Research Article

On Concircular ϕ -Recurrent K -Contact Manifold Admitting Semisymmetric Metric Connection

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Received 29 March 2012; Accepted 29 May 2012

Academic Editor: J. Dydak

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In the present paper, we have studied ϕ -recurrent and concircular ϕ -recurrent K -contact manifold with respect to semisymmetric metric connection and obtained some interesting results.

1. Introduction

The idea of semisymmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [1]. In [2], Hayden introduced idea of metric connection with torsion on a Riemannian manifold. Further, some properties of semisymmetric metric connection has been studied by Yano [3]. In [4], Golab defined and studied quarter-symmetric connection on a differentiable manifold with affine connection, which generalizes the idea of semisymmetric connection. Various properties of semisymmetric metric connection and quarter-symmetric metric connection have been studied by many geometers like Sharfuddin and Hussain [5], Amur and Pujar [6], Rastogi [7, 8], Mishra and Pandey [9], Bagewadi et al. [10–14], De et al. [15, 16], and many others.

The notion of local symmetry of a Riemannian manifold has been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, Takahashi [17] introduced the notion of local ϕ -symmetry on a Sasakian manifold. Generalizing the notion of ϕ -symmetry, De et al. [18] introduced the notion of ϕ -recurrent Sasakian manifolds.

The paper is organized as follows. Section 2 is devoted to preliminaries. In Section 3, we study semisymmetric metric connection in a K -contact manifold. In Section 4, it is proved that a ϕ -recurrent K -contact manifold with respect to semisymmetric metric connection is an

Einstein manifold. Finally, in Section 5 it is also shown that concircular ϕ -recurrent K -contact manifold admitting semisymmetric metric connection is an Einstein manifold, and the characteristic vector field ξ and the vector field ρ associated to the 1-form A are codirectional.

2. Preliminaries

An n -dimensional differentiable manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type $(1, 1)$, a vector field ξ , and a 1-form η on M , respectively, such that,

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0. \quad (2.1)$$

Thus a manifold M equipped with this structure (ϕ, ξ, η) is called an almost contact manifold and is denoted by (M, ϕ, ξ, η) . If g is a Riemannian metric on an almost contact manifold M such that,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \quad (2.2)$$

where X, Y are vector fields defined on M , then, M is said to have an almost contact metric structure (ϕ, ξ, η, g) , and M with this structure is called an almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g) .

If on (M, ϕ, ξ, η, g) the exterior derivative of 1-form η satisfies

$$d\eta(X, Y) = g(X, \phi Y), \quad (2.3)$$

then (ϕ, ξ, η, g) is said to be a contact metric structure, and M equipped with a contact metric structure is called a contact metric manifold.

If moreover ξ is killing vector field on M , then, M is called a K -contact Riemannian manifold [19, 20]. A K -contact Riemannian manifold is called Sasakian [19], if the relation

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X \quad (2.4)$$

holds, where ∇ denotes the operator of covariant differentiation with respect to g .

In a K -contact manifold M , the following relations holds:

$$\nabla_X \xi = -\phi X, \quad (2.5)$$

$$g(R(X, Y)Z, \xi) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (2.6)$$

$$S(X, \xi) = (n - 1)\eta(X), \quad (2.7)$$

for all vector fields X, Y , and Z . Here R and S are the Riemannian curvature tensor and the Ricci tensor of M , respectively.

Definition 2.1. A K -contact manifold M is said to be ϕ -recurrent if there exists a nonzero 1-form A such that,

$$\phi^2((\nabla_W R)(X, Y)Z) = A(W)R(X, Y)Z, \quad (2.8)$$

where A is defined by $A(W) = g(W, \rho)$, and ρ is a vector field associated with the 1-form A .

Definition 2.2. A K -contact manifold M is said to be concircular ϕ -recurrent [12] if there exists a non-zero 1-form A such that,

$$\phi^2((\nabla_W \bar{C})(X, Y)Z) = A(W)\bar{C}(X, Y)Z, \quad (2.9)$$

where $\bar{C}$ is a concircular curvature tensor given by [21] as follows:

$$\bar{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \quad (2.10)$$

where R is the Riemannian curvature tensor and r is the scalar curvature.

A linear connection $\tilde{\nabla}$ in an n -dimensional differentiable manifold M is said to be a semisymmetric connection if its torsion tensor T is of the form

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y] = \eta(Y)X - \eta(X)Y, \quad (2.11)$$

for all X, Y on TM . A semisymmetric connection $\tilde{\nabla}$ is called semisymmetric metric connection, if it further satisfies $\tilde{\nabla}g = 0$.

