Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2012, Article ID 736367, 23 pages
doi:10.1155/2012/736367

Research Article

Generalized Altering Distances and Common Fixed Points in Ordered Metric Spaces

Hemant Kumar Nashine[1] and Hassen Aydi[2]

[1] *Department of Mathematics, Disha Institute of Management and Technology, Satya Vihar, Vidhansabha-Chandrakhuri Marg, Mandir Hasaud, Raipur 492101, India*
[2] *Institut Superiéur d'Informatique et des Technologies de Communication de Hammam Sousse, Route GP1-4011, Hammam Sousse, Tunisia*

Correspondence should be addressed to Hassen Aydi, hassen.aydi@isima.rnu.tn

Received 27 March 2012; Revised 6 June 2012; Accepted 6 June 2012

Academic Editor: Teodor Bulboaca

Copyright © 2012 H. K. Nashine and H. Aydi. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Coincidence point and common fixed point results with the concept of generalized altering distance functions in complete ordered metric spaces are derived. These results generalize the existing fixed point results in the literature. To illustrate our results and to distinguish them from the existing ones, we equip the paper with examples. As an application, we study the existence of a common solution to a system of integral equations.

1. Introduction and Preliminaries

The study of common fixed points of mappings satisfying certain contractive conditions has been researched extensively by many mathematicians since fixed point theory plays a major role in mathematics and applied sciences (see [1–40] and others). A new category of contractive fixed point problems was addressed by Khan et al. [1]. In this work, they introduced the notion of an altering distance function, which is a control function that alters distance between two points in a metric space.

Definition 1.1 (see [1]). A function $\varphi : [0,+\infty) \to [0,+\infty)$ is called an altering distance function if and only if

(i) φ is continuous,

(ii) φ is nondecreasing,

(iii) $\varphi(t) = 0 \Leftrightarrow t = 0$.

Khan et al. [1] proved the following result.

Theorem 1.2 (see [1]). *Let (X,d) be a complete metric space, $\varphi : [0,+\infty) \to [0,+\infty)$ an altering distance function, and $T : X \to X$ a self-mapping which satisfies the following inequality:*

$$\varphi(d(Tx,Ty)) \leq c\varphi(d(x,y)) \tag{1.1}$$

for all $x,y \in X$ and for some $0 < c < 1$. Then T has a unique fixed point.

Putting $\varphi(t) = t$ in Theorem 1.2, we retrieve immediately the Banach contraction principle.

In 1997, Alber and Guerre-Delabriere [2] introduced the concept of weak contractions in Hilbert spaces.

This concept was extended to metric spaces by Rhoades in [3].

Definition 1.3. A mapping $T : X \to X$, where (X,d) is a metric space, is said to be weakly contractive if and only if

$$d(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)), \quad \forall x,y \in X, \tag{1.2}$$

where $\varphi : [0,+\infty) \to [0,+\infty)$ is an altering distance function.

Theorem 1.4 (see [3]). *Let (X,d) be a complete metric space and $T : X \to X$ a weakly contractive map. Then, T admits a unique fixed point.*

Weak inequalities of the above type have been used to establish fixed point results in a number of subsequent works, some of which are noted in [4–7, 20]. In [5], Choudhury introduced the concept of a generalized altering distance function for three variables.

Definition 1.5 (see [5]). A function $\varphi : [0,+\infty) \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is said to be a generalized altering distance function if and only if

(i) φ is continuous,

(ii) φ is nondecreasing in all the three variables,

(iii) $\varphi(x,y,z) = 0 \Leftrightarrow x = y = z = 0$.

In [5], Choudhury proved the following common fixed point theorem using altering distances for three variables.

Theorem 1.6 (see [5]). *Let (X,d) be a complete metric space and $S,T : X \to X$ two self mappings such that the following inequality is satisfied:*

$$\Phi_1(d(Sx,Ty)) \leq \psi_1(d(x,y),d(x,Sx),d(y,Ty)) - \psi_2(d(x,y),d(x,Sx),d(y,Ty)) \tag{1.3}$$

for all $x,y \in X$, where ψ_1 and ψ_2 are generalized altering distance functions and $\Phi_1(x) = \psi_1(x,x,x)$. Then S and T have a common fixed point.

In [31], Rao et al. generalized Theorem 1.6 for four mappings satisfying a generalized contractive condition and used the following generalized altering distance function for four variables.

Definition 1.7 (see [31]). A function $\varphi : [0,+\infty) \times [0,+\infty) \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is said to be a generalized altering distance function if and only if

(i) φ is continuous,

(ii) φ is nondecreasing in all the three variables,

(iii) $\varphi(t_1, t_2, t_3, t_4) = 0 \Leftrightarrow t_1 = t_2 = t_3 = t_4 = 0$.

Let Ψ_4 denote the set of all functions φ satisfying (i)–(iii) in Definition 1.7.

Recently, there have been so many exciting developments in the field of existence of fixed point in partially ordered sets (see [8–19, 21–36] and the references cited therein). The first result in this direction was given by Turinici [36], where he extended the Banach contraction principle in partially ordered sets. Ran and Reurings [33] presented some applications of Turinici's theorem to matrix equations. The obtained result by Turinici was further extended and refined in [28–32]. Subsequently, Harjani and Sadarangani [19] generalized their own results [18] by considering pair of altering functions (ψ, φ). Nashine and Altun [21] and Nashine and Altun [22] generalized the results of Harjani and Sadarangani [18, 19]. Also, Nashine and Altun [22] and Shatanawi and Samet [41] worked for a pair (T, S) of weakly increasing mappings with respect to a third mapping R. In another paper, Nashine et al. [24] prove coincidence point and common fixed point results for mappings satisfying a contractive inequality which involves two generalized altering distance functions for three variables in ordered complete metric spaces. As application, they studied the existence of a common solution to a system of integral equations.

The aim of this paper is to generalize the results of Nashine et al. [24] in the sense of four variables. We obtain coincidence point and common fixed point theorems in complete ordered metric spaces for mappings satisfying a contractive condition which involves two generalized altering distance functions in four variables. Presented theorems are ordered version of Theorem 2.1 of Rao et al. [31] for three mappings. In addition, an application to the study of the existence of a common solution to a system of integral equations is given.

2. Main Results

First we introduce some notations and definitions that will be used later.

2.1. Notations and Definitions

The following definition was introduced by Jungck in [37].

Definition 2.1 (see [37]). Let (X, d) be a metric space and $f, g : X \to X$. If $w = fx = gx$, for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The pair $\{f, g\}$ is said to be compatible if and only if $\lim_{n\to+\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} gx_n = t$ for some $t \in X$.

Let X be a nonempty set and $R : X \to X$ a given mapping. For every $x \in X$, we denote by $R^{-1}(x)$ the subset of X defined by

$$R^{-1}(x) := \{u \in X \mid Ru = x\}. \tag{2.1}$$

Definition 2.2 (see [23]). Let $(X, \preceq)$ be a partially ordered set and $T, S, R : X \to X$ are given mappings such that $TX \subseteq RX$ and $SX \subseteq RX$. We say that S and T are weakly increasing with respect to R if and only if, for all $x \in X$, we have

$$\begin{aligned} Tx &\preceq Sy, \quad \forall y \in R^{-1}(Tx), \\ Sx &\preceq Ty, \quad \forall y \in R^{-1}(Sx). \end{aligned} \tag{2.2}$$

Remark 2.3. If $R : X \to X$ is the identity mapping ($Rx = x$ for all $x \in X$), then S and T are weakly increasing with respect to R which implies that S and T are weakly increasing mappings. Note that the notion of weakly increasing mappings was introduced in [9] (also see [17, 38]).

Example 2.4. Let $X = \{1, 2, 3\}$ be endowed with the partial order $\preceq$ given by

$$\preceq := \{(1,1), (2,2), (3,3), (2,3), (3,1), (2,1)\}. \tag{2.3}$$

Define the mappings $T, S, R : X \to X$ by

$$\begin{gathered} T1 = T3 = 1, \qquad T2 = 3, \\ S1 = S2 = S3 = 1, \\ R1 = 1, \qquad R2 = R3 = 2. \end{gathered} \tag{2.4}$$

We will show that the mappings S and T are weakly increasing with respect to R.

