Research Article

Coupled Fixed Point Theorems for Nonlinear Contractions in Partial Metric Spaces

**Sharifa Al-Sharif,¹ Mohammad Al-Khaleel,¹
and Mona Khandaqji²**

¹ Department of Mathematics, Yarmouk University, Irbid 21163, Jordan

² Department of Mathematics, Al-Hashmia University, Zarqa 13115, Jordan

Correspondence should be addressed to Sharifa Al-Sharif, sharifa@yu.edu.jo

Received 28 March 2012; Revised 22 August 2012; Accepted 26 August 2012

Academic Editor: Hari Mohan Srivastava

Copyright © 2012 Sharifa Al-Sharif et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We establish some results on the existence and uniqueness of coupled fixed point involving nonlinear contractive conditions in complete-ordered partial metric spaces.

1. Introduction

The concept of partial metric which is a generalized metric space was introduced by Matthews [1] in 1994, in which the distance between two identical elements needs not be zero. The existence of fixed point for contraction-type mappings on such spaces was considered by many authors [1–12]. A modified version of a Banach contraction mapping principle, more suitable to solve certain problems arising in computer science using the concept of partial metric space is given in [1].

Gnana Bhaskar and Lakshmikantham [13] introduced the concept of coupled fixed point of a mapping $F : X \times X \rightarrow X$ and proved some interesting coupled fixed point theorems for mapping satisfying the mixed monotone property. Later in [14], Lakshmikantham and Ćirić investigated some more coupled fixed point theorems in partially ordered sets. For more on coupled fixed point theory, we refer the reader to [2, 14–20].

First, we start by recalling some definitions and properties of partial metric spaces.

Definition 1.1 (see [9]). A partial metric on a nonempty set X is a function $p : X \times X \rightarrow R^+$ such that for all $x, y, z \in X$:

$$(p_1) \quad x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y),$$

$$(p_2) \quad p(x, x) \leq p(x, y),$$

$$(p_3) \ p(x, y) = p(y, x),$$

$$(p_4) \ p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

A partial metric space is a pair (X, p) such that X is a non empty set and p is a partial metric on X . Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p -balls $\{B_p(x, \epsilon), x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$. Matthews observed in [1, page 187] that a sequence (x_n) in a partial metric space (X, p) converges to some $x \in X$ with respect to p if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n)$. It is clear that if $p(x, y) = 0$, then from (p_1) , (p_2) , and (p_3) , $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

If p is a partial metric on X , then the function $p_s : X \times X \rightarrow R^+$ given by

$$p_s(x, y) = 2p(x, y) - p(x, x) - p(y, y), \quad (1.1)$$

is a metric on X .

Example 1.2 (see, e.g., [1, 7]). Consider $X = R^+$ with $p(x, y) = \max\{x, y\}$. Then, (R^+, p) is a partial metric space.

It is clear that p is not a (usual) metric. Note that in this case $p_s(x, y) = |x - y|$.

Definition 1.3 (see [1, Definition 5.2]). Let (X, p) be a partial metric space and let (x_n) be a sequence in X . Then, $\{x_n\}$ is called a Cauchy sequence if $\lim_{n, m \rightarrow \infty} p(x_m, x_n)$ exists (and is finite).

Definition 1.4 (see [1, Definition 5.3]). A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p , to a point $x \in X$, such that $p(x, x) = \log_{n, m \rightarrow \infty} p(x_m, x_n)$.

Example 1.5 (see [12]). Let $X := [0, 1] \cup [2, 3]$ and define $p : X \times X \rightarrow [0, \infty)$ by

$$p(x, y) = \begin{cases} \max\{x, y\}, & \{x, y\} \cap [2, 3] \neq \emptyset, \\ |x - y|, & \{x, y\} \subset [0, 1], \end{cases} \quad (1.2)$$

Then, (X, p) is a complete partial metric space.

It is well known (see, e.g., [1, page 194]) that a sequence in a partial metric space (X, p) is a Cauchy sequence in (X, p) if and only if it is a Cauchy sequence in the metric space (X, p_s) , and that a partial metric space (X, p) is complete if and only if the metric space (X, p_s) is complete. Furthermore, $\lim_{n \rightarrow \infty} p_s(x, x_n) = 0$ if and only if

$$p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n) = \lim_{n, m \rightarrow \infty} p(x_m, x_n). \quad (1.3)$$

Let (X, p) be a partial metric. We endow $X \times X$ with the partial metric q defined for $(x, y), (u, v) \in X \times X$ by

$$q((x, y), (u, v)) = p(x, u) + p(y, v). \quad (1.4)$$

A mapping $F : X \times X \rightarrow X$ is said to be continuous at $(x, y) \in X \times X$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(Bq((x, y), \delta)) \subseteq Bp(F(x, y), \epsilon)$.

