the well-known Simpson rule.

Theorem 1.1. *Let $f : \mathbb{I} \subset [0, \infty) \rightarrow \mathbb{R}$ be a four-time continuous differentiable mapping on $[a, b]$ and $\|f^{(4)}\|_{\infty} = \sup_{x \in [a, b]} |f^{(4)}(x)| < \infty$. Then, the following inequality holds:*

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^4}{2880} \|f^{(4)}\|_{\infty}. \quad (1.1)$$

It is well known that the mapping f is neither four times differentiable nor is the fourth derivative $f^{(4)}$ bounded on (a, b) , then we cannot apply the classical Simpson quadrature formula.

For recent results on Simpson type inequalities, you may see the papers [1–5].

In [2, 6–8], Dragomir et al. and Park considered among others the class of mappings which are s -convex on the coordinates.

In the sequel, in this paper let $\Delta = [a, b] \times [c, d]$ be a bidimensional interval in $\mathbb{R}^2$ with $a < b$ and $c < d$.

Definition 1.2. A mapping $f : \Delta \rightarrow \mathbb{R}$ will be called s -convex in the second sense on Δ if the following inequality:

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda^s f(x, y) + (1 - \lambda)^s f(z, w), \quad (1.2)$$

holds, for all $(x, y), (z, w) \in \Delta$, $\lambda \in [0, 1]$ and $s \in [0, 1]$.

Modification for convex and s -convex mapping on Δ , which are also known as co-ordinated convex, s -convex mapping, and s - r -convex, respectively, were introduced by Dragomir, Sarikaya [5, 9, 10], and Park [4, 8, 11, 12].

Definition 1.3. A mapping $f : \Delta \rightarrow \mathbb{R}$ will be called coordinated s -convex in the second sense on Δ if the partial mappings

$$\begin{aligned} f_y : [a, b] &\longrightarrow \mathbb{R}, & f_y(u) &= f(u, y), \\ f_x : [c, d] &\longrightarrow \mathbb{R}, & f_x(v) &= f(x, v), \end{aligned} \quad (1.3)$$

are s -convex in the second sense, for all $x \in [a, b]$, $y \in [c, d]$, and $s \in [0, 1]$ [5, 9, 10].

A formal definition for coordinated s -convex mappings may be stated as follow [8].

Definition 1.4. A mapping $f : \Delta \rightarrow \mathbb{R}$ will be called coordinated s -convex in the second sense on Δ if the following inequality:

$$\begin{aligned} &f(tx + (1 - t)z, \lambda y + (1 - \lambda)w) \\ &\leq t^s \lambda^s f(x, y) + (1 - t)^s \lambda^s f(z, y) + t^s (1 - \lambda)^s f(x, w) + (1 - t)^s (1 - \lambda)^s f(z, w), \end{aligned} \quad (1.4)$$

holds, for all $t, \lambda \in [0, 1]$, $(x, y), (z, w) \in \Delta$, and $s \in [0, 1]$.

In [2], S.S. Dragomir established the following theorem.

Theorem 1.5. Let $f : \Delta \rightarrow \mathbb{R}$ be convex on the coordinates on Δ . Then, one has the inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ &\leq \frac{1}{4} [f(a, c) + f(b, c) + f(a, d) + f(b, d)]. \end{aligned} \quad (1.5)$$

In [13], Hwang et al. gave a refinement of Hadamard's inequality on the coordinates and they proved some inequalities for coordinated convex mappings.

In [1, 6, 14], Alomari and Darus proved inequalities for coordinated s -convex mappings.

In [15], Latif and Alomari defined coordinated h -convex mappings, established some inequalities for co-ordinated h -convex mappings and proved inequalities involving product of convex mappings on the coordinates.

In [3], Özdemir et al. gave the following theorems:

Theorem 1.6. Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on Δ . If $\partial^2 f / \partial t \partial \lambda$ is convex on the coordinates on Δ , then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{9} \left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \right. \\ & \quad \left. \left. + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) \right] + \frac{1}{36} \{f(a, c) + f(a, d) + f(b, c) + f(b, d)\} \right. \\ & \quad \left. + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx - A \right| \\ & \leq \left(\frac{5}{72}\right)^2 (b-a)(d-c) \\ & \quad \times \left\{ \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right| \right\}, \end{aligned} \quad (1.6)$$

where

$$\begin{aligned} A = & \frac{1}{b-a} \int_a^b \left\{ \frac{f(x, c) + 4f(x, (c+d)/2) + f(x, d)}{6} \right\} dx \\ & + \frac{1}{d-c} \int_c^d \left\{ \frac{f(a, y) + 4f((a+b)/2, y) + f(b, y)}{6} \right\} dy. \end{aligned} \quad (1.7)$$