Let $x, y \in X$ such that $y \in R^{-1}(Tx)$. By the definition of S, we have $Sy = 1$. On the other hand, $Tx \in \{1, 3\}$ and $(1,1), (3,1) \in \preceq$. Thus, we have $Tx \preceq Sy$ for all $y \in R^{-1}(Tx)$.

Let $x, y \in X$ such that $y \in R^{-1}(Sx)$. By the definitions of S and R, we have $R^{-1}(Sx) = R^{-1}(1) = \{1\}$. Then we have $y = 1$. On the other hand, $1 = Sx \preceq Ty = T1 = 1$. Then, $Sx \preceq Ty$ for all $y \in R^{-1}(Sx)$.

Thus, we proved that S and T are weakly increasing with respect to R.

Definition 2.5. Let X be a nonempty set. Then $(X, d, \preceq)$ is called an ordered metric space if and only if:

(i) (X, d) is a metric space,

(ii) $(X, \preceq)$ is a partially ordered set.

2.2. Results

Our first result is the following.

Theorem 2.6. *Let $(X, d, \preceq)$ be an ordered complete metric space. Let $T, S, R : X \to X$ be given mappings satisfying for every pair $(x, y) \in X \times X$ such that Rx and Ry are comparable,*

$$\Phi_1(d(Sx, Ty)) \leq \psi_1\left(d(Rx, Ry), d(Rx, Sx), d(Ry, Ty), \frac{1}{2}[d(Rx, Ty) + d(Ry, Sx)]\right) - \psi_2\left(d(Rx, Ry), d(Rx, Sx), d(Ry, Ty), \frac{1}{2}[d(Rx, Ty) + d(Ry, Sx)]\right), \tag{2.5}$$

where ψ_1 and ψ_2 are generalized altering distance functions (in Ψ_4) and $\Phi_1(x) = \psi_1(x, x, x, x)$. One assumes the following hypotheses:

(i) *T, S, and R are continuous,*

(ii) *$TX \subseteq RX$, $SX \subseteq RX$,*

(iii) *T and S are weakly increasing with respect to R,*

(iv) *the pairs $\{T, R\}$ and $\{S, R\}$ are compatible.*

Then, T, S, and R have a coincidence point, that is, there exists $u \in X$ such that $Ru = Tu = Su$.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $TX \subseteq RX$, there exists $x_1 \in X$ such that $Rx_1 = Tx_0$. Since $SX \subseteq RX$, there exists $x_2 \in X$ such that $Rx_2 = Sx_1$. Continuing this process, we can construct a sequence $\{Rx_n\}$ in X defined by

$$Rx_{2n+1} = Tx_{2n}, \qquad Rx_{2n+2} = Sx_{2n+1}, \quad \forall n \in \mathbb{N}. \tag{2.6}$$

We claim that

$$Rx_n \preceq Rx_{n+1}, \quad \forall n \in \mathbb{N}. \tag{2.7}$$

To this aim, we will use the increasing property of the mappings S and T with respect to R. From (2.6), we have

$$Rx_1 = Tx_0 \preceq Sy, \quad \forall y \in R^{-1}(Tx_0). \tag{2.8}$$

Since $Rx_1 = Tx_0$, then $x_1 \in R^{-1}(Tx_0)$, and we get

$$Rx_1 = Tx_0 \preceq Sx_1 = Rx_2. \tag{2.9}$$

Again,

$$Rx_2 = Sx_1 \preceq Ty, \quad \forall y \in R^{-1}(Sx_1). \tag{2.10}$$

Since $x_2 \in R^{-1}(Sx_1)$, we get

$$Rx_2 = Sx_1 \preceq Tx_2 = Rx_3. \tag{2.11}$$

Hence, by induction, (2.7) holds.

Without loss of the generality, we can assume that

$$Rx_n \neq Rx_{n+1}, \quad \forall n \in \mathbb{N}. \tag{2.12}$$

Now, we will prove our result on three steps.

Step 1. We claim that

$$\lim_{n \to +\infty} d(Rx_{n+1}, Rx_{n+2}) = 0. \tag{2.13}$$

Putting $x = x_{2n+1}$ and $y = x_{2n}$, from (2.7) and the considered contraction (2.5), we have

$$\begin{aligned}
\Phi_1(d(Rx_{2n+2}, Rx_{2n+1})) &= \Phi_1(d(Sx_{2n+1}, Tx_{2n})) \\
&\leq \psi_1\Big(d(Rx_{2n+1}, Rx_{2n}), d(Rx_{2n+1}, Sx_{2n+1}), d(Rx_{2n}, Tx_{2n}), \\
&\qquad \frac{1}{2}[d(Rx_{2n+1}, Tx_{2n}) + d(Rx_{2n}, Sx_{2n+1})]\Big) \\
&\quad - \psi_2\Big(d(Rx_{2n+1}, Rx_{2n}), d(Rx_{2n+1}, Sx_{2n+1}), d(Rx_{2n}, Tx_{2n}), \\
&\qquad \frac{1}{2}[d(Rx_{2n+1}, Tx_{2n}) + d(Rx_{2n}, Sx_{2n+1})]\Big) \\
&= \psi_1\Big(d(Rx_{2n+1}, Rx_{2n}), d(Rx_{2n+1}, Rx_{2n+2}), d(Rx_{2n}, Rx_{2n+1}), \\
&\qquad \frac{1}{2}d(Rx_{2n}, Rx_{2n+2})\Big) \\
&\quad - \psi_2\Big(d(Rx_{2n+1}, Rx_{2n}), d(Rx_{2n+1}, Rx_{2n+2}), d(Rx_{2n}, Rx_{2n+1}), \\
&\qquad \frac{1}{2}d(Rx_{2n}, Rx_{2n+2})\Big).
\end{aligned} \tag{2.14}$$

Suppose, for some $n \in \mathbb{N}$, that

$$d(Rx_{2n+1}, Rx_{2n+2}) > d(Rx_{2n}, Rx_{2n+1}). \tag{2.15}$$

Using (2.15) and a triangular inequality, we have

$$\frac{1}{2}d(Rx_{2n}, Rx_{2n+2}) \leq \frac{1}{2}(d(Rx_{2n}, Rx_{2n+1}) + d(Rx_{2n+1}, Rx_{2n+2}) < d(Rx_{2n+1}, Rx_{2n+2}). \tag{2.16}$$

Using this and (2.15) together with a property of the generalized altering function ψ_1, we get

$$\begin{aligned}&\psi_1\left(d(Rx_{2n+1},Rx_{2n}),d(Rx_{2n+1},Rx_{2n+2}),d(Rx_{2n},Rx_{2n+1}),\frac{1}{2}d(Rx_{2n},Rx_{2n+2})\right)\\&\quad\leq\Phi_1(d(Rx_{2n+2},Rx_{2n+1})).\end{aligned}\tag{2.17}$$

Hence, we obtain

$$\begin{aligned}&\Phi_1(d(Rx_{2n+2},Rx_{2n+1}))\\&\quad\leq\Phi_1(d(Rx_{2n+2},Rx_{2n+1}))\\&\qquad-\psi_2\left(d(Rx_{2n+1},Rx_{2n}),d(Rx_{2n+1},Rx_{2n+2}),d(Rx_{2n},Rx_{2n+1}),\frac{1}{2}d(Rx_{2n},Rx_{2n+2})\right).\end{aligned}\tag{2.18}$$