In this paper, we establish some results on the existence and uniqueness of a coupled fixed point involving nonlinear contractive conditions in complete-ordered partial metric spaces analogous to some other results in [17, 18].

Before presenting our main results, we recall some basic concepts.

Definition 1.6 (see [8]). An element $x, y \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Definition 1.7 (see, Gnana Bhashkar and Lakshmikantham [13]). Let $(X, \leq)$ be a partially ordered set and $F : X \times X \rightarrow X$. The mapping F is said to have the mixed monotone property if

$$\begin{aligned} x_1, x_2 \in X, \quad x_1 \leq x_2 &\implies F(x_1, y) \leq F(x_2, y) \quad \text{for any } y \in X, \\ y_1, y_2 \in X, \quad y_1 \leq y_2 &\implies F(x, y_1) \geq F(x, y_2) \quad \text{for any } x \in X. \end{aligned} \quad (1.5)$$

Now, let us denote by Φ the set of all nondecreasing continuous functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ that satisfy

- (i) $\varphi(t) = 0$ if and only if $t = 0$,
- (ii) $\varphi(t + s) \leq \varphi(t) + \varphi(s)$, for all $t, s \in [0, \infty)$.

Again, let Ψ denote all functions $\psi : [0, \infty) \rightarrow [0, \infty)$ which satisfy $\lim_{t \rightarrow r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t \rightarrow 0^+} \psi(t) = 0$. It is an easy matter to see that $\Phi \subseteq \Psi$ and $(1/2)\varphi(t) \leq \varphi(t/2)$ for any $t \in [0, \infty)$.

2. Main Results

The aim of this work is to prove the following theorem.

Theorem 2.1. Let $(X, \leq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X and assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(d(F(x, y), F(u, v))) \leq \varphi(\alpha p(x, u) + \beta p(y, v)) - \psi(\alpha p(x, u) + \beta p(y, v)), \quad (2.1)$$

for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$, and $\alpha + \beta < 1$. Suppose either F is continuous or X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n .

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$, and $F(y_0, x_0) \leq y_0$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point. Furthermore, $p(x, x) = p(y, y) = 0$.

Proof. Choose $x_0, y_0 \in X$ and set $x_1 = F(x_0, y_0)$ and $F(y_0, x_0) = y_1$. Since $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, letting $x_2 = F(x_1, y_1)$ and $y_2 = F(y_1, x_1)$, we denote

$$\begin{aligned} F^2(x_0, y_0) &= F(F(x_0, y_0), F(y_0, x_0)) = F(x_1, y_1) = x_2, \\ F^2(y_0, x_0) &= F(F(y_0, x_0), F(x_0, y_0)) = F(y_1, x_1) = y_2, \end{aligned} \quad (2.2)$$

and due to the mixed monotone property of F , we have

$$x_2 = F(x_1, y_1) \geq F(x_0, y_0) = x_1, \quad y_2 = F(y_1, x_1) \leq F(y_0, x_0) = y_1. \quad (2.3)$$

Further, for $n = 1, 2, \dots$, we can easily verify that

$$\begin{aligned} x_0 \leq F(x_0, y_0) = x_1 \leq F^2(x_0, y_0) = x_2 \leq \dots \leq F(F^n(x_0, y_0), F^n(y_0, x_0)) = x_{n+1}, \\ y_0 \geq F(y_0, x_0) = y_1 \geq F^2(y_0, x_0) = y_2 \geq \dots \geq F(F^n(y_0, x_0), F^n(x_0, y_0)) = y_{n+1}. \end{aligned} \quad (2.4)$$

Since $x_n \geq x_{n-1}$ and $y_n \leq y_{n-1}$, from (2.1), we have

$$\begin{aligned} \varphi(p(x_n, x_{n+1})) &= \varphi(p(F(x_{n-1}, y_{n-1}), F(x_n, y_n))) \\ &\leq \varphi(\alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n)) - \varphi(\alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n)) \\ &\leq \varphi(\alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n)). \end{aligned} \quad (2.5)$$

Similarly, since $y_{n-1} \geq y_n$ and $x_{n-1} \leq x_n$, from (2.1), we also have

$$\begin{aligned} \varphi(p(y_n, y_{n+1})) &= \varphi(p(F(y_{n-1}, x_{n-1}), F(y_n, x_n))) \\ &\leq \varphi(\alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n)) - \varphi(\alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n)) \\ &\leq \varphi(\alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n)). \end{aligned} \quad (2.6)$$