3. Semisymmetric Metric Connection in a K -Contact Manifold

A semisymmetric metric connection $\tilde{\nabla}$ in a K -contact manifold can be defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi, \quad (3.1)$$

where ∇ is the Levi-Civita connection on M [3].

A relation between the curvature tensor of M , with respect to the semisymmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection, ∇ is given by

$$\begin{aligned} \tilde{R}(X, Y)Z &= R(X, Y)Z + [g(\phi Y, Z)X - g(\phi X, Z)Y] + [g(Y, Z)\phi X - g(X, Z)\phi Y] \\ &\quad + [g(\phi X, \phi Z)Y - g(\phi Y, \phi Z)X] + [\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\xi, \end{aligned} \quad (3.2)$$

where $\tilde{R}$ and R are the Riemannian curvatures of the connections $\tilde{\nabla}$ and ∇ , respectively.

From (3.2), it follows that

$$\tilde{S}(Y, Z) = S(Y, Z) - (n-2)g(Y, Z) + (n-2)g(\phi Y, Z) + (n-2)\eta(Y)\eta(Z), \quad (3.3)$$

where $\tilde{S}$ and S are the Ricci tensors of the connections $\tilde{\nabla}$ and ∇ , respectively.

Contracting (3.3), we get

$$\tilde{r} = r - (n-1)(n-2), \quad (3.4)$$

where $\tilde{r}$ and r are the scalar curvatures of the connections $\tilde{\nabla}$ and ∇ , respectively.

4. ϕ -Recurrent K -Contact Manifold with respect to Semisymmetric Metric Connection

A K -contact manifold is called ϕ -recurrent with respect to the semisymmetric metric connection if its curvature tensor $\tilde{R}$ satisfies the following condition:

$$\phi^2 \left((\tilde{\nabla}_W \tilde{R})(X, Y)Z \right) = A(W) \tilde{R}(X, Y)Z. \quad (4.1)$$

By virtue of (2.1) and (4.1), we have

$$-(\tilde{\nabla}_W \tilde{R})(X, Y)Z + \eta \left((\tilde{\nabla}_W \tilde{R})(X, Y)Z \right) \xi = A(W) \tilde{R}(X, Y)Z, \quad (4.2)$$

from which, it follows that

$$-g \left((\tilde{\nabla}_W \tilde{R})(X, Y)Z, U \right) + \eta \left((\tilde{\nabla}_W \tilde{R})(X, Y)Z \right) g(\xi, U) = A(W) g(\tilde{R}(X, Y)Z, U). \quad (4.3)$$

Let $\{e_i\}$, $i = 1, 2, \dots, n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (4.3) and taking summation over i , $1 \leq i \leq n$, we get

$$-(\tilde{\nabla}_W \tilde{S})(Y, Z) + \sum_{i=1}^n \eta \left((\tilde{\nabla}_W \tilde{R})(e_i, Y)Z \right) \eta(e_i) = A(W) \tilde{S}(Y, Z). \quad (4.4)$$

Put $Z = \xi$, then the second term of (4.4) takes the following form:

$$\begin{aligned} g \left((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi \right) &= g \left(\tilde{\nabla}_W \tilde{R}(e_i, Y)\xi, \xi \right) - g \left(\tilde{R}(\tilde{\nabla}_W e_i, Y)\xi, \xi \right) \\ &\quad - g \left(\tilde{R}(e_i, \tilde{\nabla}_W Y)\xi, \xi \right) - g \left(\tilde{R}(e_i, Y)\tilde{\nabla}_W \xi, \xi \right). \end{aligned} \quad (4.5)$$

On simplification, we obtain $g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = 0$.

Now (4.4) implies that

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = -A(W) \tilde{S}(Y, \xi). \quad (4.6)$$

We know that

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi). \quad (4.7)$$

Using (3.3), (2.5), and (2.7) in the above relation, we get

$$\begin{aligned} \left(\tilde{\nabla}_W \tilde{S} \right) (Y, \xi) &= S(Y, \phi W) - S(Y, W) - (n-1)g(Y, \phi W) \\ &\quad + (n-1)g(Y, W) + 2(n-2)g(\phi Y, \phi W). \end{aligned} \quad (4.8)$$

In view of (4.6) and (4.8), we have

$$S(Y, W) - S(Y, \phi W) + (n-1)g(Y, \phi W) - (n-1)g(Y, W) - 2(n-2)g(\phi Y, \phi W) = (n-1)A(W)\eta(Y). \quad (4.9)$$

Again putting $Y = \phi Y$ in (4.9), we get

$$S(\phi Y, W) - S(\phi Y, \phi W) + (n-1)g(\phi Y, \phi W) - (n-1)g(\phi Y, W) + 2(n-2)g(Y, \phi W) = 0. \quad (4.10)$$

Interchanging Y and W in (4.10), we obtain

$$S(\phi W, Y) - S(\phi W, \phi Y) + (n-1)g(\phi W, \phi Y) - (n-1)g(\phi W, Y) + 2(n-2)g(W, \phi Y) = 0. \quad (4.11)$$

Adding (4.10) and (4.11) which on simplification, we have

$$S(Y, W) = (n-1)g(Y, W). \quad (4.12)$$

Therefore, we can state the following.