This implies that

$$\psi_2\left(d(Rx_{2n+1},Rx_{2n}),d(Rx_{2n+1},Rx_{2n+2}),d(Rx_{2n},Rx_{2n+1}),\frac{1}{2}d(Rx_{2n},Rx_{2n+2})\right)=0,\tag{2.19}$$

which yields that

$$d(Rx_{2n+1},Rx_{2n})=0.\tag{2.20}$$

Hence, we obtain a contradiction with (2.12). We deduce that

$$d(Rx_{2n+1},Rx_{2n+2})\leq d(Rx_{2n},Rx_{2n+1}),\quad\forall n\in\mathbb{N}.\tag{2.21}$$

Similarly, putting $x=x_{2n+1}$ and $y=x_{2n+2}$, from (2.7) and the considered contraction (2.5), we have

$$\begin{aligned}&\Phi_1(d(Rx_{2n+2},Rx_{2n+3}))\\&\quad\leq\psi_1\left(d(Rx_{2n+1},Rx_{2n+2}),d(Rx_{2n+1},Rx_{2n+2}),d(Rx_{2n+2},Rx_{2n+3}),\frac{1}{2}d(Rx_{2n+1},Rx_{2n+3})\right)\\&\qquad-\psi_2\left(d(Rx_{2n+1},Rx_{2n+2}),d(Rx_{2n+1},Rx_{2n+2}),d(Rx_{2n+2},Rx_{2n+3}),\frac{1}{2}d(Rx_{2n+1},Rx_{2n+3})\right).\end{aligned}\tag{2.22}$$

Suppose, for some $n\in\mathbb{N}$, that

$$d(Rx_{2n+2},Rx_{2n+3})>d(Rx_{2n+1},Rx_{2n+2}).\tag{2.23}$$

Then, by a triangular inequality, we have

$$\frac{1}{2}d(Rx_{2n+1}, Rx_{2n+3}) \leq \frac{1}{2}(d(Rx_{2n+1}, Rx_{2n+2}) + d(Rx_{2n+2}, Rx_{2n+3})) < d(Rx_{2n+2}, Rx_{2n+3}). \quad (2.24)$$

Hence, from this, (2.22), and (2.23), we obtain

$$\begin{aligned}
&\Phi_1(d(Rx_{2n+2}, Rx_{2n+3})) \\
&\quad \leq \Phi_1(d(Rx_{2n+2}, Rx_{2n+3})) \\
&\qquad - \psi_2\left(d(Rx_{2n+1}, Rx_{2n+2}), d(Rx_{2n+1}, Rx_{2n+2}), d(Rx_{2n+2}, Rx_{2n+3}), \frac{1}{2}d(Rx_{2n+1}, Rx_{2n+3})\right).
\end{aligned} \quad (2.25)$$

This implies that

$$\psi_2\left(d(Rx_{2n+1}, Rx_{2n+2}), d(Rx_{2n+1}, Rx_{2n+2}), d(Rx_{2n+2}, Rx_{2n+3}), \frac{1}{2}d(Rx_{2n+1}, Rx_{2n+3})\right) = 0, \quad (2.26)$$

which leads to

$$d(Rx_{2n+1}, Rx_{2n+2}) = 0. \quad (2.27)$$

Hence, we obtain a contradiction with (2.12). We deduce that

$$d(Rx_{2n+1}, Rx_{2n+2}) \geq d(Rx_{2n+2}, Rx_{2n+3}), \quad \forall n \in \mathbb{N}. \quad (2.28)$$

Combining (2.21) and (2.28), we obtain

$$d(Rx_{n+1}, Rx_{n+2}) \geq d(Rx_{n+2}, Rx_{n+3}), \quad \forall n \in \mathbb{N}. \quad (2.29)$$

Then, $\{d(Rx_{n+1}, Rx_{n+2})\}$ is a nonincreasing sequence of positive real numbers. This implies that there exists $r \geq 0$ such that

$$\lim_{n \to +\infty} d(Rx_{n+1}, Rx_{n+2}) = r. \quad (2.30)$$

By (2.14), we have

$$\begin{aligned}
\Phi_1(d(Rx_{2n+2}, Rx_{2n+1})) &\leq \psi_1\Big(d(Rx_{2n+1}, Rx_{2n}), d(Rx_{2n+1}, Rx_{2n+2}), d(Rx_{2n}, Rx_{2n+1}), \\
&\qquad \frac{1}{2}d(Rx_{2n}, Rx_{2n+2})\Big) \\
&\quad - \psi_2\Big(d(Rx_{2n+1}, Rx_{2n}), d(Rx_{2n+1}, Rx_{2n+2}), d(Rx_{2n}, Rx_{2n+1}), \\
&\qquad \frac{1}{2}d(Rx_{2n}, Rx_{2n+2})\Big) \\
&\leq \psi_1(d(Rx_{2n+1}, Rx_{2n}), d(Rx_{2n}, Rx_{2n+1}), d(Rx_{2n}, Rx_{2n+1}), \\
&\qquad d(Rx_{2n}, Rx_{2n+1})) \\
&\quad - \psi_2(d(Rx_{2n+1}, Rx_{2n}), d(Rx_{2n+1}, Rx_{2n+2}), d(Rx_{2n}, Rx_{2n+1}), 0) \\
&= \Phi_1(d(Rx_{2n}, Rx_{2n+1})) \\
&\quad - \psi_2(d(Rx_{2n+1}, Rx_{2n}), d(Rx_{2n+1}, Rx_{2n+2}), \mathrm{d}(Rx_{2n}, Rx_{2n+1}), 0).
\end{aligned} \tag{2.31}$$

Letting $n \to +\infty$ in (2.31) and using the continuities of Φ_1 and ψ_2, we obtain

$$\Phi_1(r) \leq \Phi_1(r) - \psi_2(r, r, r, 0), \tag{2.32}$$

which implies that $\psi_2(r, r, r, 0) = 0$, so $r = 0$. Hence

$$\lim_{n \to +\infty} d(Rx_{n+1}, Rx_{n+2}) = 0. \tag{2.33}$$

Hence, (2.13) is proved.

Step 2. We claim that $\{Rx_n\}$ is a Cauchy sequence.

From (2.13), it will be sufficient to prove that $\{Rx_{2n}\}$ is a Cauchy sequence. We proceed by negation and suppose that $\{Rx_{2n}\}$ is not a Cauchy sequence. Then, there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $\{m(i)\}$ and $\{n(i)\}$ such that, for all positive integer i,

$$n(i) > m(i) > i, \quad d\big(Rx_{2m(i)}, Rx_{2n(i)}\big) \geq \varepsilon, \qquad d\big(Rx_{2m(i)}, Rx_{2n(i)-2}\big) < \varepsilon. \tag{2.34}$$

From (2.34) and using a triangular inequality, we get

$$\begin{aligned}
\varepsilon &\leq d\big(Rx_{2m(i)}, Rx_{2n(i)}\big) \\
&\leq d\big(Rx_{2m(i)}, Rx_{2n(i)-2}\big) + d\big(Rx_{2n(i)-2}, Rx_{2n(i)-1}\big) + d\big(Rx_{2n(i)-1}, Rx_{2n(i)}\big) \\
&< \varepsilon + d\big(Rx_{2n(i)-2}, Rx_{2n(i)-1}\big) + d\big(Rx_{2n(i)-1}, Rx_{2n(i)}\big).
\end{aligned} \tag{2.35}$$

Letting $i \to +\infty$ in the previous inequality and using (2.13), we obtain

$$\lim_{i\to+\infty} d\left(Rx_{2m(i)}, Rx_{2n(i)}\right) = \varepsilon. \tag{2.36}$$

Again, a triangular inequality gives us

$$\left|d\left(Rx_{2n(i)}, Rx_{2m(i)-1}\right) - d\left(Rx_{2n(i)}, Rx_{2m(i)}\right)\right| \le d\left(Rx_{2m(i)-1}, Rx_{2m(i)}\right). \tag{2.37}$$

Letting $i \to +\infty$ in the above inequality and using (2.13) and (2.36), we get

$$\lim_{i\to+\infty} d\left(Rx_{2n(i)}, Rx_{2m(i)-1}\right) = \varepsilon. \tag{2.38}$$

