Research Article

On Generalized Approximative Properties of Systems

Turab Mursal Ahmedov and Zahira Vahid Mamedova

Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan, 9 B. Vaxabzade, Baku AZ1141, Azerbaijan

Correspondence should be addressed to Zahira Vahid Mamedova, zahira_eng13@hotmail.com

Received 5 January 2012; Accepted 15 February 2012

Academic Editor: Siegfried Gottwald

Copyright © 2012 T. Mursal Ahmedov and Z. Vahid Mamedova. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Generalized concepts of b -completeness, b -independence, b -minimality, and b -basicity are introduced. Corresponding concept of a space of coefficients is defined, and some of its properties are stated.

1. Introduction

Theory of classical basis (including Schauder basis) is sufficiently well developed, and quite a good number of monographs such as Day [1], Singer [2, 3], Young [4], Bilalov, Veliyev [5], Ch. Heil [6], and so forth have been dedicated to it so far. There are different versions and generalizations of the concept of classical basis (for more details see [2, 3]). One of such generalizations is proposed in [5, 7]. In these works, the concept of b_Y -invariance is first introduced and then used to obtain main results. It should be noted that the condition of b_Y -invariance implies the fact that the corresponding mapping is tensorial.

In our work, we neglect the b_Y -invariance condition. We give more detailed consideration to the space of coefficients. We also state a criterion of basicity.

2. Needful Notations and Concepts

Let X, Y, Z be some Banach spaces, and let $\|\cdot\|_X, \|\cdot\|_Y, \|\cdot\|_Z$ be the corresponding norms. Assume that we are given some bounded bilinear mapping $b : X \times Y \rightarrow Z$, that is, $\|b(x; y)\|_Z \leq c \|x\|_X \|y\|_Y$, for all $x \in X$, for all $y \in Y$, where c is an absolute constant. For simplicity, we denote $xy \equiv b(x; y)$. Let $M \subset Y$ be some set. By $L^b[M]$ we denote the b -span of M . So, by definition, $L^b[M] \equiv \{z \in Z : \exists \{x_k\}_1^n \subset X, \exists \{y_k\}_1^n \subset M, z = \sum_{k=1}^n x_k y_k\}$. By $(\bar{\cdot})$ we denote the closure in the corresponding space.

System $\{y_n\}_{n \in \mathbb{N}} \subset Y$ is called *b-linearly independent* if $\sum_{n=1}^{\infty} x_n y_n = 0$ in Z implies that $x_n = 0$, for all $n \in \mathbb{N}$.

In the context of the above mentioned, the concept of usual completeness is stated as follows.

Definition 2.1. System $\{y_n\}_{n \in \mathbb{N}} \subset Y$ is called *b-complete* in Z if $\overline{L^b[\{y_n\}_{n \in \mathbb{N}}]} \equiv Z$. We will also need the concepts of *b-biorthogonal* system and *b-basis*.

Definition 2.2. System $\{y_n^*\}_{n \in \mathbb{N}} \subset L(Z; X)$ is called *b-biorthogonal* to the system $\{y_n\}_{n \in \mathbb{N}} \subset Y$ if $y_n^*(x y_k) = \delta_{nk} x$, for all $n, k \in \mathbb{N}$, for all $x \in X$, where δ_{nk} is the Kronecker symbol.

Definition 2.3. System $\{y_n\}_{n \in \mathbb{N}} \subset Y$ is called *b-basis* in Z if for all $z \in Z$, $\exists! \{x_n\}_{n \in \mathbb{N}} \subset X : z = \sum_{n=1}^{\infty} x_n y_n$.

To obtain our main results we will use the following concept.

Definition 2.4. System $\{y_n\}_{n \in \mathbb{N}} \subset Y$ is called *nondegenerate* if $\exists c_n > 0 : \|x\|_X \leq c_n \|x y_n\|_Z$, for all $x \in X$, for all $n \in \mathbb{N}$.