Theorem 4.1. *A ϕ -recurrent K -contact manifold with respect to semisymmetric metric connection is an Einstein manifold.*

5. Conircular ϕ -Recurrent K -Contact Manifold with respect to Semisymmetric Metric Connection

Let us consider a conircular ϕ -recurrent K -contact manifold with respect to the semisymmetric metric connection defined by

$$\phi^2 \left(\left(\tilde{\nabla}_W \tilde{\tilde{C}} \right) (X, Y) Z \right) = A(W) \tilde{\tilde{C}}(X, Y) Z, \quad (5.1)$$

where $\tilde{\tilde{C}}$ is a conircular curvature tensor with respect to the semisymmetric metric connection given by

$$\tilde{\tilde{C}}(X, Y) Z = \tilde{R}(X, Y) Z - \frac{\tilde{r}}{n(n-1)} [g(Y, Z) X - g(X, Z) Y]. \quad (5.2)$$

By virtue of (2.1) and (5.1), we have

$$-\left(\tilde{\nabla}_W \tilde{\tilde{C}}\right)(X, Y)Z + \eta\left(\left(\tilde{\nabla}_W \tilde{\tilde{C}}\right)(X, Y)Z\right)\xi = A(W)\tilde{\tilde{C}}(X, Y)Z, \quad (5.3)$$

from which, it follows that

$$-g\left(\left(\tilde{\nabla}_W \tilde{\tilde{C}}\right)(X, Y)Z, U\right) + \eta\left(\left(\tilde{\nabla}_W \tilde{\tilde{C}}\right)(X, Y)Z\right)g(\xi, U) = A(W)g\left(\tilde{\tilde{C}}(X, Y)Z, U\right), \quad (5.4)$$

where

$$\begin{aligned} \left(\tilde{\nabla}_W \tilde{\tilde{C}}\right)(X, Y)Z &= ((\nabla_W R)(X, Y)Z) + 3[g(Y, W)\eta(Z)X - g(X, W)\eta(Z)Y] \\ &\quad + 3[g(Y, Z)g(W, X) - g(X, Z)g(W, Y)]\xi \\ &\quad + 2[\eta(X)g(\phi W, Z)Y - \eta(Y)g(\phi W, Z)X] \\ &\quad + 2[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)]\phi W + [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]W \\ &\quad + 2\eta(W)[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)]\xi \\ &\quad + 2\eta(Z)\eta(W)[\eta(X)Y - \eta(Y)X] + g(Z, W)[\eta(Y)X - \eta(X)Y] \\ &\quad - g(W, R(X, Y)Z)\xi - \eta(X)R(W, Y)Z \\ &\quad - \eta(Y)R(X, W)Z - \eta(Z)R(X, Y)W \\ &\quad - \frac{\nabla_W r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]. \end{aligned} \quad (5.5)$$

Let $\{e_i\}$, $i = 1, 2, \dots, n$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = e_i$ in (5.4) and taking summation over i , $1 \leq i \leq n$, we get

$$\left(\tilde{\nabla}_W \tilde{\tilde{S}}\right)(Y, Z) - \frac{\tilde{\nabla}_W \tilde{r}}{n}g(Y, Z) = -\frac{\tilde{\nabla}_W \tilde{r}}{n(n-1)}[g(Y, Z) - \eta(Y)\eta(Z)] - A(W)\left[\tilde{\tilde{S}}(Y, Z) - \frac{\tilde{r}}{n}g(Y, Z)\right]. \quad (5.6)$$

Replacing Z by ξ in (5.6), we obtain

$$\left(\tilde{\nabla}_W \tilde{\tilde{S}}\right)(Y, \xi) = \frac{\tilde{\nabla}_W \tilde{r}}{n}\eta(Y) - A(W)\left[\tilde{\tilde{S}}(Y, \xi) - \frac{\tilde{r}}{n}\eta(Y)\right]. \quad (5.7)$$

We know that

$$\left(\tilde{\nabla}_W \tilde{\tilde{S}}\right)(Y, \xi) = \tilde{\nabla}_W \tilde{\tilde{S}}(Y, \xi) - \tilde{\tilde{S}}(\tilde{\nabla}_W Y, \xi) - \tilde{\tilde{S}}(Y, \tilde{\nabla}_W \xi). \quad (5.8)$$