Consequently, since φ is nondecreasing, using (2.5) and (2.6), we get

$$p(x_n, x_{n+1}) \leq \alpha p(x_{n-1}, x_n) + \beta p(y_{n-1}, y_n), \quad (2.7)$$

$$p(y_n, y_{n+1}) \leq \alpha p(y_{n-1}, y_n) + \beta p(x_{n-1}, x_n). \quad (2.8)$$

By adding (2.7) to (2.8), we have

$$p(x_n, x_{n+1}) + p(y_n, y_{n+1}) \leq (\alpha + \beta)(p(x_{n-1}, x_n) + p(y_{n-1}, y_n)). \quad (2.9)$$

Now, we will show that both x_n and y_n are Cauchy sequences. Note that

$$\begin{aligned}
 p(x_n, x_{n+1}) + p(y_n, y_{n+1}) &\leq (\alpha + \beta)(p(x_{n-1}, x_n) + p(y_{n-1}, y_n)) \\
 &\leq (\alpha + \beta)^2(p(x_{n-2}, x_{n-1}) + p(y_{n-2}, y_{n-1})) \\
 &\vdots \\
 &\leq (\alpha + \beta)^n(p(x_0, x_1) + p(y_0, y_1)).
 \end{aligned} \tag{2.10}$$

Consequently, if $p(x_0, x_1) + p(y_0, y_1) = 0$, then $x_1 = x_0 = F(x_0, y_0)$ and $y_1 = y_0 = F(y_0, x_0)$, that means (x_0, y_0) is a coupled fixed point of F . If $p(x_0, x_1) + p(y_0, y_1) > 0$, for each $m > n$, combining (2.7) and (2.8) using property p_4 , we have

$$\begin{aligned}
 p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) - \sum_{k=n+1}^{m-1} p(x_k, x_k) \\
 &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m), \\
 p(y_n, y_m) &\leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m) - \sum_{k=n+1}^{m-1} p(y_k, y_k) \\
 &\leq p(y_n, y_{n+1}) + p(y_{n+1}, y_{n+2}) + \cdots + p(y_{m-1}, y_m).
 \end{aligned} \tag{2.11}$$

Thus,

$$\begin{aligned}
 p(x_n, x_m) + p(y_n, y_m) &\leq p(x_n, x_{n+1}) + p(y_n, y_{n+1}) + p(x_{n+1}, x_{n+2}) + p(y_{n+1}, y_{n+2}) \\
 &\quad + \cdots + p(x_{m-1}, x_m) + p(y_{m-1}, y_m) \\
 &\leq \left((\alpha + \beta)^n + (\alpha + \beta)^{n+1} + \cdots + (\alpha + \beta)^{m-1} \right) (p(x_0, x_1) + p(y_0, y_1)) \\
 &\leq \frac{(\alpha + \beta)^n}{1 - (\alpha + \beta)} (p(x_0, x_1) + p(y_0, y_1)).
 \end{aligned} \tag{2.12}$$

By definition of p_s , we have $p_s(x, y) \leq 2p(x, y)$, so

$$\begin{aligned}
 p_s(x_n, x_m) + p_s(y_n, y_m) &\leq 2p(x_n, x_m) + 2p(y_n, y_m) \\
 &\leq \frac{2(\alpha + \beta)^n}{1 - (\alpha + \beta)} (p(x_0, x_1) + p(y_0, y_1)).
 \end{aligned} \tag{2.13}$$

Consequently,

$$\begin{aligned} p_s(x_n, x_m) &\leq p_s(x_n, x_m) + p_s(y_n, y_m) \leq \frac{2(\alpha + \beta)^n}{1 - (\alpha + \beta)} (p(x_0, x_1) + p(y_0, y_1)), \\ p_s(y_n, y_m) &\leq p_s(x_n, x_m) + p_s(y_n, y_m) \leq \frac{2(\alpha + \beta)^n}{1 - (\alpha + \beta)} (p(x_0, x_1) + p(y_0, y_1)), \end{aligned} \quad (2.14)$$

which implies that both (x_n) and (y_n) are Cauchy sequences in the metric space (X, p_s) . Since the metric space (X, p_s) is complete, it follows that there exist $x, y \in X$ such that

$$\lim_{n \rightarrow \infty} p_s(x_n, x) = \lim_{n \rightarrow \infty} p_s(y_n, y) = 0. \quad (2.15)$$