Theorem 1.7. Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on Δ . If $\partial^2 f / \partial t \partial \lambda$ is bounded, that is,

$$\begin{aligned} & \left\| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right\|_{\infty} \\ & = \sup_{(x, y) \in (a, b) \times (c, d)} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right| < \infty \end{aligned} \quad (1.8)$$

for all $(t, \lambda) \in [0, 1]^2$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{1}{9} \left[f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) \right] \right. \\ & \quad \left. + \frac{1}{36} \{f(a, c) + f(a, d) + f(b, c) + f(b, d)\} \right| \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx - A \Bigg| \\
& \leq \left(\frac{5}{72} \right)^2 (b-a)(d-c) \left\| \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) \right\|_{\infty},
\end{aligned} \tag{1.9}$$

where A is defined in Theorem 1.6.

In [3], Özdemir et al. proved a new equality and, by using this equality, established some inequalities on coordinated convex mappings.

In this paper the author give a generalized identity for some partial differentiable mappings on a bidimensional interval and, by using this result, establish a generalizations of Simpson-like type inequalities for coordinated s -convex mappings in the second sense.

2. Main Results

To prove our main results, we need the following lemma.

Lemma 2.1. Let $f : \Delta \rightarrow \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$. If $(\partial^2 f / \partial t \partial \lambda) \in L_1(\Delta)$, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0, 1)$ with $(1/r_1) \leq h_1 \leq (r_1 - 1/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2)$, the following equality holds:

$$\begin{aligned}
I(f)(h_1, h_2, r_1, r_2) &= \left\{ \frac{(r_1 - 2)(r_2 - 2)}{r_1 r_2} \right\} f(h_1 a + (1 - h_1)b, h_2 c + (1 - h_2)d) \\
&+ \left\{ \frac{(r_1 - 2)}{r_1 r_2} \right\} \{ f(h_1 a + (1 - h_1)b, c) + f(h_1 a + (1 - h_1)b, d) \} \\
&+ \left\{ \frac{(r_2 - 2)}{r_1 r_2} \right\} \{ f(a, h_2 c + (1 - h_2)d) + f(b, h_2 c + (1 - h_2)d) \} \\
&+ \frac{1}{r_1 r_2} \{ f(a, c) + f(a, d) + f(b, c) + f(b, d) \} \\
&- \frac{1}{r_2(b-a)} \int_a^b \{ f(x, c) + (r_2 - 2)f(x, h_2 c + (1 - h_2)d) + f(x, d) \} dx \\
&- \frac{1}{r_1(d-c)} \int_c^d \{ f(a, y) + (r_1 - 2)f(h_1 a + (1 - h_1)b, y) + f(b, y) \} dy \\
&+ \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
&= (b-a)(d-c) \int_0^1 \int_0^1 p(h_1, r_1, t) q(h_2, r_2, \lambda) \\
&\quad \times \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt d\lambda,
\end{aligned} \tag{2.1}$$

where

$$p(h_1, r_1, t) = \begin{cases} t - \frac{1}{r_1}, & t \in [0, h_1], \\ t - \frac{r_1 - 1}{r_1}, & t \in (h_1, 1], \end{cases}$$

$$q(h_2, r_2, \lambda) = \begin{cases} \lambda - \frac{1}{r_2}, & \lambda \in [0, h_2], \\ \lambda - \frac{r_2 - 1}{r_2}, & \lambda \in (h_2, 1]. \end{cases} \quad (2.2)$$

Proof. By the definitions of $p(h_1, r_1, t)$ and $q(h_2, r_2, \lambda)$, we can write

$$\begin{aligned} I &= \int_0^1 \int_0^1 p(h_1, r_1, t) q(h_2, r_2, \lambda) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt d\lambda \\ &= \int_0^1 q(h_2, r_2, \lambda) \left[\int_0^1 p(h_1, r_1, t) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt \right] d\lambda \\ &= \int_0^1 q(h_2, r_2, \lambda) \left[\int_0^{h_1} \left(t - \frac{1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt \right. \\ &\quad \left. + \int_{h_1}^1 \left(t - \frac{r_1 - 1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt \right] d\lambda \\ &= \int_0^1 q(h_2, r_2, \lambda) [I_{11} + I_{12}] d\lambda, \end{aligned} \quad (2.3)$$

where

$$\begin{aligned} I_{11} &= \int_0^{h_1} \left(t - \frac{1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt, \\ I_{12} &= \int_{h_1}^1 \left(t - \frac{r_1 - 1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt. \end{aligned} \quad (2.4)$$