On the other hand, we have

$$\begin{aligned} d\left(Rx_{2n(i)}, Rx_{2m(i)}\right) &\le d\left(Rx_{2n(i)}, Rx_{2n(i)+1}\right) + d\left(Rx_{2n(i)+1}, Rx_{2m(i)}\right) \\ &= d\left(Rx_{2n(i)}, Rx_{2n(i)+1}\right) + d\left(Tx_{2n(i)}, Sx_{2m(i)-1}\right). \end{aligned} \tag{2.39}$$

Then, from (2.13), (2.36), and the continuity of Φ_1, we get by letting $i \to +\infty$ in the above inequality

$$\Phi_1(\varepsilon) \le \lim_{i\to+\infty} \Phi_1\left(d\left(Sx_{2m(i)-1}, Tx_{2n(i)}\right)\right). \tag{2.40}$$

Now, using the considered contractive condition (2.5) for $x = x_{2m(i)-1}$ and $y = x_{2n(i)}$, we have

$$\begin{aligned} \Phi_1\left(d\left(Sx_{2m(i)-1}, Tx_{2n(i)}\right)\right) \le{}& \psi_1\Big(d\left(Rx_{2m(i)-1}, Rx_{2n(i)}\right), d\left(Rx_{2m(i)-1}, Rx_{2m(i)}\right), \\ & d\left(Rx_{2n(i)}, Rx_{2n(i)+1}\right), \\ & \frac{1}{2}\left[d\left(Rx_{2m(i)-1}, Rx_{2n(i)+1}\right) + d\left(Rx_{2n(i)}, Rx_{2m(i)}\right)\right]\Big) \\ & - \psi_2\Big(d\left(Rx_{2m(i)-1}, Rx_{2n(i)}\right), d\left(Rx_{2m(i)-1}, Rx_{2m(i)}\right), \\ & d\left(Rx_{2n(i)}, Rx_{2n(i)+1}\right), \\ & \frac{1}{2}\left[d\left(Rx_{2m(i)-1}, Rx_{2n(i)+1}\right) + d\left(Rx_{2n(i)}, Rx_{2m(i)}\right)\right]\Big). \end{aligned} \tag{2.41}$$

Then, from (2.13), (2.38), and the continuities of ψ_1 and ψ_2, we get by letting $i \to +\infty$ in the above inequality

$$\lim_{i\to+\infty} \Phi_1\left(d\left(Sx_{2m(i)-1}, Tx_{2n(i)}\right)\right) \le \psi_1(\varepsilon, 0, 0, \varepsilon) - \psi_2(\varepsilon, 0, 0, \varepsilon) \le \Phi_1(\varepsilon) - \psi_2(\varepsilon, 0, 0, \varepsilon). \tag{2.42}$$

Now, combining (2.40) with the previous inequality, we get

$$\Phi_1(\varepsilon) \leq \Phi_1(\varepsilon) - \psi_2(\varepsilon, 0, 0, \epsilon), \tag{2.43}$$

which implies that $\psi_2(\varepsilon, 0, 0, \epsilon) = 0$, that is a contradiction since $\varepsilon > 0$. We deduce that $\{Rx_n\}$ is a Cauchy sequence.

Step 3. We claim existence of a coincidence point.

Since $\{Rx_n\}$ is a Cauchy sequence in the complete metric space (X, d), there exists $u \in X$ such that

$$\lim_{n \to +\infty} Rx_n = u. \tag{2.44}$$

From (2.44) and the continuity of R, we get

$$\lim_{n \to +\infty} R(Rx_n) = Ru. \tag{2.45}$$

By the triangular inequality, we have

$$d(Ru, Tu) \leq d(Ru, R(Rx_{2n+1})) + d(R(Tx_{2n}), T(Rx_{2n})) + d(T(Rx_{2n}), Tu). \tag{2.46}$$

On the other hand, we have

$$Rx_{2n} \longrightarrow u, \qquad Tx_{2n} \longrightarrow u \quad \text{as} \quad n \to +\infty. \tag{2.47}$$

Since R and T are compatible mappings, this implies that

$$\lim_{n \to +\infty} d(R(Tx_{2n}), T(Rx_{2n})) = 0. \tag{2.48}$$

Now, from the continuity of T and (2.44), we have

$$\lim_{n \to +\infty} d(T(Rx_{2n}), Tu) = 0. \tag{2.49}$$

Combining (2.45), (2.48), and (2.49) and letting $n \to +\infty$ in (2.46), we obtain

$$d(Ru, Tu) \leq 0, \tag{2.50}$$

that is,

$$Ru = Tu. \tag{2.51}$$

Again, by a triangular inequality, we have

$$d(Ru, Su) \leq d(Ru, R(Rx_{2n+2})) + d(R(Sx_{2n+1}), S(Rx_{2n+1})) + d(S(Rx_{2n+1}), Su). \tag{2.52}$$

On the other hand, we have

$$Rx_{2n+1} \longrightarrow u, \qquad Sx_{2n+1} \longrightarrow u \quad \text{as } n \longrightarrow +\infty. \tag{2.53}$$

Since R and S are compatible mappings, this implies that

$$\lim_{n\to+\infty} d(R(Sx_{2n+1}), S(Rx_{2n+1})) = 0. \tag{2.54}$$

Now, from the continuity of S and (2.44), we have

$$\lim_{n\to+\infty} d(S(Rx_{2n+1}), Su) = 0. \tag{2.55}$$

Combining (2.45), (2.54), and (2.55) and letting $n \to +\infty$ in (2.52), we obtain

$$d(Ru, Su) \leq 0, \tag{2.56}$$

that is,

$$Ru = Su. \tag{2.57}$$

Finally, from (2.51) and (2.57), we have

$$Tu = Ru = Su, \tag{2.58}$$

that is, u is a coincidence point of T, S, and R. This makes end to the proof. □

In the next theorem, we omit the continuity hypotheses satisfied by T, S, and R.

Definition 2.7. Let $(X, d, \preceq)$ be a partially ordered metric space. We say that X is regular if and only if the following hypothesis holds: if $\{z_n\}$ is a nondecreasing sequence in X with respect to $\preceq$ such that $z_n \to z \in X$ as $n \to +\infty$, then $z_n \preceq z$ for all $n \in \mathbb{N}$.

Now, our second result is the following.

Theorem 2.8. *Let $(X, d, \preceq)$ be an ordered complete metric space. Let $T, S, R : X \to X$ be given mappings satisfying for every pair $(x, y) \in X \times X$ such that Rx and Ry are comparable,*

$$\begin{aligned}\Phi_1(d(Sx, Ty)) \leq{}& \psi_1\left(d(Rx, Ry), d(Rx, Sx), d(Ry, Ty), \frac{1}{2}[d(Rx, Ty) + d(Ry, Sx)]\right)\\ &- \psi_2\left(d(Rx, Ry), d(Rx, Sx), d(Ry, Ty), \frac{1}{2}[d(Rx, Ty) + d(Ry, Sx)]\right),\end{aligned} \tag{2.59}$$

where ψ_1 and ψ_2 are generalized altering distance functions and $\Phi_1(x) = \psi_1(x,x,x,x)$. We assume the following hypotheses:

(i) *X is regular,*

(ii) *T and S are weakly increasing with respect to R,*

(iii) *RX is a complete subspace of (X,d),*

(iv) *$TX \subseteq RX$, $SX \subseteq RX$.*

Then, T, S, and R have a coincidence point.