Using (3.3), (2.5) and (2.7), the above relation becomes

$$\left(\tilde{\nabla}_W \tilde{S}\right)(Y, \xi) = S(Y, \phi W) - S(Y, W) - (n-1)g(Y, \phi W) + (n-1)g(Y, W) + 2(n-2)g(\phi Y, \phi W). \quad (5.9)$$

In view of (5.7) and (5.9), we obtain

$$\begin{aligned} S(Y, \phi W) - S(Y, W) - (n-1)g(Y, \phi W) + (n-1)g(Y, W) + 2(n-2)g(Y, W) - 2(n-2)\eta(Y)\eta(W) \\ = \frac{\nabla_W r}{n}\eta(Y) - A(W)\left[\frac{2(n-1)^2 - r}{n}\eta(Y)\right]. \end{aligned} \quad (5.10)$$

Replacing Y by ϕY in (5.10), we have

$$S(\phi Y, \phi W) - S(\phi Y, W) - (n-1)g(\phi Y, \phi W) + (n-1)g(\phi Y, W) + 2(n-2)g(\phi Y, W) = 0. \quad (5.11)$$

Interchanging Y and W in (5.11), we get

$$S(\phi W, \phi Y) - S(\phi W, Y) - (n-1)g(\phi W, \phi Y) + (n-1)g(\phi W, Y) + 2(n-2)g(\phi W, Y) = 0. \quad (5.12)$$

Adding (5.11) and (5.12), which on simplification, we have

$$S(Y, W) = (n-1)g(Y, W). \quad (5.13)$$

Thus, we obtain the following theorem.

Theorem 5.1. *A Concircular ϕ -recurrent K -contact manifold with respect to semisymmetric metric connection is an Einstein manifold.*

Next, from (5.3), one has

$$\left(\tilde{\nabla}_W \tilde{\tilde{C}}\right)(X, Y)Z = \eta\left(\left(\tilde{\nabla}_W \tilde{\tilde{C}}\right)(X, Y)Z\right)\xi - A(W)\tilde{\tilde{C}}(X, Y)Z. \quad (5.14)$$

Now, using (3.2), (3.4), (5.5), and Bianchi's identity in (5.14), one obtains

$$\begin{aligned} A(W)\eta(R(X, Y)Z) + A(X)\eta(R(Y, W)Z) + A(Y)\eta(R(W, X)Z) \\ = -A(W)[g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)] \\ - A(X)[g(\phi W, Z)\eta(Y) - g(\phi Y, Z)\eta(W)] \\ - A(Y)[g(\phi X, Z)\eta(W) - g(\phi W, Z)\eta(X)] \end{aligned}$$

$$\begin{aligned}
& + \frac{r - (n-1)(n-2)}{n(n-1)} A(W) [g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] \\
& + \frac{r - (n-1)(n-2)}{n(n-1)} A(X) [g(W, Z)\eta(Y) - g(Y, Z)\eta(W)] \\
& + \frac{r - (n-1)(n-2)}{n(n-1)} A(Y) [g(X, Z)\eta(W) - g(W, Z)\eta(X)].
\end{aligned} \tag{5.15}$$

Putting $Y = Z = e_i$ in (5.15) and taking summation over i , $1 \leq i \leq n$, one gets

$$\begin{aligned}
& \left[\frac{-n(n-1)(n-2) + r(n-2) - (n-1)(n-2)^2}{n(n-1)} \right] A(X)\eta(W) \\
& + \left[\frac{n(n-1)(n-2) - r(n-2) + (n-1)(n-2)^2}{n(n-1)} \right] A(W)\eta(X) \\
& = A(\phi W)\eta(X) - A(\phi X)\eta(W).
\end{aligned} \tag{5.16}$$

Replacing X by ξ in (5.16), one gets

$$\left[\frac{[r(n-2) + 2(n-1)(n-2)]^2 + n^2(n-1)^2}{n(n-1)[r(n-2) + 2(n-1)(n-2)]} \right] [A(W) - A(\xi)\eta(W)] = 0, \tag{5.17}$$

therefore

$$A(W) = \eta(W)\eta(\rho), \tag{5.18}$$

for any vector field W .

Hence, one states the following.

Theorem 5.2. *In a concircular ϕ -recurrent K -contact manifold admitting semisymmetric metric connection the characteristic vector field ξ and the vector field ρ associated to the 1-form A are co-directional and the 1-form A is given by (5.18).*

Acknowledgment

The authors express their thanks to DST (Department of Science and Technology), Government of India for providing financial assistance under major research project (no. SR/S4/MS:482/07).

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