Therefore, using property p_2 and the fact that (X, p) is complete if and only if (X, p_s) is complete, using (1.3), we have

$$\begin{aligned} p(x, x) &= \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(x_n, x_n), \\ p(y, y) &= \lim_{n \rightarrow \infty} p(y_n, y) = \lim_{n \rightarrow \infty} p(y_n, y_n). \end{aligned} \quad (2.16)$$

From p_2 and (2.10), we have

$$p(x_n, x_n) \leq p(x_n, x_{n+1}) \leq (\alpha + \beta)^n (p(x_0, x_1) + p(y_0, y_1)). \quad (2.17)$$

Since $\alpha + \beta < 1$, we get, $\lim_{n \rightarrow \infty} p(x_n, x_n) = 0$. Similarly, one can show that $\lim_{n \rightarrow \infty} p(y_n, y_n) = 0$. Therefore,

$$\begin{aligned} p(x, x) &= \lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(x_n, x_n) = 0, \\ p(y, y) &= \lim_{n \rightarrow \infty} p(y_n, y) = \lim_{n \rightarrow \infty} p(y_n, y_n) = 0. \end{aligned} \quad (2.18)$$

Finally, we will show that $x = F(x, y)$ and $y = F(y, x)$.

(a) Assume that F is continuous on X . In particular, F is continuous at (x, y) , hence for any $\epsilon > 0$, there exists $\delta > 0$ such that if $(u, v) \in X \times X$ verifying $q((x, y), (u, v)) < q((x, y), (x, y)) + \delta$, meaning that

$$p(x, u) + p(y, v) < p(x, x) + p(y, y) + \delta = \delta, \quad (2.19)$$

because $p(x, x) = p(y, y) = 0$, then we have

$$p(F(x, y), F(u, v)) < p(F(x, y), F(x, y)) + \frac{\epsilon}{2}. \quad (2.20)$$

Since $\lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(y_n, y) = 0$, for $\gamma = \min\{\delta/2, \epsilon/2\} > 0$, there exist $n_0, m_0 \in \mathbb{N}$ such that, for $n \geq n_0, m \geq m_0$, $p(x_n, x) < \gamma$ and $p(y_n, y) < \gamma$. Therefore, for $n \in \mathbb{N}, n \geq \max(n_0, m_0)$, we have

$$p(x_n, x) + p(y_n, y) < 2\gamma < \delta, \quad (2.21)$$

so we get

$$p(F(x, y), F(x_n, y_n)) < p(F(x, y), F(x, y)) + \frac{\epsilon}{2}. \quad (2.22)$$

Now, for any $n \geq \max(n_0, m_0)$,

$$\begin{aligned} p(F(x, y), x) &\leq p(F(x, y), x_{n+1}) + p(x_{n+1}, x) \\ &= p(F(x, y), F(x_n, y_n)) + p(x_{n+1}, x) \\ &< p(F(x, y), F(x, y)) + \frac{\epsilon}{2} + \gamma \\ &< p(F(x, y), F(x, y)) + \epsilon. \end{aligned} \quad (2.23)$$

On the other hand, inserting $p(x, x) = p(y, y) = 0$ in (2.1), we get

$$\begin{aligned} \varphi(p(F(x, y), F(x, y))) &\leq \varphi(\alpha p(x, x) + \beta p(y, y)) - \psi(\alpha p(x, x) + \beta p(y, y)) \\ &= \varphi(0) - \psi(0) = -\psi(0) \leq 0, \end{aligned} \quad (2.24)$$

which implies $p(F(x, y), F(x, y)) = 0$, so for any $\epsilon > 0$, $p(F(x, y), x) < \epsilon$. This implies that $F(x, y) = x$. Similarly, we can show that $F(y, x) = y$.

(b) Assume that X satisfies the two conditions given by (i) and (ii). Since $(x_n), (y_n)$ are a nondecreasing sequences and $x_n \rightarrow x, y_n \rightarrow y$, we have $x_n \leq x$ and $y_n \geq y$ for all n . By the condition (p_4) , we have

$$p(x, F(x, y)) \leq p(x, x_{n+1}) + p(x_{n+1}, F(x, y)) = p(x, x_{n+1}) + p(F(x_n, y_n), F(x, y)). \quad (2.25)$$

Therefore,

$$\begin{aligned} \varphi(p(x, F(x, y))) &\leq \varphi(p(x, x_{n+1})) + \varphi(p(F(x_n, y_n), F(x, y))) \\ &\leq \varphi(p(x, x_{n+1})) + \varphi(\alpha p(x_n, x) + \beta p(y_n, y)) \\ &\quad - \psi(\alpha p(x_n, x) + \beta p(y_n, y)). \end{aligned} \quad (2.26)$$