3. Main Results

3.1. Space of Coefficients

Let $\{y_n\}_{n \in \mathbb{N}} \subset Y$ be some system. Assume that

$$\mathcal{K}_{\bar{y}} \equiv \left\{ \{x_n\}_{n \in \mathbb{N}} \subset X : \text{the series } \sum_{n=1}^{\infty} x_n y_n \text{ converges in } Z \right\}. \quad (3.1)$$

With regard to the ordinary operations of addition and multiplication by a complex number, $\mathcal{K}_{\bar{y}}$ is a linear space. We introduce a norm $\|\cdot\|_{\mathcal{K}_{\bar{y}}}$ in $\mathcal{K}_{\bar{y}}$ as follows:

$$\|\bar{x}\|_{\mathcal{K}_{\bar{y}}} = \sup_m \left\| \sum_{n=1}^m x_n y_n \right\|_Z, \quad (3.2)$$

where $\bar{x} \equiv \{x_n\}_{n \in \mathbb{N}} \in \mathcal{K}_{\bar{y}}$. In fact, it is clear that

$$\begin{aligned} \|\lambda \bar{x}\|_{\mathcal{K}_{\bar{y}}} &= |\lambda| \|\bar{x}\|_{\mathcal{K}_{\bar{y}}}, \quad \forall \lambda \in \mathbb{C}; \\ \|\bar{x}_1 + \bar{x}_2\|_{\mathcal{K}_{\bar{y}}} &\leq \|\bar{x}_1\|_{\mathcal{K}_{\bar{y}}} + \|\bar{x}_2\|_{\mathcal{K}_{\bar{y}}}, \quad \forall \bar{x}_k \in \mathcal{K}_{\bar{y}}, \quad k = 1, 2. \end{aligned} \quad (3.3)$$

Assume that $\|\bar{x}\|_{\mathcal{K}_{\bar{y}}} = 0$ for some $\bar{x} \equiv \{x_n\}_{n \in \mathbb{N}} \in \mathcal{K}_{\bar{y}}$. Let $n_0 = \inf \{n : x_k = 0, \text{ for all } k \leq n-1\}$. In what follows we will suppose that the system $\{y_n\}_{n \in \mathbb{N}}$ is nondegenerate. Let $n_0 < +\infty$. We have

$$\sup_m \left\| \sum_{n=1}^m x_n y_n \right\|_Z \geq \left\| \sum_{n=1}^{n_0} x_n y_n \right\|_Z = \|x_{n_0} y_{n_0}\|_Z \geq \frac{1}{c_{n_0}} \|x_{n_0}\|_X > 0. \quad (3.4)$$

But this is contrary to $\|\bar{x}\|_{\mathcal{K}_{\bar{y}}} = 0$. Consequently, $n_0 = +\infty$, that is, $\bar{x} = 0$. Thus, $(\mathcal{K}_{\bar{y}}; \|\cdot\|_{\mathcal{K}_{\bar{y}}})$ is a normed space. Let us show that it is complete. Let $\{\bar{x}_n\}_{n \in \mathbb{N}} \subset \mathcal{K}_{\bar{y}}$ be some fundamental sequence with $\bar{x}_n \equiv \{x_k^{(n)}\}_{k \in \mathbb{N}} \subset X$. For arbitrary fixed number $k \in \mathbb{N}$. We have

$$\begin{aligned} \|x_k^{(n)} - x_k^{(n+p)}\|_X &\leq c_k \left\| (x_k^{(n)} - x_k^{(n+p)}) y_k \right\|_Z \\ &= c_k \left\| \sum_{i=1}^k (x_i^{(n)} - x_i^{(n+p)}) y_i - \sum_{i=1}^{k-1} (x_i^{(n)} - x_i^{(n+p)}) y_i \right\|_Z \\ &\leq 2c_k \sup_m \left\| \sum_{i=1}^m (x_i^{(n)} - x_i^{(n+p)}) y_i \right\| \\ &= 2c_k \|\bar{x}_n - \bar{x}_{n+p}\|_{\mathcal{K}_{\bar{y}}} \longrightarrow 0, \quad \text{as } n, p \longrightarrow \infty. \end{aligned} \quad (3.5)$$

As a result, for all fixed $k \in \mathbb{N}$ the sequence $\{x_k^{(n)}\}_{n \in \mathbb{N}}$ is fundamental in X . Let $x_k^{(n)} \rightarrow x_k$, $n \rightarrow \infty$. Let us take arbitrary positive ε . Then $\exists n_0$: for all $n \geq n_0$, for all $p \in \mathbb{N}$, we have $\|\bar{x}_n - \bar{x}_{n+p}\|_{\mathcal{K}_{\bar{y}}} < \varepsilon$. Thus

$$\left\| \sum_{k=1}^m (x_k^{(n)} - x_k^{(n+p)}) y_k \right\|_Z < \varepsilon, \quad \forall n \geq n_0, \forall p, m \in \mathbb{N}. \quad (3.6)$$