Proof. Following the proof of Theorem 2.6, we have $\{Rx_n\}$ is a Cauchy sequence in (RX,d). Since RX is a complete, there exists $u = Rv$, $v \in X$ such that

$$\lim_{n\to+\infty} Rx_n = u = Rv. \tag{2.60}$$

Since $\{Rx_n\}$ is a nondecreasing sequence and X is regular, it follows from (2.60) that $Rx_n \preceq Rv$ for all $n \in \mathbb{N}$. Hence, we can apply the contractive condition (2.5). Then, for $x = v$ and $y = x_{2n}$, we obtain

$$\begin{aligned}
\Phi_1(d(Sv,Rx_{2n+1})) &= \Phi_1(d(Sv,Tx_{2n})) \\
&\le \psi_1\Big(d(Rv,Rx_{2n}), d(Rv,Sv), d(Rx_{2n},Rx_{2n+1}), \\
&\qquad \frac{1}{2}[d(Rv,Tx_{2n}) + d(Rx_{2n},Sv)]\Big) \\
&\quad - \psi_2\Big(d(Rv,Rx_{2n}), d(Rv,Sv), d(Rx_{2n},Rx_{2n+1}), \\
&\qquad \frac{1}{2}[d(Rv,Tx_{2n}) + d(Rx_{2n},Sv)]\Big).
\end{aligned} \tag{2.61}$$

Letting $n \to +\infty$ in the above inequality and using (2.13), (2.60), and the properties of ψ_1 and ψ_2, we obtain

$$\begin{aligned}
\Phi_1(d(Sv,Rv)) &\le \psi_1\Big(0, d(Rv,Sv), 0, \frac{1}{2}d(Rv,Sv)\Big) - \psi_2\Big(0, d(Rv,Sv), 0, \frac{1}{2}d(Rv,Sv)\Big) \\
&\le \Phi_1(d(Sv,Rv)) - \psi_2\Big(0, d(Rv,Sv), 0, \frac{1}{2}d(Rv,Sv)\Big).
\end{aligned} \tag{2.62}$$

This implies that $\psi_2(0, d(Rv,Sv), 0, (1/2)d(Rv,Sv)) = 0$, which gives us that $d(Rv,Sv) = 0$, that is,

$$Rv = Sv. \tag{2.63}$$

Similarly, for $x = x_{2n+1}$ and $y = v$, we obtain

$$\begin{aligned}\Phi_1(d(Rx_{2n+2}, Tv)) &= \Phi_1(d(Sx_{2n+1}, Tv)) \\ &\leq \psi_1\Big(d(Rx_{2n+2}, Rv), d(Rx_{2n+1}, Rx_{2n+2}), d(Rv, Tv), \\ &\qquad \frac{1}{2}[d(Rx_{2n+1}, Tv) + d(Rv, Sx_{2n+1})]\Big) \\ &\quad - \psi_2\Big(d(Rx_{2n+2}, Rv), d(Rx_{2n+1}, Rx_{2n+2}), d(Rv, Tv), \\ &\qquad \frac{1}{2}[d(Rx_{2n+1}, Tv) + d(Rv, Sx_{2n+1})]\Big).\end{aligned} \tag{2.64}$$

Letting $n \to +\infty$ in the above inequality, we get

$$\begin{aligned}\Phi_1(d(Rv, Tv)) &\leq \psi_1\left(0, 0, d(Rv, Tv), \frac{1}{2}d(Rv, Tv)\right) - \psi_2\left(0, 0, d(Rv, Tv), \frac{1}{2}d(Rv, Tv)\right) \\ &\leq \Phi_1(d(Rv, Tv)) - \psi_2\left(0, 0, d(Rv, Tv), \frac{1}{2}d(Rv, Tv)\right).\end{aligned} \tag{2.65}$$

This implies that $\psi_2(0, 0, d(Rv, Tv), (1/2)d(Rv, Tv)) = 0$, and then,

$$Rv = Tv. \tag{2.66}$$

Now, combining (2.63) and (2.66), we obtain

$$Rv = Tv = Sv. \tag{2.67}$$

Hence, v is a coincidence point of T, S, and R. This makes end to the proof. □

Now, we give a sufficient condition that assures the uniqueness of the common coincidence point of $\{T, R\}$ and $\{S, R\}$.

Theorem 2.9. *Under the hypotheses of Theorem 2.6 (resp., Theorem 2.8) and suppose that* $(SX, \preceq)$ *is a totally ordered set and* S *is one-to-one mapping, then one obtains a unique common coincidence point of* $\{T, R\}$ *and* $\{S, R\}$.

Proof. Following the proof of Theorem 2.6 (resp., Theorem 2.8), the set of common coincidence points of $\{T, R\}$ and $\{S, R\}$ is nonempty. Let $\mu, \nu \in X$be two common coincidence points of $\{T, R\}$ and $\{S, R\}$, that is,

$$T\mu = R\mu = S\mu, \qquad T\nu = R\nu = S\nu. \tag{2.68}$$

This implies that $R\mu, R\nu \in SX$, and then, $R\mu$ and $R\nu$ are comparable with respect to $\preceq$. Then, we can apply the contractive condition (2.5). We have

$$\begin{aligned}\Phi_1(d(S\mu,T\nu)) &\leq \psi_1\left(d(R\mu,R\nu), d(R\mu,S\mu), d(R\nu,T\nu), \frac{1}{2}[d(R\mu,T\nu)+d(R\nu,S\mu)]\right)\\ &\quad - \psi_2\left(d(R\mu,R\nu), d(R\mu,S\mu), d(R\nu,T\nu), \frac{1}{2}[d(R\mu,T\nu)+d(R\nu,S\mu)]\right)\\ &= \psi_1(d(S\mu,T\nu),0,0,d(S\mu,T\nu)) - \psi_2(d(S\mu,T\nu),0,0,d(S\mu,T\nu))\\ &\leq \Phi_1(d(S\mu,T\nu)) - \psi_2(d(S\mu,T\nu),0,0,d(S\mu,T\nu)).\end{aligned} \tag{2.69}$$

This implies that $\psi_2(d(S\mu,T\nu),0,0,d(S\mu,T\nu))$, which gives us that $d(S\mu,T\nu) = 0$, that is, $S\mu = T\nu$. From (2.68), we get $S\mu = S\nu$. Since S is one-to-one, we have $\mu = \nu$. This makes end to the proof. □

Now, it is easy to state a corollary of Theorem 2.6 or Theorem 2.8 involving contractions of integral type.

Corollary 2.10. *Let R, S and T satisfy the conditions of Theorem 2.6 or Theorem 2.8, except that condition* (2.5) *is replaced by the following: there exists a positive Lebesgue integrable function u on $\mathbb{R}_+$ such that $\int_0^\varepsilon u(t)dt > 0$ for each $\varepsilon > 0$ and that*

$$\begin{aligned}\int_0^{\Phi_1(d(Sx,Ty))} u(t)dt &\leq \int_0^{\psi_1(d(Rx,Ry),d(Rx,Sx),d(Ry,Ty),(1/2)[d(Rx,Ty)+d(Ry,Sx)])} u(t)dt\\ &\quad - \int_0^{\psi_2(d(Rx,Ry),d(Rx,Sx),d(Ry,Ty),(1/2)[d(Rx,Ty)+d(Ry,Sx)])} u(t)dt.\end{aligned} \tag{2.70}$$

Then, T, S, and R have a coincidence point.

If $R : X \to X$ is the identity mapping, we can deduce easily the following common fixed point results.

The next result is an immediate consequence of Theorem 2.6.

Corollary 2.11. *Let $(X,d,\preceq)$ be an ordered complete metric space. Let $T,S : X \to X$ be given mappings satisfying for every pair $(x,y) \in X \times X$ such that x and y are comparable,*

$$\begin{aligned}\Phi_1(d(Sx,Ty)) &\leq \psi_1\left(d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Sx)]\right)\\ &\quad - \psi_2\left(d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Sx)]\right),\end{aligned} \tag{2.71}$$

where ψ_1 and ψ_2 are generalized altering distance functions and $\Phi_1(x) = \psi_1(x,x,x,x)$. One assumes the following hypotheses:

(i) *T and S are continuous,*

(ii) *T and S are weakly increasing.*

Then, T and S have a common fixed point, that is, there exists $u \in X$ such that $u = Tu = Su$.

The following result is an immediate consequence of Theorem 2.8.