Taking the limit as $n \rightarrow +\infty$ in the above inequality, using (2.18), and the properties of φ and ψ , we get $\varphi(p(x, F(x, y))) = 0$. Thus, $p(x, F(x, y)) = 0$. Hence, $x = F(x, y)$. Similarly, one can show that $y = F(y, x)$. Thus, we proved that F has a coupled fixed point. $\square$

Corollary 2.2. Let $(X, \leq)$ be a partially ordered set and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Supposed that

$$p(F(x, y), F(u, v)) \leq \alpha p(x, u) + \beta p(y, v) \quad (2.27)$$

for all $x, y, u, v \in X$ with $x \geq u, y \leq v$ and $\alpha + \beta < 1$. Suppose either F is continuous or X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n . If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$ then, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point. Also, $p(x, x) = p(y, y) = 0$.

Proof. For $\alpha + \beta < 1$, taking $\varphi(t) = t$ and $\psi(t) = 0$, we get the result. $\square$

The following main theorem for Gnana Bhaskar and Lakshmikantham in [13] proved the next theorem.

Theorem 2.3 (see Gnana Bhaskar and Lakshmikantham [13]). Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Assume that there exists a $k \in [0, 1)$ with

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}(d(x, u) + d(y, v)), \quad (2.28)$$

for all $x, y, u, v \in X$ with $x \leq u$ and $v \leq y$. suppose either F is continuous or X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n .

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$ then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Note that for $k \in [0, 1)$, and $\alpha = \beta = k/2$ in Corollary 2.2, we get analogous to Theorem 2.3 in complete-ordered partial metric space.

Theorem 2.4. Let $(X, \leq)$ be a partially ordered set having the property that for every $(x, y), (z, t) \in X \times X$, there exists a $(u, v) \in X \times X$ such that $u \leq x, z$ and $v \leq y, t$ and suppose that there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X and assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(p(F(x, y), F(u, v))) \leq \varphi(\alpha p(x, u) + \beta p(y, v)) - \psi(\alpha p(x, u) + \beta p(y, v)) \quad (2.29)$$

for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$, and $\alpha + \beta < 1$. Suppose either F is continuous or X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n .

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$ then F has a unique coupled fixed point.

Proof. From Theorem 2.1, the set of coupled fixed points of F is non-empty. Suppose (x, y) and (z, t) are coupled fixed points of F , that is, $x = F(x, y)$, $y = F(y, x)$, $z = F(z, t)$ and $t = F(t, z)$. We shall show that $x = z$ and $y = t$.

By assumption, there exists $(u, v) \in X \times X$ such that $u \leq x, z$ and $v \leq y, t$. We define sequences $(u_n), (v_n)$ as follows:

$$u_0 = u, \quad v_0 = v, \quad u_{n+1} = F(u_n, v_n), \quad v_{n+1} = F(v_n, u_n) \quad \forall n. \quad (2.30)$$

Since we may assume that $(u_0, v_0) = (u, v)$. By using the mathematical induction, it is easy to prove that $u_n \leq x$ and $v_n \leq y$ for any $n \in N$. From (2.1), we have

$$\begin{aligned} \varphi(p(x, u_{n+1})) &= \varphi(p(F(x, y), F(u_n, v_n))) \\ &\leq \varphi(\alpha p(x, u_n) + \beta p(y, v_n)) - \varphi(\alpha p(x, u_n) + \beta p(y, v_n)) \\ &\leq \varphi(\alpha p(x, u_n) + \beta p(y, v_n)), \\ \varphi(p(v_{n+1}, y)) &= \varphi(p(F(v_n, u_n), F(y, x))) \\ &\leq \varphi(\alpha p(v_n, y) + \beta p(u_n, x)) - \varphi(\alpha p(v_n, y) + \beta p(u_n, x)) \\ &\leq \varphi(\alpha p(v_n, y) + \beta p(u_n, x)). \end{aligned} \quad (2.31)$$

Since φ is nondecreasing, from the above inequalities, we have

$$p(x, u_{n+1}) \leq \alpha p(x, u_n) + \beta p(y, v_n), \quad (2.32)$$

$$p(v_{n+1}, y) \leq \alpha p(v_n, y) + \beta p(u_n, x). \quad (2.33)$$

Adding (2.32) to (2.33), we get

$$p(x, u_{n+1}) + p(y, v_{n+1}) \leq (\alpha + \beta)(p(y, v_n) + p(x, u_n)). \quad (2.34)$$