Passing to the limit as $p \rightarrow \infty$ yields

$$\left\| \sum_{k=1}^m (x_k^{(n)} - x_k) y_k \right\|_Z \leq \varepsilon, \quad \forall n \geq n_0, \forall m \in \mathbb{N}. \quad (3.7)$$

It is easy to see that

$$\left\| \sum_{k=m}^{m+p} (x_k^{(n)} - x_k) y_k \right\|_Z \leq 2\varepsilon, \quad \forall n \geq n_0, \forall m, p \in \mathbb{N}. \quad (3.8)$$

Since the series $\sum_{k=1}^{\infty} x_k^{(n)} y_k$ converges in Z , it is clear that $\exists m_0^{(n)}$: for all $m \geq m_0^{(n)}$, for all $p \in \mathbb{N}$, we have

$$\left\| \sum_{k=m}^{m+p} x_k^{(n)} y_k \right\|_Z < \varepsilon. \quad (3.9)$$

Then it follows from the previous inequality that

$$\begin{aligned} \left\| \sum_{k=m}^{m+p} x_k y_k \right\|_Z &\leq \left\| \sum_{k=m}^{m+p} (x_k^{(n)} - x_k) y_k \right\|_Z + \left\| \sum_{k=m}^{m+p} x_k^{(n)} y_k \right\|_Z \leq 3\varepsilon, \\ &\forall m \geq m_0^{(n)}, \quad \forall p \in \mathbb{N}. \end{aligned} \quad (3.10)$$

Consequently, the series $\sum_{k=1}^{\infty} x_k y_k$ converges in Z , and therefore $\bar{x} \equiv \{x_k\}_{k \in \mathbb{N}} \in \mathcal{K}_{\bar{y}}$. From (3.7) it follows directly that $\|\bar{x}_n - \bar{x}\|_{\mathcal{K}_{\bar{y}}} \rightarrow 0$, $n \rightarrow \infty$. Thus, $\mathcal{K}_{\bar{y}}$ is a Banach space.

Let us consider the operator $K : \mathcal{K}_{\bar{y}} \rightarrow Z$, defined by the expression

$$K\bar{x} = \sum_{n=1}^{\infty} x_n y_n, \bar{x} \equiv \{x_n\}_{n \in \mathbb{N}}. \quad (3.11)$$

It is obvious that K is a linear operator. Let $z = K\bar{x}$. We have

$$\|K\bar{x}\|_Z = \|z\|_Z = \left\| \sum_{n=1}^{\infty} x_n y_n \right\|_Z \leq \sup_m \left\| \sum_{n=1}^m x_n y_n \right\|_Z = \|\bar{x}\|_{\mathcal{K}_{\bar{y}}}. \quad (3.12)$$

It follows that $K \in L(\mathcal{K}_{\bar{y}}; Z)$ and $\|K\| \leq 1$. Let $\bar{x}_0 \equiv \{x; 0; \dots\}$, $x \in X$. It is clear that $\|K\bar{x}_0\|_Z = \|\bar{x}_0\|_{\mathcal{K}_{\bar{y}}}$. Consequently, $\|K\| = 1$. It is absolutely obvious that if the system $\{y_n\}_{n \in \mathbb{N}} \subset Y$ is *b-linearly independent*, then $\text{Ker} K = \{0\}$. In this case, $\exists K^{-1} : Z \rightarrow \mathcal{K}_{\bar{y}}$. If $\text{Im } K$ is closed, then, by the Banach's theorem on the inverse operator, we obtain that $K^{-1} \in L(\text{Im } K; \mathcal{K}_{\bar{y}})$. The same considerations are valid in the case when the system $\{y_n\}_{n \in \mathbb{N}}$ has a *b-biorthogonal* system. We will call operator K a coefficient operator. Thus, we have proved the following.

Theorem 3.1. *Every nondegenerate system $S_{\bar{y}} \equiv \{y_n\}_{n \in \mathbb{N}} \subset Y$ is corresponded by a Banach space of coefficients $\mathcal{K}_{\bar{y}}$ and coefficient operator $K \in L(\mathcal{K}_{\bar{y}}; Z)$, $\|K\| = 1$. If the system $S_{\bar{y}}$ is b-linearly independent or has a b-biorthogonal system, then $\exists K^{-1}$. Moreover, if $\text{Im } K$ is closed, then $K^{-1} \in L(\text{Im } K; \mathcal{K}_{\bar{y}})$.*

In what follows, we will need the concept of *b-basis* in the space of coefficients $\mathcal{K}_{\bar{y}}$.