Corollary 2.12. *Let $(X,d,\preceq)$ be an ordered complete metric space. Let $T,S : X \to X$ be given mappings satisfying for every pair $(x,y) \in X \times X$ such that x and y are comparable,*

$$\begin{aligned}\Phi_1(d(Sx,Ty)) \le{}& \psi_1\left(d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Sx)]\right)\\ &-\psi_2\left(d(x,y), d(x,Sx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Sx)]\right),\end{aligned} \tag{2.72}$$

where ψ_1 and ψ_2 are generalized altering distance functions and $\Phi_1(x) = \psi_1(x,x,x,x)$. One assume the following hypotheses:

(i) *X is regular,*

(ii) *T and S are weakly increasing.*

Then, T and S have a common fixed point.

Now, we give some examples to support our results.

Example 2.13. Let $X = \{4,5,6\}$ be endowed with the usual metric $d(x,y) = |x-y|$ for all $x,y \in X$, and $\preceq := \{(4,4),(5,5),(6,6),(6,4)\}$. Consider the mappings

$$S = T = \begin{pmatrix} 4 & 5 & 6 \\ 4 & 6 & 4 \end{pmatrix}, \qquad R = \begin{pmatrix} 4 & 5 & 6 \\ 4 & 5 & 6 \end{pmatrix}. \tag{2.73}$$

We define the functions $\psi_1, \psi_2 : [0,+\infty)^4 \to [0,+\infty)$ by

$$\begin{aligned}\psi_1(t_1,t_2,t_3,t_4) &= \frac{1}{4}(t_1+t_2+t_3+t_4),\\ \psi_2(t_1,t_2,t_3,t_4) &= \frac{1}{16}(t_1+t_2+t_3+t_4).\end{aligned} \tag{2.74}$$

Clearly, $\Phi_1(t) = t$ for all $t \ge 0$. Now, we will check that all the hypotheses required by Theorem 2.8 are satisfied.

(i) X is regular.

Let $\{z_n\}$ be a nondecreasing sequence in X with respect to $\preceq$ such that $z_n \to z \in X$ as $n \to +\infty$. We have $z_n \preceq z_{n+1}$ for all $n \in \mathbb{N}$.

(a) If $z_0 = 4$, then $z_0 = 4 \preceq z_1$. From the definition of $\preceq$, we have $z_1 = 4$. By induction, we get $z_n = 4$ for all $n \in \mathbb{N}$ and $z = 4$. Then, $z_n \preceq z$ for all $n \in \mathbb{N}$.

(b) If $z_0 = 5$, then $z_0 = 5 \preceq z_1$. From the definition of $\preceq$, we have $z_1 = 5$. By induction, we get $z_n = 5$ for all $n \in \mathbb{N}$ and $z = 5$. Then, $z_n \preceq z$ for all $n \in \mathbb{N}$.

(c) If $z_0 = 6$, then $z_0 = 6 \preceq z_1$. From the definition of $\preceq$, we have $z_1 \in \{6, 4\}$. By induction, we get $z_n \in \{6, 4\}$ for all $n \in \mathbb{N}$. Suppose that there exists $p \geq 1$ such that $z_p = 4$. From the definition of $\preceq$, we get $z_n = z_p = 4$ for all $n \geq p$. Thus, we have $z = 4$ and $z_n \preceq z$ for all $n \in \mathbb{N}$. Now, suppose that $z_n = 6$ for all $n \in \mathbb{N}$. In this case, we get $z = 6$ and $z_n \preceq z$ for all $n \in \mathbb{N}$.

Thus, we proved that, in all cases, we have $z_n \preceq z$ for all $n \in \mathbb{N}$. Then, X is regular.

(ii) T and S are weakly increasing.

Since $S = T$, we have to check that $Tx \preceq T(Tx)$ for all $x \in X$.

For $x = 4$, we have

$$T4 = 4 \preceq T(T4) = T4 = 4. \tag{2.75}$$

For $x = 5$, we have

$$T5 = 6 \preceq T(T5) = T6 = 4. \tag{2.76}$$

For $x = 6$, we have

$$T6 = 4 \preceq T(T6) = T4 = 4. \tag{2.77}$$

Thus, we proved that T and S are weakly increasing.

On the other hand, it is very easy to show that (2.5) is satisfied for all $x, y \in X$ such that $Rx \preceq Ry$.

Now, all the hypotheses of Theorem 2.8 are satisfied. Then R, S, and T have a coincidence point $u = 4$.

Note that inequality (2.5) is not satisfied for $x = 4$ and $y = 5$. Indeed,

$$\begin{aligned}
&\psi_1\left(d(R4,R5), d(R4,S4), d(R5,T5), \frac{1}{2}[d(R4,T5) + d(R5,S4)]\right) \\
&\quad - \psi_2\left(d(R4,R5), d(R4,S4), d(R5,T5), \frac{1}{2}[d(R4,T5) + d(R5,S4)]\right) \\
&\quad = \psi_1\left(1, 0, 1, \frac{3}{2}\right) - \psi_2\left(1, 0, 1, \frac{3}{2}\right) \\
&\quad = \frac{21}{32} \\
&\quad < \Phi_1(d(S4,T5)) = \Phi_1(2) = 2.
\end{aligned} \tag{2.78}$$

Then, Theorem 2.1 of Rao et al. [31] cannot be applied (for three maps) in this case.

Example 2.14. Let $X = \{(0,1), (1,0), (1,1)\} \subset \mathbb{R}^2$ with the Euclidean distance d_2. (X, d_2) is, obviously, a complete metric space. Moreover, we consider the order $\preceq$ in X given by

$\mathcal{R} = \{(x,x), x \in X\}$. Notice that the elements in X are only comparable to themselves, so $(X,\preceq)$ is regular. Also we consider $R,S,T : X \to X$ given by

$$\begin{gathered} T(1,0) = S(1,0) = R(1,0) = (1,0), \\ T(0,1) = S(0,1) = (1,0), \quad R(0,1) = (0,1), \\ T(1,1) = S(1,1) = R(1,1) = (1,1). \end{gathered} \tag{2.79}$$

It is easy that, for all $x \in X$, $TSx \preceq Sx$, and $STx \preceq Tx$, so the pair $\{S,T\}$ is weakly increasing.

Define $\psi_1, \psi_2 : [0,+\infty)^4 \to [0,+\infty)$ by

$$\begin{gathered} \psi_1(t_1,t_2,t_3,t_4) = \max\{t_1,t_2,t_3,t_4\}, \\ \psi_2(t_1,t_2,t_3,t_4) = \frac{1}{2}\max\{t_1,t_2,t_3,t_4\}. \end{gathered} \tag{2.80}$$

Then, $\psi_1(t) = t$ for all $t \geq 0$.

As the elements in X are only comparable to themselves, condition (2.5) appearing in Theorem 2.8 is, obviously, satisfied. Now, all the hypotheses of Theorem 2.8 are satisfied. $(1,1)$ and $(1,0)$ are the coincidence points of the mappings R, S, and T.

On the other hand, the inequality (2.5) is not satisfied for $x = (1,0)$ and $y = (1,1)$. Indeed,

$$\begin{aligned} &\psi_1\Big(d_2(R(1,0),R(1,1)), d_2(R(1,0),S(1,0)), d_2(R(1,1),T(1,1)), \\ &\qquad \frac{1}{2}[d_2(R(1,0),T(1,1)) + d_2(R(1,1),S(1,0))]\Big) \\ &\quad - \psi_2\Big(d_2(R(1,0),R(1,1)), d_2(R(1,0),S(1,0)), d_2(R(1,1),T(1,1)), \\ &\qquad\qquad \frac{1}{2}[d_2(R(1,0),T(1,1)) + d_2(R(1,1),S(1,0))]\Big) \\ &\quad = \psi_1(1,0,0,1) - \psi_2(1,0,0,1) \\ &\quad = \frac{1}{2} \\ &\quad < \Phi_1(d_2(S(1,0),T(1,1))) = \Phi_1(1) = 1. \end{aligned} \tag{2.81}$$

Then, again Theorem 2.1 of Rao et al. [31] cannot be applied (for three maps) in this case.