Therefore,

$$\begin{aligned}
 p(x, u_{n+1}), p(y, v_{n+1}) &\leq p(x, u_{n+1}) + p(y, v_{n+1}) \\
 &\leq (\alpha + \beta)(p(x, u_n) + p(y, v_n)) \\
 &\leq (\alpha + \beta)^2(p(x, u_{n-1}) + p(y, v_{n-1})) \\
 &\vdots \\
 &\leq (\alpha + \beta)^n(p(x, u_1) + p(y, v_1)),
 \end{aligned} \tag{2.35}$$

that is, since $\alpha + \beta < 1$, the sequences $\{p(y, v_n)$ and $p(x, u_n)\}$ are convergent and

$$\lim_{n \rightarrow \infty} p(x, u_n) = \lim_{n \rightarrow \infty} p(y, v_n) = 0. \tag{2.36}$$

Similarly, one can show that $\lim_{n \rightarrow \infty} p(z, u_n) = \lim_{n \rightarrow \infty} p(t, v_n) = 0$. Since

$$\begin{aligned}
 p(x, z) &\leq p(x, u_n) + p(u_n, z), \\
 p(y, t) &\leq p(y, v_n) + p(v_n, t),
 \end{aligned} \tag{2.37}$$

letting $n \rightarrow \infty$, we obtain $p(x, z) = p(y, t) = 0$, so $x = z$ and $y = t$. $\square$

Theorem 2.5. Let $(X, \leq)$ be a partially ordered set such that for every $(x, y), (z, t) \in X \times X$, there exists a (u, v) in $X \times X$ such that $u \leq x, z$ and $v \leq y, t$ and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X and assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that

$$\varphi(p(F(x, y), F(u, v))) \leq \varphi(\alpha p(x, u) + \beta p(y, v)) - \psi(\alpha p(x, u) + \beta p(y, v)), \tag{2.38}$$

for all $x, y, u, v \in X$ with $x \geq u, y \leq v$ and $\alpha + \beta < 1$. Suppose either F is continuous or X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n .

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then F has a unique coupled fixed point. In addition, if $x_0 \leq y_0$ or $y_0 \leq x_0$, then $x = F(x, y) = F(y, x) = y$ where (x, y) is a coupled fixed point of F .

Proof. Following the proof of Theorem 2.4, F has a unique coupled fixed point (x, y) . We only have to show that $x = y$. Assume $y_0 \leq x_0$. Using the mathematical induction, one can show that $x_n \geq y_n$ for any $n \in \mathbb{N}$. Note that, by (p_4) ,

$$\begin{aligned}
 p(x, y) &\leq p(x, x_{n+1}) + p(x_{n+1}, y_{n+1}) + p(y_{n+1}, y) \\
 &= p(x, x_{n+1}) + p(y_{n+1}, y) + p(F(x_n, y_n), F(y_n, x_n)).
 \end{aligned} \tag{2.39}$$

Therefore, using the condition (p_3) , (2.1), and a property of φ ,

$$\begin{aligned}\varphi(p(x, y)) &\leq \varphi(p(x, x_{n+1})) + \varphi(p(y_{n+1}, y)) + \varphi(p(F(x_n, y_n), F(y_n, x_n))) \\ &\leq \varphi(p(x, x_{n+1}) + \varphi(p(y_{n+1}, y)) + \varphi((\alpha + \beta)p(x_n, y_n)) - \varphi((\alpha + \beta)p(x_n, y_n)).\end{aligned}\quad (2.40)$$

From $\lim_{n \rightarrow \infty} p(x_n, x) = \lim_{n \rightarrow \infty} p(y_n, y) = 0$, we have $\lim_{n \rightarrow \infty} p(x_n, y_n) = p(x, y)$. Assume that $p(x, y) \neq 0$. Letting $n \rightarrow +\infty$ in (2.40) we get

$$\begin{aligned}\varphi(p(x, y)) &\leq 2\varphi(0) + \varphi((\alpha + \beta)p(x, y)) - \lim_{n \rightarrow \infty} \varphi((\alpha + \beta)p(x_n, y_n)) \\ &= \varphi((\alpha + \beta)p(x, y)) - \lim_{p(x_n, y_n) \rightarrow p(x, y)} \varphi((\alpha + \beta)p(x_n, y_n)).\end{aligned}\quad (2.41)$$

Since $\alpha + \beta < 1$, and φ is nondecreasing it follows that

$$\varphi(p(x, y)) \leq \varphi(p(x, y)) - \lim_{p(x_n, y_n) \rightarrow p(x, y)} \varphi((\alpha + \beta)p(x_n, y_n)), \quad (2.42)$$

that is, $\lim_{p(x_n, y_n) \rightarrow p(x, y)} \varphi((\alpha + \beta)p(x_n, y_n)) \leq 0$, which is a contradiction. Thus, $p(x, y) = 0$, so $x = y$. $\square$