By integration by parts, we have

$$\begin{aligned} I_{11} &= \int_0^{h_1} \left(t - \frac{1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt \\ &= \frac{1}{a-b} \left[\left(h_1 - \frac{1}{r_1} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1-h_1)b, \lambda c + (1-\lambda)d) \right. \\ &\quad \left. + \frac{1}{r_1} \frac{\partial f}{\partial \lambda} (b, \lambda c + (1-\lambda)d) \right. \\ &\quad \left. - \int_0^{h_1} \frac{\partial f}{\partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt \right], \end{aligned} \quad (2.5)$$

$$\begin{aligned}
I_{12} &= \int_{h_1}^1 \left(t - \frac{r_1 - 1}{r_1} \right) \frac{\partial^2 f}{\partial t \partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt \\
&= \frac{1}{a-b} \left[\left(\frac{1}{r_1} \right) \frac{\partial f}{\partial \lambda} (a, \lambda c + (1-\lambda)d) \right. \\
&\quad \left. - \left(h_1 - \frac{r_1 - 1}{r_1} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1-h_1)b, \lambda c + (1-\lambda)d) \right. \\
&\quad \left. - \int_{h_1}^1 \frac{\partial f}{\partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) dt \right].
\end{aligned} \tag{2.6}$$

By using the equalities (2.5) and (2.6) in (2.3), we have

$$\begin{aligned}
I &= \left(\frac{1}{a-b} \right) \left\{ \left(\frac{r_1 - 2}{r_1} \right) \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (h_1 a + (1-h_1)b, \lambda c + (1-\lambda)d) d\lambda \right. \\
&\quad + \left(\frac{1}{r_1} \right) \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (b, \lambda c + (1-\lambda)d) d\lambda \\
&\quad + \left(\frac{1}{r_1} \right) \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (a, \lambda c + (1-\lambda)d) d\lambda \\
&\quad \left. - \int_0^1 q(h_2, r_2, \lambda) \int_0^1 \frac{\partial f}{\partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) d\lambda dt \right\} \\
&= \left(\frac{1}{a-b} \right) \left\{ \left(\frac{r_1 - 2}{r_1} \right) I_{21} + \left(\frac{1}{r_1} \right) I_{22} + \left(\frac{1}{r_1} \right) I_{23} - I_{24} \right\},
\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
I_{21} &= \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (h_1 a + (1-h_1)b, \lambda c + (1-\lambda)d) d\lambda, \\
I_{22} &= \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (b, \lambda c + (1-\lambda)d) d\lambda, \\
I_{23} &= \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (a, \lambda c + (1-\lambda)d) d\lambda, \\
I_{24} &= \int_0^1 q(h_2, r_2, \lambda) \int_0^1 \frac{\partial f}{\partial \lambda} (ta + (1-t)b, \lambda c + (1-\lambda)d) d\lambda dt.
\end{aligned} \tag{2.8}$$

Note that

$$\begin{aligned}
\text{(i)} \quad &\int_0^{h_2} \left(\lambda - \frac{1}{r_2} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1-h_1)b, \lambda c + (1-\lambda)d) d\lambda \\
&= \frac{1}{c-d} \left\{ \left(h_2 - \frac{1}{r_2} \right) f(h_1 a + (1-h_1)b, h_2 c + (1-h_2)d) \right. \\
&\quad + \frac{1}{r_2} f(h_1 a + (1-h_1)b, d) \\
&\quad \left. - \int_0^{h_2} f(h_1 a + (1-h_1)b, \lambda c + (1-\lambda)d) d\lambda \right\},
\end{aligned} \tag{2.9}$$

$$\begin{aligned}
(ii) \int_{h_2}^1 \left(\lambda - \frac{r_2 - 1}{r_2} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \\
= \frac{1}{c - d} \left\{ \left(\frac{1}{r_2} \right) f(h_1 a + (1 - h_1) b, c) \right. \\
\quad - \left(h_2 - \frac{r_2 - 1}{r_2} \right) f(h_1 a + (1 - h_1) b, h_2 c + (1 - h_2) d) \\
\quad \left. - \int_{h_2}^1 f(h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \right\}.
\end{aligned} \tag{2.10}$$

By the equalities (2.9) and (2.10), we have

$$\begin{aligned}
I_{21} &= \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \\
&= \int_0^{h_2} \left(\lambda - \frac{1}{r_2} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \\
&\quad + \int_{h_2}^1 \left(\lambda - \frac{r_2 - 1}{r_2} \right) \frac{\partial f}{\partial \lambda} (h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \\
&= \frac{1}{c - d} \left\{ \frac{1}{r_2} f(h_1 a + (1 - h_1) b, c) + \frac{1}{r_2} f(h_1 a + (1 - h_1) b, d) \right. \\
&\quad + \frac{r_2 - 2}{r_2} f(h_1 a + (1 - h_1) b, h_2 c + (1 - h_2) d) \\
&\quad \left. - \int_0^1 f(h_1 a + (1 - h_1) b, \lambda c + (1 - \lambda) d) d\lambda \right\}.
\end{aligned} \tag{2.11}$$