A number of fixed point results may be obtained by assuming different forms for the functions ψ_1 and ψ_2. In particular, fixed point results under various contractive conditions may be derived from the above theorems. For example, if we consider

$$\begin{aligned} \psi_1(x,y,z,t) &= k_1x^s + k_2y^s + k_3z^s + k_4t^4, \\ \psi_2(x,y,z,t) &= (1-k)\left[k_1x^s + k_2y^s + k_3z^s + k_4t^4\right], \end{aligned} \tag{2.82}$$

where $s > 0$ and $0 < k = k_1 + k_2 + k_3 + k_4 < 1$, we obtain the following results.

The next result is an immediate consequence of Corollaries 2.11 and 2.12.

Corollary 2.15. *Let $(X, d, \preceq)$ be an ordered complete metric space. Let $T, S : X \to X$ be given mappings satisfying for every pair $(x, y) \in X \times X$ such that x and y are comparable,*

$$[d(Sx,Ty)]^s \le k_1[d(x,y)]^s + k_2[d(x,Sx)]^s + k_3[d(y,Ty)]^s + k_4\left[\frac{1}{2}(d(x,Ty) + d(y,Sx))\right]^s, \tag{2.83}$$

where $s > 0$ and $0 < k = k_1 + k_2 + k_3 + k_4 < 1$. One assumes the following hypotheses:

(i) *T and S are continuous or X is regular,*

(ii) *T and S are weakly increasing.*

Then, T and S have a common fixed point, that is, there exists $u \in X$ such that $u = Tu = Su$.

Remark 2.16. Other fixed point results may also be obtained under specific choices of ψ_1 and ψ_2.

3. Application

Consider the integral equations:

$$\begin{aligned} u(t) &= \int_0^T K_1(t,s,u(s))ds + g(t), \quad t \in [0,T], \\ u(t) &= \int_0^T K_2(t,s,u(s))ds + g(t), \quad t \in [0,T], \end{aligned} \tag{3.1}$$

where $T > 0$.

The purpose of this section is to give an existence theorem for common solution of (3.1) using Corollary 2.15. This application is inspired by [9].

Previously, we consider the space $X = C(I)(I = [0,T])$ of continuous functions defined on I. Obviously, this space with the metric given by

$$d(x,y) = \sup_{t\in I}|x(t) - y(t)|, \quad \forall x, y \in C(I), \tag{3.2}$$

is a complete metric space. $C(I)$ can also be equipped with the partial order $\preceq$ given by

$$x, y \in C(I), \qquad x \preceq y \Longleftrightarrow x(t) \leq y(t), \quad \forall t \in I. \tag{3.3}$$

Moreover, in [28], it is proved that $(C(I), \preceq)$ is regular.

Now, we will prove the following result.

Theorem 3.1. *Suppose that the following hypotheses hold:*

(i) $K_1, K_2 : I \times I \times \mathbb{R} \to \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous,*

(ii) *for all* $t, s \in I$,

$$\begin{aligned} K_1(t,s,u(t)) &\leq K_2\left(t, s, \int_0^T K_1(s,\tau,u(\tau))d\tau + g(s)\right), \\ K_2(t,s,u(t)) &\leq K_1\left(t, s, \int_0^T K_2(s,\tau,u(\tau))d\tau + g(s)\right), \end{aligned} \tag{3.4}$$

(iii) *there exist* $k_1, k_2, k_3 \geq 0$ *such that*

$$\left|G_x(t) - F_y(t)\right| \leq k_1\left(y(t) - x(t)\right) + k_2\left|x(t) - G_x(t) - g(t)\right| + k_3\left|y(t) - F_y(t) - g(t)\right|, \tag{3.5}$$

where

$$F_x(t) = \int_0^T K_1(t,s,x(s))ds, \quad t \in I, \qquad G_x(t) = \int_0^T K_2(t,s,x(s))ds, \quad t \in I, \tag{3.6}$$

and $k_1 + k_2 + k_3 < 1$, *for every* $x, y \in X$ *and* $x \preceq y$ *and* $t \in I$.

Then, the integral equations (3.1) *have a solution* $u^* \in C(I)$.

Proof. Define $T, S : C(I) \to C(I)$ by

$$\begin{aligned} Tx(t) &= F_x(t) + g(t), \quad t \in I, \\ Sx(t) &= G_x(t) + g(t), \quad t \in I. \end{aligned} \tag{3.7}$$

Now, we will prove that T and S are weakly increasing. From (ii), for all $t \in I$, we have

$$\begin{aligned} Tx(t) &= \int_0^T K_1(t,s,x(s))ds + g(t) \\ &\leq \int_0^T K_2\left(t, s, \int_0^T K_1(s,\tau,x(\tau))d\tau + g(s)\right)ds + g(t) \\ &= \int_0^T K_2(t,s,Tx(s))ds + g(t) \\ &= STx(t). \end{aligned} \tag{3.8}$$

Similarly,

$$\begin{aligned} Sx(t) &= \int_0^T K_2(t,s,x(s))ds + g(t) \\ &\leq \int_0^T K_1\left(t,s,\int_0^T K_2(s,\tau,x(\tau))d\tau + g(s)\right)ds + g(t) \\ &= \int_0^T K_1(t,s,Sx(s))ds + g(t) \\ &= TSx(t). \end{aligned} \tag{3.9}$$

Then, we have $Tx \preceq STx$ and $Sx \preceq TSx$ for all $x \in C(I)$. This implies that T and S are weakly increasing.

Now, for all $x, y \in C(I)$ such that $x \preceq y$, by (iii), we have

$$\begin{aligned} \left|Sx(t) - Ty(t)\right| &= \left|G_x(t) - F_y(t)\right| \\ &\leq k_1\left(y(t) - x(t)\right| + k_2\left|x(t) - G_x(t) - g(t)\right| + k_3\left|y(t) - F_y(t) - g(t)\right|. \end{aligned} \tag{3.10}$$

Hence

$$\begin{aligned} d(Sx,Ty) &= \sup_{t\in[0,T]}\left|Sx(t) - Ty(t)\right| \\ &\leq k_1 \sup_{t\in[0,T]}\left|x(t) - y(t)\right| + k_2 \sup_{t\in[0,T]}\left|x(t) - G_x(t) - g(t)\right| + k_3 \sup_{t\in[0,T]}\left|y(t) - F_y(t) - g(t)\right| \\ &= k_1 \sup_{t\in[0,T]}\left|x(t) - y(t)\right| + k_2 \sup_{t\in[0,T]}\left|x(t) - Sx(t)\right| + k_3 \sup_{t\in[0,T]}\left|y(t) - Ty(t)\right|. \end{aligned} \tag{3.11}$$

Then

$$d(Sx,Ty) \leq k_1 d(x,y) + k_2 d(x,Sx) + k_3 d(y,Ty) \tag{3.12}$$

for all $x, y \in X$ such that $y \preceq x$.

This implies that, for all $x, y \in C(I)$ such that $x \preceq y$,

$$d(Sx,Ty) \leq k_1 d(x,y) + k_2 d(x,Sx) + k_3 d(y,Ty). \tag{3.13}$$

Hence the contractive condition required by Corollary 2.15 is satisfied with $s = 1$, $k_4 = 0$, and $k_1 + k_2 + k_3 < 1$.

Now, all the required hypotheses of Corollary 2.15 are satisfied. Then, there exists $u^* \in C(I)$, a common fixed point of T and S, that is, u^* is a solution to (3.1). □

References

[1] M. S. Khan, M. Swaleh, and S. Sessa, "Fixed point theorems by altering distances between the points," *Bulletin of the Australian Mathematical Society*, vol. 30, no. 1, pp. 1–9, 1984.

[2] I. Y. Alber and S. Guerre-Delabriere, "Principles of weakly contractive maps in Hilbert spaces," *Operator Theory*, vol. 98, pp. 7–22, 1997.

[3] B. E. Rhoades, "Some theorems on weakly contractive maps," *Nonlinear Analysis*, vol. 47, pp. 2683–2693, 2001.