Example 2.6 (see [17]). Let $X = [0, 1]$ with usual order. Define $p : [0, 1] \times [0, 1] \rightarrow R^+$ by $p(x, y) = \max\{x, y\}$ and $F : [0, 1] \times [0, 1] \rightarrow [0, 1]$ by $F(x, y) = (1/8)x$. Then,

- (i) $(X, \leq, p)$ is a complete partially ordered partial metric space,
- (ii) F has the mixed monotone property,
- (iii) for $x, u, v \in X$ with $x \geq u$ and $y \leq v$, we have

$$p(F(x, y), F(u, v)) \leq \frac{1}{8}(p(x, u) + p(y, v)), \quad (2.43)$$

- (iv) F is continuous.

Proof. The proofs of (i), (ii), and (iii) are clear. To prove (iv), letting $(x, y) \in X \times X$ and $\epsilon > 0$, we claim $F(B_q((x, y), 8\epsilon)) \subseteq B_q(F(x, y), \epsilon)$. To prove our claim, let $(s, t) \in B_q((x, y), 8\epsilon)$, then

$$q((s, t), (x, y)) < q((x, y), (x, y)) + 8\epsilon. \quad (2.44)$$

So,

$$p(s, x) + p(t, y) < p(x, x) + p(y, y) + 8\epsilon. \quad (2.45)$$

Since $y \leq p(t, y)$, $p(x, x) = x$ and $p(y, y) = y$, we have $p(s, x) < x + 8\epsilon$. Therefore, and hence $p(F(s, t), F(x, y)) \leq p(F(x, y), p(x, y))$. So, $F(s, t) \in Bp(F(x, y), \epsilon)$. We deduce that all the

hypotheses of Theorem 2.1 are satisfied with $\varphi(t) = t$, $\psi(t) = 0$ and $\alpha = \beta = 1/8$. Therefore, F has a coupled fixed point. Here, $(0, 0)$ is the coupled fixed point of F . $\square$

3. Application

In this part, from previous obtained results, we will deduce some coupled fixed point results for mappings satisfying a contraction of integral type in a complete partial metric space.

Let Γ be the set of all functions $\alpha : [0, +\infty) \rightarrow [0, +\infty)$ satisfying the following conditions:

- (i) α is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$,
- (ii) for all $\epsilon > 0$, we have $\int_0^\epsilon \alpha(s) ds > 0$,
- (iii) α is subadditive on each $[a, b] \subset [0, +\infty)$, that is,

$$\int_0^{a+b} \alpha(s) ds \leq \int_0^a \alpha(s) ds + \int_0^b \alpha(s) ds. \quad (3.1)$$

Let $N \in \mathbb{N}^*$ be fixed. Let $\{\alpha_i\}_{1 \leq i \leq N}$ be a family of N functions that belong to Γ . For all $t \geq 0$, we denote (I_i) , $i = 1, \dots, N$ as follows:

$$I_1(t) = \int_0^t \alpha_1(s) ds, \quad I_2(t) = \int_0^{I_1(t)} \alpha_2(s) ds, \dots, \quad I_N(t) = \int_0^{I_{N-1}(t)} \alpha_N(s) ds. \quad (3.2)$$

We have the following result.

Theorem 3.1. *Let $(X, \leq)$ be a partially ordered set and suppose there is a partial metric p on X such that (X, p) is a complete partial metric space. Let $F : X \times X \rightarrow X$ be a mapping having the mixed monotone property on X . Assume that there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$I_N(\varphi(p(F(x, y), F(u, v)))) \leq I_N(\varphi((\alpha p(x, u) + \beta p(y, v)))) - I_N(\psi(\alpha p(x, u) + \beta p(y, v))), \quad (3.3)$$

for all $x, y, u, v \in X$ with $x \geq u$, $y \leq v$, and $\alpha + \beta < 1$. Suppose either F is continuous, or X has the following properties:

- (i) if a nondecreasing sequence $x_n \rightarrow x$, then $x_n \leq x$ for all n ,
- (ii) if a nonincreasing sequence $x_n \rightarrow x$, then $x \leq x_n$ for all n .

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $F(y_0, x_0) \leq y_0$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$, that is, F has a coupled fixed point.