By the similar way, we get the following:

$$\begin{aligned}
I_{22} &= \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (b, \lambda c + (1 - \lambda) d) d\lambda \\
&= \frac{1}{c - d} \left\{ \frac{1}{r_2} f(b, c) + \frac{1}{r_2} f(b, d) + \left(\frac{r_2 - 2}{r_2} \right) f(b, h_2 c + (1 - h_2) d) \right. \\
&\quad \left. - \int_0^1 f(b, \lambda c + (1 - \lambda) d) d\lambda \right\},
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
I_{23} &= \int_0^1 q(h_2, r_2, \lambda) \frac{\partial f}{\partial \lambda} (a, \lambda c + (1 - \lambda) d) d\lambda \\
&= \frac{1}{c - d} \left\{ \frac{1}{r_2} f(a, c) + \frac{1}{r_2} f(a, d) + \left(\frac{r_2 - 2}{r_2} \right) f(a, h_2 c + (1 - h_2) d) \right. \\
&\quad \left. - \int_0^1 f(a, \lambda c + (1 - \lambda) d) d\lambda \right\},
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
I_{24} &= \int_0^1 q(h_2, r_2, \lambda) \int_0^1 \frac{\partial f}{\partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) d\lambda dt \\
&= \frac{1}{c-d} \left\{ \frac{1}{r_2} \int_0^1 f(ta + (1-t)b, c) dt + \frac{1}{r_2} \int_0^1 f(ta + (1-t)b, d) dt \right. \\
&\quad + \left(\frac{r_2 - 2}{r_2} \right) \int_0^1 f(ta + (1-t)b, h_2 c + (1-h_2)d) dt \\
&\quad \left. - \iint_0^1 f(ta + (1-t)b, \lambda c + (1-\lambda)d) d\lambda dt \right\}.
\end{aligned} \tag{2.14}$$

By the equalities (2.7) and (2.11)–(2.14) and using the change of the variables $x = ta + (1-t)b$ and $y = \lambda c + (1-\lambda)d$ for $(t, \lambda) \in [0, 1]^2$, then multiplying both sides with $(b-a)(d-c)$, we have the required result (2.1), which completes the proof. $\square$

Remark 2.2. Lemma 2.1 is a generalization of the results which proved by Sarikaya, Set, Özdemir, and Dragomir [3, 5, 9, 10].

Theorem 2.3. Let $f : \Delta \rightarrow \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$. If $\partial^2 f / \partial t \partial \lambda$ is in $L_1(\Delta)$ and is a coordinated s -convex mapping in the second sense on Δ , then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0, 1)$ with $(1/r_1) \leq h_1 \leq ((r_1 - 1)/r_1)$, and $(1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2)$ the following inequality holds:

$$\begin{aligned}
&\frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| \\
&\leq \mu_1(r, s) \left\{ \mu_2(r, s) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right| + \nu_2(r, s) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right| \right\} \\
&\quad + \nu_1(r, s) \left\{ \mu_2(r, s) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right| + \nu_2(r, s) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right| \right\},
\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}
\mu_1(r, s) &= M(r_1, s) + N(h_1, s), \\
\mu_2(r, s) &= M(r_2, s) + N(h_2, s), \\
\nu_1(r, s) &= M(r_1, s) + N(1-h_1, s), \\
\nu_2(r, s) &= M(r_2, s) - N(1-h_2, s)
\end{aligned} \tag{2.16}$$

for

$$\begin{aligned}
M(r, s) &= \frac{2 + 2(r-1)^{s+2} + r^{s+1}(s-r+2)}{(s+1)(s+2)r^{s+2}}, \\
N(h, s) &= \frac{h^{s+1}((2h-1)s + 2(h-1))}{(s+1)(s+2)}.
\end{aligned} \tag{2.17}$$