[4] M. Abbas and M. A. Khan, "Common fixed point theorem of two mappings satisfying a generalized weak contractive condition," *International Journal of Mathematics and Mathematical Sciences*, vol. 2009, Article ID 131068, 9 pages, 2009.

[5] B. S. Choudhury, "A common unique fixed point result in metric spaces involving generalised altering distances," *Mathematical Communications*, vol. 10, no. 2, pp. 105–110, 2005.

[6] P. N. Dutta and B. S. Choudhury, "A generalisation of contraction principle in metric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 406368, 8 pages, 2008.

[7] Q. Zhang and Y. Song, "Fixed point theory for generalized φ-weak contractions," *Applied Mathematics Letters*, vol. 22, no. 1, pp. 75–78, 2009.

[8] R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, "Generalized contractions in partially ordered metric spaces," *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.

[9] I. Altun and H. Simsek, "Some fixed point theorems on ordered metric spaces and application," *Fixed Point Theory and Applications*, vol. 2010, Article ID 621469, 17 pages, 2010.

[10] A. Amini-Harandi and H. Emami, "A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 72, no. 5, pp. 2238–2242, 2010.

[11] H. Aydi, H. K. Nashine, B. Samet, and H. Yazidi, "Coincidence and common fixed point results in partially ordered cone metric spaces and applications to integral equations," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 74, no. 17, pp. 6814–6825, 2011.

[12] I. Beg and A. R. Butt, "Fixed point for set-valued mappings satisfying an implicit relation in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 71, no. 9, pp. 3699–3704, 2009.

[13] I. Beg and A. R. Butt, "Fixed points for weakly compatible mappings satisfying an implicit relation in partially ordered metric spaces," *Carpathian Journal of Mathematics*, vol. 25, no. 1, pp. 1–12, 2009.

[14] T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 65, no. 7, pp. 1379–1393, 2006.

[15] L. Ćirić, N. Cakić, M. Rajović, and J. S. Ume, "Monotone generalized nonlinear contractions in partially ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2008, Article ID 131294, 11 pages, 2008.

[16] V. Lakshmikantham and L. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 70, no. 12, pp. 4341–4349, 2009.

[17] B. C. Dhage, D. O'Regan, and R. P. Agarwal, "Common fixed point theorems for a pair of countably condensing mappings in ordered Banach spaces," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 16, no. 3, pp. 243–248, 2003.

[18] J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 71, no. 7-8, pp. 3403–3410, 2009.

[19] J. Harjani and K. Sadarangani, "Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 72, no. 3-4, pp. 1188–1197, 2010.

[20] H. K. Nashine, "New Fixed point theorems for mappings satisfying generalized weakly contractive condition with weaker control functions," *Annales Polonici Mathematici*, vol. 104, pp. 109–119, 2012.

[21] H. K. Nashine and I. Altun, "Fixed point theorems for generalized weakly contractive condition in ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2011, Article ID 132367, 20 pages, 2011.

[22] H. K. Nashine and I. Altun, "Fixed point theorems for generalized weakly contractive condition in ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2011, Article ID 132367, 2011.

[23] H. K. Nashine and B. Samet, "Fixed point results for mappings satisfying ψ, ϕ-weakly contractive condition in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 74, no. 6, pp. 2201–2209, 2011.

[24] H. K. Nashine, B. Samet, and J. K. Kim, "Fixed point results for contractions involving generalized altering distances in ordered metric spaces," *Fixed Point Theory and Applications*, vol. 2011, article 5, 2011.

[25] H. K. Nashine, B. Samet, and C. Vetro, "Monotone generalized nonlinear contractions and fixed point theorems in ordered metric spaces," *Mathematical and Computer Modelling*, vol. 54, no. 1-2, pp. 712–720, 2011.

[26] H. K. Nashine, B. Samet, and C. Vetro, "Fixed point theorems in partially ordered metric spaces and existence results for integral equations ," *Numerical Functional Analysis and Optimization*. In press.

[27] H. K. Nashine and W. Shatanawi, "Coupled common fixed point theorems for a pair of commuting mappings in partially ordered complete metric spaces," *Computers and Mathematics with Applications*, vol. 62, no. 4, pp. 1984–1993, 2011.

[28] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[29] J. J. Nieto and R. Rodríguez-López, "Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations," *Acta Mathematica Sinica*, vol. 23, no. 12, pp. 2205–2212, 2007.

[30] J. J. Nieto, R. L. Pouso, and R. Rodríguez-López, "Fixed point theorems in ordered abstract spaces," *Proceedings of the American Mathematical Society*, vol. 135, no. 8, pp. 2505–2517, 2007.

[31] K. P. R. Rao, G. R. Babu, and D. V. Babu, "Common fixed point theorems through generalized altering distance functions," *Mathematical Communications*, vol. 13, no. 1, pp. 67–73, 2008.

[32] D. O'Regan and A. Petruşel, "Fixed point theorems for generalized contractions in ordered metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1241–1252, 2008.

[33] A. C. M. Ran and M. C. B. Reurings, "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proceedings of the American Mathematical Society*, vol. 132, no. 5, p. 1435–1443 (electronic), 2004.

[34] B. Samet, "Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods and Applications A*, vol. 72, no. 12, pp. 4508–4517, 2010.

[35] B. Samet and H. Yazidi, "Coupled fixed point theorems in partially ordered ε-chainable metric spaces," *The Journal of Mathematics and Computer Science*, vol. 1, no. 3, pp. 142–151, 2010.

[36] M. Turinici, "Abstract comparison principles and multivariable Gronwall-Bellman inequalities," *Journal of Mathematical Analysis and Applications*, vol. 117, no. 1, pp. 100–127, 1986.

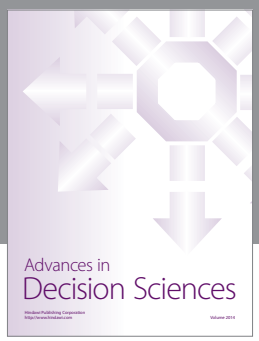
[37] G. Jungck, "Compatible mappings and common fixed points," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771–779, 1986.

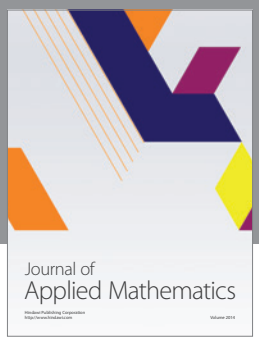
[38] B. C. Dhage, "Condensing mappings and applications to existence theorems for common solution of differential equations," *Bulletin of the Korean Mathematical Society*, vol. 36, no. 3, pp. 565–578, 1999.

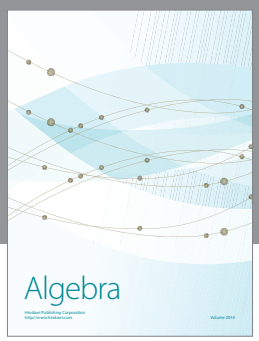
[39] D. W. Boyd and J. S. W. Wong, "On nonlinear contractions," *Proceedings of the American Mathematical Society*, vol. 20, pp. 458–464, 1969.

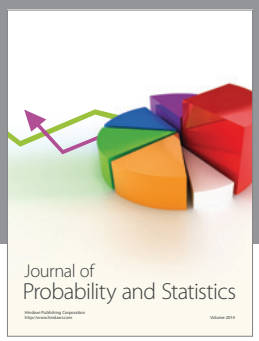
[40] S. Reich, "Some fixed point problems," *Atti della Accademia Nazionale dei Lincei*, vol. 57, no. 3-4, pp. 194–198, 1974.

[41] W. Shatanawi and B. Samet, "On ψ, φ-weakly contractive condition in partially ordered metric spaces," *Computers and Mathematics with Applications*, vol. 62, no. 8, pp. 3204–3214, 2011.









Hindawi

Submit your manuscripts at
http://www.hindawi.com