Proof. Take $\tilde{\varphi} = I_N \circ \varphi$ and $\tilde{\psi} = I_N \circ \psi$. Note that the (α_i) , $i = 1, \dots, N$ are taken to be sub-additive on each $[a, b] \subset [0, \infty)$ in order to get $\tilde{\varphi}(a + b) \leq \tilde{\varphi}(a) + \tilde{\varphi}(b)$. Moreover, it is easy to

show that $\tilde{\varphi}$ is continuous, nondecreasing and verifies $\tilde{\varphi}(t) = 0 \Leftrightarrow t = 0$. We get that $\tilde{\varphi} \in \Phi$. Also, we can find that $\tilde{\varphi} \in \Psi$. From (3.3), we have

$$\tilde{\varphi}(p(F(x, y), F(u, v))) \leq \tilde{\varphi}(\alpha p(x, u) + \beta p(y, v)) - \tilde{\varphi}(\alpha p(x, u) + \beta p(y, v)). \quad (3.4)$$

Now, applying Theorem 2.1, we obtain the desired result. $\square$

References

- [1] S. G. Matthews, "Partial metric topology," in *Proceedings of the 8th Summer Conference on General Topology and Applications*, vol. 728 of *Annals of the New York Academy of Sciences*, pp. 183–197, 1994.
- [2] W. Shatanawi, B. Samet, and M. Abbas, "Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces," *Mathematical and Computer Modelling*, vol. 55, no. 3–4, pp. 680–687, 2012.
- [3] N. Hussain, Z. Kadelburg, S. Radenović, and F. Al-Solamy, "Comparison functions and fixed point results in partial metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 605781, 15 pages, 2012.
- [4] H. K. Nashine, Z. Kadelburg, and S. Radenović, "Common fixed point theorems for weakly isotone increasing mappings in ordered partial metricspaces," *Mathematical and Computer Modelling*. In press.
- [5] Z. Kadelburg, H. K. Nashine, and S. Radenović, "Fixed point results under various contractive conditions in partial metric spaces," *Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales*, vol. 105, no. 1, 2012.
- [6] Z. Golubović, Z. Kadelburg, and S. Radenović, "Coupled coincidence points of mappings in ordered partial metric spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 192581, 18 pages, 2012.
- [7] T. Abdeljawad, E. Karapinar, and K. Taş, "Existence and uniqueness of a common fixed point on partial metric spaces," *Applied Mathematics Letters*, vol. 24, no. 11, pp. 1900–1904, 2011.
- [8] I. Altun, F. Sola, and H. Simsek, "Generalized contractions on partial metric spaces," *Topology and Its Applications*, vol. 157, no. 18, pp. 2778–2785, 2010.
- [9] H. Aydi, "Fixed point results for weakly contractive mappings in ordered partial metric spaces," *Journal of Advanced Mathematical Studies*, vol. 4, no. 2, pp. 1–12, 2011.
- [10] M. P. Schellekens, "The correspondence between partial metrics and semivaluations," *Theoretical Computer Science*, vol. 315, no. 1, pp. 135–149, 2004.
- [11] O. Valero, "On Banach fixed point theorems for partial metric spaces," *Applied General Topology*, vol. 6, no. 2, pp. 229–240, 2005.
- [12] D. Ilić, V. Pavlović, and V. Rakočević, "Some new extensions of Banach's contraction principle to partial metric space," *Applied Mathematics Letters*, vol. 24, no. 8, pp. 1326–1330, 2011.
- [13] T. Gnana Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis*, vol. 65, no. 7, pp. 1379–1393, 2006.
- [14] V. Lakshmikantham and Lj. Ćirić, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Analysis*, vol. 70, no. 12, pp. 4341–4349, 2009.
- [15] N. Van Luong and N. Xuan Thuan, "Coupled fixed point theorems in partially ordered metric spaces," *Bulletin of Mathematical Analysis and Applications*, vol. 2, no. 4, pp. 16–24, 2010.
- [16] N. V. Van Luong and N. Xuan Thuan, "Coupled fixed points in partially ordered metric spaces and application," *Nonlinear Analysis*, vol. 74, no. 3, pp. 983–992, 2011.
- [17] H. Aydi, E. Karapinar, and W. Shatanawi, "Coupled fixed point results for (ψ, φ) -weakly contractive condition in ordered partial metric spaces," *Computers & Mathematics with Applications*, vol. 62, no. 12, pp. 4449–4460, 2011.
- [18] H. Aydi, "Some coupled fixed point results on partial metric spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2011, Article ID 647091, 11 pages, 2011.
- [19] D. Đukić, Z. Kadelburg, and S. Radenović, "Coupled fixed point results for mappings without mixed monotone property," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1803–1808, 2012.
- [20] B. Samet, "Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces," *Nonlinear Analysis*, vol. 72, no. 12, pp. 4508–4517, 2010.