Proof. From Lemma 2.1 and by the coordinated s -convexity in the second sense of $\partial^2 f / \partial t \partial \lambda$, we can write

$$\begin{aligned}
 & \frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| \\
 & \leq \int_0^1 |p(h_1, r_1, t) q(h_2, r_2, \lambda)| \\
 & \quad \times \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right| dt d\lambda \\
 & \leq \int_0^1 |p(h_1, r_1, t) q(h_2, r_2, \lambda)| \\
 & \quad \times \left\{ t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right| + t^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right| \right. \\
 & \quad \left. + (1-t)^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right| + (1-t)^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right| \right\} dt d\lambda \quad (2.18) \\
 & = \left\{ \int_0^1 |p(h_1, r_1, t)| t^s dt \right\} \left\{ \int_0^1 |q(h_2, r_2, \lambda)| \lambda^s d\lambda \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right| \right. \\
 & \quad \left. + \int_0^1 |q(h_2, r_2, \lambda)| (1-\lambda)^s d\lambda \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right| \right\} \\
 & \quad + \left\{ \int_0^1 |p(h_1, r_1, t)| (1-t)^s dt \right\} \left\{ \int_0^1 |q(h_2, r_2, \lambda)| \lambda^s d\lambda \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right| \right. \\
 & \quad \left. + \int_0^1 |q(h_2, r_2, \lambda)| (1-\lambda)^s d\lambda \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right| \right\}.
 \end{aligned}$$

Note that

$$\begin{aligned}
 & \text{(i)} \int_0^1 |p(h_1, r_1, t)| t^s dt = \mu_1(r, s), \\
 & \text{(ii)} \int_0^1 |p(h_1, r_1, t)| (1-t)^s dt = \nu_1(r, s), \\
 & \text{(iii)} \int_0^1 |q(h_2, r_2, \lambda)| \lambda^s d\lambda = \mu_2(r, s), \\
 & \text{(iv)} \int_0^1 |q(h_2, r_2, \lambda)| (1-\lambda)^s d\lambda = \nu_2(r, s). \quad (2.19)
 \end{aligned}$$

By (2.18) and (2.19), we get the inequality (2.15) by the simple calculations. $\square$

Remark 2.4. In Theorem 2.3,

(i) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 6$, and $s = 1$ in (2.15), then we get

$$\left| I(f) \left(\frac{1}{2}, \frac{1}{2}, 6, 6 \right) \right| \leq \left(\frac{5}{72} \right)^2 M(b-a)(d-c), \quad (2.20)$$

(ii) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 2$, and $s = 1$ in (2.15), then we get

$$\left| I(f) \left(\frac{1}{2}, \frac{1}{2}, 2, 2 \right) \right| \leq \left(\frac{1}{8} \right)^2 M(b-a)(d-c), \quad (2.21)$$

where

$$M = \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right| + \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|, \quad (2.22)$$

which implies that Theorem 2.3 is a generalization of Theorem 1.6.

Theorem 2.5. Let $f : \Delta \rightarrow \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$. If $\partial^2 f / \partial t \partial \lambda$ is bounded, that is,

$$\begin{aligned} \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_{\infty} &= \left\| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right\|_{\infty} \\ &= \sup_{(x,y) \in (a,b) \times (c,d)} \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right| < \infty \end{aligned} \quad (2.23)$$

for all $(t, \lambda) \in [0, 1]^2$, then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0, 1)$ with $(1/r_1) \leq h_1 \leq ((r_1 - 1)/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2)$, the following inequality holds:

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| \\ &\leq \left\{ \frac{1}{2} - h_1 + h_1^2 + \frac{(2-r_1)}{r_1^2} \right\} \left\{ \frac{1}{2} - h_2 + h_2^2 + \frac{(2-r_2)}{r_2^2} \right\} \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_{\infty}. \end{aligned} \quad (2.24)$$

Proof. From Lemma 2.1, using the property of modulus and the boundedness of $\partial^2 f / \partial t \partial s$, we get

$$\begin{aligned} &\frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| \\ &\leq \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_{\infty} \int_0^1 \int_0^1 |p(h_1, r_1, t)q(h_2, r_2, \lambda)| dt d\lambda. \end{aligned} \quad (2.25)$$

By the simple calculations, we have

$$(i) \int_0^1 |p(h_1, r_1, t)| dt = \frac{1}{2} - h_1 + h_1^2 + \frac{(2-r_1)}{r_1^2}, \quad (2.26)$$

$$(ii) \int_0^1 |q(h_2, r_2, \lambda)| d\lambda = \frac{1}{2} - h_2 + h_2^2 + \frac{(2-r_2)}{r_2^2}. \quad (2.27)$$

By using the inequality (2.25) and the equalities (2.26)-(2.27), the assertion (2.24) holds. $\square$

Remark 2.6. In Theorem 2.5,

(i) if we choose $h_1 = h_2 = \frac{1}{2}$ and $r_1 = r_2 = 6$, then we get

$$\left| I(f) \left(\frac{1}{2}, \frac{1}{2}, 6, 6 \right) \right| \leq \left(\frac{5}{36} \right)^2 \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_{\infty} (b-a)(d-c), \quad (2.28)$$

(ii) if we choose $h_1 = h_2 = 1/2$ and $r_1 = r_2 = 2$, then we get

$$\left| I(f) \left(\frac{1}{2}, \frac{1}{2}, 2, 2 \right) \right| \leq \left(\frac{1}{4} \right)^2 \left\| \frac{\partial^2 f}{\partial t \partial \lambda} \right\|_{\infty} (b-a)(d-c), \quad (2.29)$$

which implies that Theorem 2.5 is a generalization of Theorem 1.7.