Definition 3.2. System $\{T_n\}_{n \in \mathbb{N}} \subset L(X; \mathcal{K}_{\bar{y}})$ is called *b-basis* in $\mathcal{K}_{\bar{y}}$ if for for all $\bar{x} \in \mathcal{K}_{\bar{y}}$, $\exists! \{x_n\}_{n \in \mathbb{N}} \subset X : \bar{x} = \sum_{n=1}^{\infty} T_n x_n$ (convergence in $\mathcal{K}_{\bar{y}}$).

Consider the operators $E_n : X \rightarrow \mathcal{K}_{\bar{y}} : E_n x = \{\delta_{nk} x\}_{k \in \mathbb{N}}$, $n \in \mathbb{N}$. We have

$$\|E_n x\|_{\mathcal{K}_{\bar{y}}} = \|x y_n\|_Z \leq \|y_n\|_Y \|x\|_X, \quad \forall x \in X. \quad (3.13)$$

Thus, $E_n \in L(X; \mathcal{K}_{\bar{y}})$, for all $n \in \mathbb{N}$. Take $\bar{x} \equiv \{x_n\}_{n \in \mathbb{N}} \in \mathcal{K}_{\bar{y}}$. Then

$$\left\| \bar{x} - \sum_{n=1}^m E_n x_n \right\|_{\mathcal{K}_{\bar{y}}} = \left\| \left\{ \underbrace{0; \dots; 0}_m; x_{m+1}; x_{m+2}; \dots \right\} \right\|_{\mathcal{K}_{\bar{y}}} = \sup_p \left\| \sum_{n=m+1}^{m+p} x_n y_n \right\|_Z \rightarrow 0, \quad (3.14)$$

as $m \rightarrow \infty$, since the series $\sum_{n=1}^{\infty} x_n y_n$ converges in Z . As a result, we obtain that $\bar{x} = \sum_{n=1}^{\infty} E_n x_n$. Consider the operators $P_n : \mathcal{K}_{\bar{y}} \rightarrow X : P_n \bar{x} = x_n$, $n \in \mathbb{N}$. We have

$$\|P_n \bar{x}\|_X = \|x_n\|_X \leq c_n \|x_n y_n\|_Z \leq c_n \sup_m \left\| \sum_{k=1}^m x_k y_k \right\|_Z = c_n \|\bar{x}\|_{\mathcal{K}_{\bar{y}}}. \quad (3.15)$$

Consequently, $P_n \in L(\mathcal{K}_{\bar{y}}; X)$, $n \in \mathbb{N}$. Let us show that the expansion $\bar{x} = \sum_{n=1}^{\infty} E_n x_n$ is unique. Let $\sum_{n=1}^{\infty} E_n x_n = 0$. We have $0 = P_k(\sum_{n=1}^{\infty} E_n x_n) = \sum_{n=1}^{\infty} P_k(E_n x_n) = x_k$, for all $k \in \mathbb{N}$. As a result, we obtain that the system $\{E_n\}_{n \in \mathbb{N}}$ forms a *b-basis* for $\mathcal{K}_{\bar{y}}$. We will call this system a *canonical system*. So we have proved the following.

Theorem 3.3. *Let $\mathcal{K}_{\bar{y}}$ be a space of coefficients of nondegenerate system $\{y_n\}_{n \in \mathbb{N}} \subset Y$. Then the canonical system $\{E_n\}_{n \in \mathbb{N}}$ forms a *b-basis* for $\mathcal{K}_{\bar{y}}$.*

Let the nondegenerate system $\{y_n\}_{n \in \mathbb{N}} \subset Y$ form a *b-basis* for Z . Consider the coefficient operator $K : \mathcal{K}_{\bar{y}} \rightarrow Z$. By definition of *b-basis*, the equation $K \bar{x} = z$ is solvable with regard to $\bar{x} \in \mathcal{K}_{\bar{y}}$ for all $z \in Z$. It is absolutely clear that $\text{Ker } K = \{0\}$. Then it follows from Theorem 3.1 and Banach theorem that $K^{-1} \in L(Z; \mathcal{K}_{\bar{y}})$. Consequently, operator K performs isomorphism between $\mathcal{K}_{\bar{y}}$ and Z .