The following theorem is a generalization of Theorem 1.6.

Theorem 2.7. Let $f : \Delta = [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d]$. If $|\partial^2 f / \partial t \partial \lambda|^q (q > 1)$ is in $L_1(\Delta)$ and is a coordinated s -convex mapping in the second sense on Δ , then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0, 1)$ with $(1/r_1) \leq h_1 \leq ((r_1 - 1)/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2)$, the following inequality holds:

$$\begin{aligned} \frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| &\leq \mu_3^{1/p} \nu_3^{1/p} \\ &\times \left\{ \frac{|\partial^2 f / \partial t \partial \lambda(a, c)|^q + |\partial^2 f / \partial t \partial \lambda(a, d)|^q + |\partial^2 f / \partial t \partial \lambda(b, c)|^q + |\partial^2 f / \partial t \partial \lambda(b, d)|^q}{(s+1)^2} \right\}^{1/p}, \end{aligned} \quad (2.30)$$

where

$$\begin{aligned}\mu_3 &= \frac{2 + (r_1 - r_1 h_1 - 1)^{p+1} + (r_1 h_1 - 1)^{p+1}}{r_1^{p+1}(p+1)}, \\ \nu_3 &= \frac{2 + (r_2 - r_2 h_2 - 1)^{p+1} + (r_2 h_2 - 1)^{p+1}}{r_2^{p+1}(p+1)}.\end{aligned}\quad (2.31)$$

Proof. From Lemma 2.1, we can write

$$\begin{aligned}& \frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| \\& \leq \iint_0^1 |p(h_1, r_1, t)q(h_2, r_2, \lambda)| \\& \quad \times \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right| dt d\lambda \\& \leq \left\{ \iint_0^1 |p(h_1, r_1, t)q(h_2, r_2, \lambda)|^p dt d\lambda \right\}^{1/p} \\& \quad \times \left\{ \iint_0^1 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^q dt d\lambda \right\}^{1/q}.\end{aligned}\quad (2.32)$$

Hence, by the inequality (2.32) and the coordinated s -convexity in the second sense of $|\partial^2 f / \partial t \partial \lambda|^q$, it follows that

$$\begin{aligned}& \frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| \\& \leq \left\{ \iint_0^1 |p(h_1, r_1, t)q(h_2, r_2, \lambda)|^p dt d\lambda \right\}^{1/p} \\& \quad \times \left\{ \frac{|\partial^2 f / \partial t \partial \lambda(a, c)|^q + |\partial^2 f / \partial t \partial \lambda(a, d)|^q + |\partial^2 f / \partial t \partial \lambda(b, c)|^q + |\partial^2 f / \partial t \partial \lambda(b, d)|^q}{(s+1)^2} \right\}^{1/q}.\end{aligned}\quad (2.33)$$

Note that

$$(i) \int_0^1 |p(h_1, r_1, t)|^p dt = \frac{2 + (r_1 - r_1 h_1 - 1)^{p+1} + (r_1 h_1 - 1)^{p+1}}{r_1^{p+1}(p+1)}, \quad (2.34)$$

$$(ii) \int_0^1 |q(h_2, r_2, \lambda)|^p d\lambda = \frac{2 + (r_2 - r_2 h_2 - 1)^{p+1} + (r_2 h_2 - 1)^{p+1}}{r_2^{p+1}(p+1)}. \quad (2.35)$$

By the inequality (2.33) and the equalities (2.34) and (2.35), the assertion (2.30) holds. $\square$

Remark 2.8. In Theorem 2.7,

(i) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 6$, and $s = 1$, then we get

$$\left| I(f) \left(\frac{1}{2}, \frac{1}{2}, 6, 6 \right) \right| \leq \left\{ \frac{2(1+2^{p+1})}{6^{p+1}(p+1)} \right\}^{2/p} (b-a)(d-c) M_q^{1/q}, \quad (2.36)$$

(ii) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 2$, and $s = 1$, then we get

$$\left| I(f) \left(\frac{1}{2}, \frac{1}{2}, 2, 2 \right) \right| \leq \left\{ \frac{1}{2^p(p+1)} \right\}^{2/p} (b-a)(d-c) M_q^{1/q}, \quad (2.37)$$

(iii) if we choose $h_1 = h_2 = 1/2$, $r_1 = r_2 = 6$, $s = 1$, and $q = 1$, then we get

$$\left| I(f) \left(\frac{1}{2}, \frac{1}{2}, 6, 6 \right) \right| \leq \left(\frac{5}{36} \right)^2 (b-a)(d-c) M_1, \quad (2.38)$$

where

$$M_q = \frac{|\partial^2 f / \partial t \partial \lambda(a, c)|^q + |\partial^2 f / \partial t \partial \lambda(a, d)|^q + |\partial^2 f / \partial t \partial \lambda(b, c)|^q + |\partial^2 f / \partial t \partial \lambda(b, d)|^q}{4}. \quad (2.39)$$

Theorem 2.9. Let $f : \Delta \rightarrow \mathbb{R}^2$ be a partial differentiable mapping on $\Delta = [a, b] \times [c, d] \subset \mathbb{R}^2$. If $|\partial^2 f / \partial t \partial \lambda|^q$ ($q \geq 1$) is in $L_1(\Delta)$ and is a coordinated s -convex mapping in the second sense on Δ , then, for $r_1, r_2 \geq 2$ and $h_1, h_2 \in (0, 1)$ with $(1/r_1) \leq h_1 \leq ((r_1 - 1)/r_1)$ and $(1/r_2) \leq h_2 \leq ((r_2 - 1)/r_2)$, the following inequality holds:

$$\begin{aligned} & \frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| \\ & \leq \left\{ \left(\frac{1}{2} - h_1 + h_1^2 + \frac{(2-r_1)}{r_1^2} \right) \left(\frac{1}{2} - h_2 + h_2^2 + \frac{(2-r_2)}{r_2^2} \right) \right\}^{1-(1/q)} \\ & \quad \times \left\{ \mu_1 \left(\mu_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right|^q + \nu_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right|^q \right) \right. \\ & \quad \left. + \nu_1 \left(\mu_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right|^q + \nu_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|^q \right) \right\}^{1/q}, \end{aligned} \quad (2.40)$$

where μ_i and ν_i ($i = 1, 2$) are as given in Theorem 2.3.

Proof. From Lemma 2.1, we can write

$$\begin{aligned}
 & \frac{1}{(b-a)(d-c)} |I(f)(h_1, h_2, r_1, r_2)| \\
 & \leq \int_0^1 |p(h_1, r_1, t)q(h_2, r_2, \lambda)| \\
 & \quad \times \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right| dt d\lambda \\
 & \leq \left\{ \int_0^1 \int_0^1 |p(h_1, r_1, t)q(h_2, r_2, \lambda)| dt d\lambda \right\}^{1-(1/q)} \\
 & \quad \times \left\{ \int_0^1 \int_0^1 |p(h_1, r_1, t)q(h_2, r_2, \lambda)| \right. \\
 & \quad \times \left. \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^q dt d\lambda \right\}^{1/q}.
 \end{aligned} \tag{2.41}$$

By the simple calculations, we have

$$(i) \int_0^1 |p(h_1, r_1, t)| dt = \frac{1}{2} - h_1 + h_1^2 + \frac{(2-r_1)}{r_1^2}, \tag{2.42}$$

$$(ii) \int_0^1 |q(h_2, r_2, \lambda)| d\lambda = \frac{1}{2} - h_2 + h_2^2 + \frac{(2-r_2)}{r_2^2}. \tag{2.43}$$

Since $|\partial^2 f / \partial t \partial \lambda|^q$ is a coordinated s -convex mapping in the second sense on $\Delta = [a, b] \times [c, d]$, we have that, for $t \in [0, 1]$,

$$\begin{aligned}
 & \int_0^1 \int_0^1 |p(h_1, r_1, t)q(h_2, r_2, \lambda)| \left| \frac{\partial^2 f}{\partial t \partial \lambda}(ta + (1-t)b, \lambda c + (1-\lambda)d) \right|^q dt d\lambda \\
 & \leq \int_0^1 \int_0^1 |p(h_1, r_1, t)q(h_2, r_2, \lambda)| \\
 & \quad \times \left\{ t^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right|^q + t^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right|^q \right. \\
 & \quad \times \left. + (1-t)^s \lambda^s \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right|^q + (1-t)^s (1-\lambda)^s \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|^q \right\} dt d\lambda
 \end{aligned}$$