Vice versa, let $\{y_n\}_{n \in \mathbb{N}} \subset Y$ be a nondegenerate system and let $\mathcal{K}_{\bar{y}}$ be a corresponding space of coefficients. Assume that a coefficient operator $K \in L(\mathcal{K}_{\bar{y}}; Z)$ is an isomorphism. Take for all $z \in Z$. It is clear that $\exists \bar{x} \equiv \{x_n\}_{n \in \mathbb{N}} \in \mathcal{K}_{\bar{y}} : K \bar{x} = z$, that is $z = \sum_{n=1}^{\infty} x_n y_n$ in Z . Consequently, z can be expanded in a series with respect to the system $\{y_n\}_{n \in \mathbb{N}}$. Let us show that such an expansion is unique. Let $\sum_{n=1}^{\infty} x_n^0 y_n = 0$ for some $\bar{x}^0 \equiv \{x_n^0\}_{n \in \mathbb{N}} \in \mathcal{K}_{\bar{y}}$. This means that $K \bar{x}^0 = 0 \Rightarrow \bar{x}^0 = 0 \Rightarrow x_n^0 = 0$, for all $n \in \mathbb{N}$. Thus, the system $\{y_n\}_{n \in \mathbb{N}}$ forms a *b-basis* for Z . So the following theorem is proved.

Theorem 3.4. *Let $\mathcal{K}_{\bar{y}}$ be a space of coefficients of nondegenerate system $\{y_n\}_{n \in \mathbb{N}}$ and let K be a corresponding coefficient operator. Then this system forms a *b-basis* for Z if and only if K is an isomorphism in $L(\mathcal{K}_{\bar{y}}; Z)$.*

3.2. Criterion of Basicity

Let the systems $\{y_n\}_{n \in \mathbb{N}} \subset Y$ and $\{y_n^*\}_{n \in \mathbb{N}} \subset L(Z; X)$ be *b-biorthogonal*. Take for all $z \in Z$ and consider the partial sums

$$S_n z = \sum_{k=1}^n y_k^*(z) y_k, \quad n \in \mathbb{N}. \quad (3.16)$$

We have

$$\begin{aligned} S_n(S_m z) &= \sum_{k=1}^n y_k^*(S_m z) y_k = \sum_{k=1}^n y_k^* \left[\sum_{i=1}^m y_i^*(z) y_i \right] y_k \\ &= \sum_{k=1}^n \sum_{i=1}^m \delta_{ki} y_i^*(z) y_k = \sum_{k=1}^{\min\{n; m\}} y_k^*(z) y_k = S_{\min\{n; m\}} z, \quad \forall n, m \in \mathbb{N}. \end{aligned} \quad (3.17)$$

Hence, $S_n^2 = S_n$, for all $n \in \mathbb{N}$, that is, S_n is a projector in Z . It follows directly from the estimate

$$\|y_k^*(z) y_k\|_Z \leq c \|y_k^*(z)\|_X \|y_k\|_Y \leq c \|y_k^*\| \|y_k\|_Y \|z\|_Z, \quad \forall z \in Z, \quad (3.18)$$

that S_n is a continuous projector. Suppose that the nondegenerate system $\{y_n\}_{n \in \mathbb{N}} \subset Y$ forms a b -basis for Z . Then for all $z \in Z$ has a unique expansion $z = \sum_{n=1}^{\infty} x_n y_n$ in Z . We denote the correspondence $z \rightarrow x_n$ by $y_n^* : y_n^*(z) = x_n$, for all $n \in \mathbb{N}$. It is obvious that $y_n^* : Z \rightarrow X$ is a linear operator. Let $\mathcal{K}_{\bar{y}}$ be a space of coefficients of basis $\{y_n\}_{n \in \mathbb{N}}$, and let $K : \mathcal{K}_{\bar{y}} \rightarrow Z$ be the corresponding coefficient operator. By Theorem 3.4, K is an isomorphism. We have

$$\begin{aligned} \|y_n^*(z)\|_X &= \|x_n\|_X \leq c_n \|x_n y_n\| \leq c_n \sup_m \left\| \sum_{k=1}^m x_k y_k \right\|_Z = c_n \|\bar{x}\|_{\mathcal{K}_{\bar{y}}} \\ &= c_n \|K^{-1}z\|_{\mathcal{K}_{\bar{y}}} \leq c_n \|K^{-1}\| \|z\|_Z, \end{aligned} \quad (3.19)$$