$$\begin{aligned}
&= \left\{ \int_0^1 |p(h_1, r_1, t)| t^s dt \right\} \left\{ \left(\int_0^1 |q(h_2, r_2, \lambda)| \lambda^s d\lambda \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right|^q \right. \\
&\quad \left. + \left(\int_0^1 |q(h_2, r_2, \lambda)| (1-\lambda)^s d\lambda \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right|^q \right\} \\
&\quad + \left\{ \int_0^1 |p(h_1, r_1, t)| (1-t)^s dt \right\} \left\{ \left(\int_0^1 |q(h_2, r_2, s)| \lambda^s d\lambda \right) \left| \frac{\partial^2 f}{\partial t \partial s}(b, c) \right|^q \right. \\
&\quad \left. + \left(\int_0^1 |q(h_2, r_2, \lambda)| (1-\lambda)^s d\lambda \right) \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|^q \right\} \\
&= \mu_1 \left\{ \mu_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, c) \right|^q + \nu_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(a, d) \right|^q \right\} \\
&\quad + \nu_1 \left\{ \mu_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, c) \right|^q + \nu_2 \left| \frac{\partial^2 f}{\partial t \partial \lambda}(b, d) \right|^q \right\}. \tag{2.44}
\end{aligned}$$

By (2.41)–(2.44), the assertion (2.40) holds. $\square$

References

- [1] M. Alomari, M. Darus, and S. S. Dragomir, "New inequalities of Simpson's type for s -convex functions with applications," *RGMLA Research Report Collection*, vol. 12, no. 4, article 9, 2009.
- [2] S. S. Dragomir, R. P. Agarwal, and P. Cerone, "On Simpson's inequality and applications," *Journal of Inequalities and Applications*, vol. 5, no. 6, pp. 533–579, 2000.
- [3] M. E. Özdemir, A. O. Akdemir, H. Kavurmaci, and M. Avci, "On the Simpson's inequality for co-ordinated convex functions," *Classical Analysis and ODEs*, <http://arxiv.org/abs/1101.0075>.
- [4] J. Park, "Generalization of some Simpson-like type inequalities via differentiable s -convex mappings in the second sense," *International Journal of Mathematics and Mathematical Sciences*, vol. 2011, Article ID 493531, 13 pages, 2011.
- [5] M. Z. Sarikaya, E. Set, and M. E. Ozdemir, "On new inequalities of Simpson's type for s -convex functions," *Computers & Mathematics with Applications*, vol. 60, no. 8, pp. 2191–2199, 2010.
- [6] M. Alomari and M. Darus, "Co-ordinated s -convex function in the first sense with some Hadamard-type inequalities," *International Journal of Contemporary Mathematical Sciences*, vol. 3, no. 29-32, pp. 1557–1567, 2008.
- [7] S. S. Dragomir, "On the Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane," *Taiwanese Journal of Mathematics*, vol. 5, no. 5, pp. 775–788, 2001.
- [8] J. Park, "Generalizations of the Simpson-like type inequalities for coordinated s -convex mappings," *Far East Journal of Mathematical Sciences*, vol. 54, no. 2, pp. 225–236, 2011.
- [9] M. Z. Sarikaya, E. Set, M. E. Ozdemir, and S. S. Dragomir, "New some Hadamard's type inequalities for co-ordinated convex functions," *Classical Analysis and ODEs*, <http://arxiv.org/abs/1005.0700>.
- [10] E. Set, M. E. Ozdemir, and M. Z. Sarikaya, "Inequalities of Hermite-Hadamard's type for functions whose derivatives absolute values are m -convex," *AIP Conference Proceedings*, vol. 1309, no. 1, pp. 861–873, 2010.
- [11] J. Park, "New generalizations of Simpson's type inequalities for twice differentiable convex mappings," *Far East Journal of Mathematical Sciences*, vol. 52, no. 1, pp. 43–55, 2011.
- [12] J. Park, "Some Hadamard's type inequalities for co-ordinated (s, m) -convex mappings in the second sense," *Far East Journal of Mathematical Sciences*, vol. 51, no. 2, pp. 205–216, 2011.
- [13] D.-Y. Hwang, K.-L. Tseng, and G.-S. Yang, "Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane," *Taiwanese Journal of Mathematics*, vol. 11, no. 1, pp. 63–73, 2007.

- [14] M. Alomari and M. Darus, "The Hadamard's inequality for s -convex function of 2-variables on the co-ordinates," *International Journal of Mathematical Analysis*, vol. 2, no. 13-16, pp. 629–638, 2008.
- [15] M. A. Latif and M. Alomari, "Hadamard-type inequalities for product two convex functions on the co-ordinates," *International Mathematical Forum*, vol. 4, no. 45-48, pp. 2327–2338, 2009.