where $\bar{x} \equiv \{x_n\}_{n \in \mathbb{N}}$. Consequently, $\{y_n^*\}_{n \in \mathbb{N}} \subset L(Z; X)$. It follows directly from the uniqueness of the expansion that $y_n^*(x y_k) = \delta_{nk} x$; for all $n, k \in \mathbb{N}$, for all $x \in X$. As a result, we obtain that the system $\{y_n^*\}_{n \in \mathbb{N}}$ is b -biorthogonal to $\{y_n\}_{n \in \mathbb{N}}$. Let us consider the projectors $S_m \in L(Z)$ for all $z \in Z$:

$$S_m z = \sum_{n=1}^m y_n^*(z) y_n, \quad m \in \mathbb{N}. \quad (3.20)$$

As the series (3.20) converges for all $z \in Z$, it follows from Banach-Steinhaus theorem that

$$M = \sup_m \|S_m\| < +\infty. \quad (3.21)$$

It is absolutely obvious that the system $\{y_n\}_{n \in \mathbb{N}}$ is b -complete in Z . Thus, if the system $\{y_n\}_{n \in \mathbb{N}}$ forms a b -basis for Z , then (1) it is b -complete in Z ; (2) it has a b -biorthogonal system; (3) the corresponding family of projectors is uniformly bounded.

Vice versa, let the system $\{y_n\}_{n \in \mathbb{N}}$ be b -complete in Z and have a b -biorthogonal system $\{y_n^*\}_{n \in \mathbb{N}}$. Assume that the corresponding family of projectors $\{S_m\}_{m \in \mathbb{N}}$ is uniformly bounded, that is, relation (3.21) holds. Let $z \in Z$ be an arbitrary element. Let us take arbitrary positive ε . It is clear that $\exists \{x_n\}_{n=1}^{m_0} \subset X : \|z - \sum_{n=1}^{m_0} x_n y_n\|_Z < \varepsilon$. Let $z_0 = \sum_{n=1}^{m_0} x_n y_n$. We have

$$y_n^*(z_0) = \sum_{k=1}^{m_0} y_n^*(x_k y_k) = x_n, \quad n = \overline{1, m_0}; \quad y_n^*(z_0) = 0, \quad \forall n > m_0. \quad (3.22)$$

As a result, we obtain for $m \geq m_0$ that

$$\|z - S_m z\|_Z \leq \|z - z_0\|_Z + \|z_0 - S_m z\|_Z \leq \varepsilon + \left\| \sum_{n=1}^m y_n^*(z - z_0) y_n \right\|_Z \leq (M + 1) \varepsilon. \quad (3.23)$$

From the arbitrariness of ε we get $\lim_{m \rightarrow \infty} S_m z = z$. Consequently, for all $z \in Z$ can be expanded in a series with respect to the system $\{y_n\}_{n \in \mathbb{N}}$. The existence of b -biorthogonal system implies the uniqueness of the expansion. Thus, the following theorem is valid.

Theorem 3.5. *Nondegenerate system $\{y_n\}_{n \in \mathbb{N}} \subset Y$ forms a b -basis for Z if and only if the following conditions are satisfied:*

- (1) *the system is b -complete in Z ;*
- (2) *it has a b -biorthogonal system $\{y_n^*\}_{n \in \mathbb{N}} \subset L(Z; X)$;*
- (3) *the family of projectors (3.21) is uniformly bounded.*

References

- [1] M. M. Day, *Normed Linear Spaces*, Izdat. Inostr. Lit., Moscow, Russia, 1961.
- [2] I. Singer, *Bases in Banach Spaces*, vol. 1, Springer, New York, NY, USA, 1970.
- [3] I. Singer, *Bases in Banach Spaces*, vol. 2, Springer, New York, NY, USA, 1981.
- [4] R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic Press, New York, NY, USA, 1980.
- [5] B. T. Bilalov and S. G. Veliyev, *Some problems of Basis*, Elm, Baku, Azerbaijan, 2010.
- [6] Ch. Heil, *A Basis Theory Primer*, Springer, New York, NY, USA, 2011.
- [7] B. T. Bilalov and A. I. Ismailov, "Bases and tensor product," *Transactions of National Academy of Sciences of Azerbaijan*, vol. 25, no. 4, pp. 15–20, 2005.