Research Article

Combined Algebraic Properties of IP^* and Central* Sets Near 0

Dibyendu De¹ and Ram Krishna Paul²

¹ Department of Mathematics, University of Kalyani, West Bengal, Kalyani 741235, India

² Department of Mathematics, National Institute of Technology, South Sikkim, Ravangla 737139, India

Correspondence should be addressed to Dibyendu De, dibyendude@klyuniv.ac.in

Received 25 March 2012; Accepted 26 June 2012

Academic Editor: Manfred Moller

Copyright © 2012 D. De and R. K. Paul. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

It is known that for an IP^* set A in $\mathbb{N}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ there exists a sum subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that $FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A$. Similar types of results also have been proved for central* sets. In this present work we will extend the results for dense subsemigroups of $((0, \infty), +)$.

1. Introduction

One of the famous Ramsey theoretic results is Hindman's Theorem.

Theorem 1.1. *Given a finite coloring $\mathbb{N} = \bigcup_{i=1}^r A_i$ there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ and $i \in \{1, 2, \dots, r\}$ such that*

$$FS(\langle x_n \rangle_{n=1}^\infty) = \left\{ \sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N}) \right\} \subseteq A_i, \quad (1.1)$$

where for any set X , $\mathcal{P}_f(X)$ is the set of finite nonempty subsets of X .

The original proof of this theorem was combinatorial in nature. But later using algebraic structure of $\beta\mathbb{N}$ a very elegant proof of this theorem was established in [1, Corollary 5.10]. First we give a brief description of algebraic structure of βS_d for a discrete semigroup $(S, \cdot)$.

We take the points of βS_d to be the ultrafilters on S , identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S_d$. Given $A \subseteq S$,

$$c\ell A = \overline{A} = \{p \in \beta S_d : A \in p\} \quad (1.2)$$

is a basis for the closed sets of βS_d . The operation $\cdot$ on S can be extended to the Stone-Čech compactification βS_d of S so that $(\beta S_d, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S_d$, the function $\rho_p : \beta S_d \rightarrow \beta S_d$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S_d \rightarrow \beta S_d$ defined by $\lambda_x(q) = x \cdot q$ is continuous). A nonempty subset I of a semigroup T is called a *left ideal* of S if $TI \subset I$, a *right ideal* if $IT \subset I$, and a *two-sided ideal* (or simply an *ideal*) if it is both a left and right ideal. A *minimal left ideal* is the left ideal that does not contain any proper left ideal. Similarly, we can define *minimal right ideal* and *smallest ideal*.

Any compact Hausdorff right topological semigroup T has a smallest two-sided ideal:

$$\begin{aligned} K(T) &= \bigcup \{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup \{R : R \text{ is a minimal right ideal of } T\}. \end{aligned} \quad (1.3)$$

Given a minimal left ideal L and a minimal right ideal R , $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is a *minimal* idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal.

Given $p, q \in \beta S$, and $A \subseteq S$, $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. See [1] for an elementary introduction to the algebra of βS and for any unfamiliar details.

$A \subseteq \mathbb{N}$ is called an IP^* set if it belongs to every idempotent in $\beta\mathbb{N}$. Given a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$, we let $FP(\langle x_n \rangle_{n=1}^\infty)$ be the product analogue of Finite Sum. Given a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$, we say that $\langle y_n \rangle_{n=1}^\infty$ is a *sum subsystem* of $\langle x_n \rangle_{n=1}^\infty$ provided there is a sequence $\langle H_n \rangle_{n=1}^\infty$ of nonempty finite subsets of $\mathbb{N}$ such that $\max H_n < \min H_{n+1}$ and $y_n = \sum_{t \in H_n} x_t$ for each $n \in \mathbb{N}$.

Theorem 1.2. *Let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in $\mathbb{N}$ and let A be an IP^* set in $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that*

$$FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A. \quad (1.4)$$

Proof. See [2, Theorem 2.6] or see [1, Corollary 16.21]. □

Definition 1.3. A subset $C \subseteq S$ is called *central* if and only if there is an idempotent $p \in K(\beta S)$ such that $C \in p$.

The algebraic structure of the smallest ideal of βS has played a significant role in Ramsey Theory. It is known that any central subset of $(\mathbb{N}, +)$ is guaranteed to have substantial additive structure. But Theorem 16.27 of [1] shows that central sets in $(\mathbb{N}, +)$ need not have any multiplicative structure at all. On the other hand, in [2] we see that sets which belong

to every minimal idempotent of $\mathbb{N}$, called central* sets, must have significant multiplicative structure. In fact central* sets in any semigroup $(S, \cdot)$ are defined to be those sets which meet every central set.

Theorem 1.4. *If A is a central* set in $(\mathbb{N}, +)$ then it is central in $(\mathbb{N}, \cdot)$.*

Proof. See [2, Theorem 2.4]. □

In case of central* sets a similar result has been proved in [3] for a restricted class of sequences called minimal sequences, where a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ is said to be a minimal sequence if

$$\bigcap_{m=1}^{\infty} \overline{FS(\langle x_n \rangle_{n=m}^\infty)} \cap K(\beta\mathbb{N}) \neq \emptyset. \quad (1.5)$$

Theorem 1.5. *Let $\langle y_n \rangle_{n=1}^\infty$ be a minimal sequence and let A be a central* set in $(\mathbb{N}, +)$. Then there exists a sum subsystem $\langle x_n \rangle_{n=1}^\infty$ of $\langle y_n \rangle_{n=1}^\infty$ such that*

$$FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A. \quad (1.6)$$

Proof. See [3, Theorem 2.4]. □

A strongly negative answer to the partition analogue of Hindman's theorem was presented in [4]. Given a sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$, let us denote $PS(\langle x_n \rangle_{n=1}^\infty) = \{x_m + x_n : m, n \in \mathbb{N} \text{ and } m \neq n\}$ and $PP(\langle x_n \rangle_{n=1}^\infty) = \{x_m \cdot x_n : m, n \in \mathbb{N} \text{ and } m \neq n\}$.

Theorem 1.6. *There exists a finite partition $\mathcal{R}$ of $\mathbb{N}$ with no one-to-one sequence $\langle x_n \rangle_{n=1}^\infty$ in $\mathbb{N}$ such that $PS(\langle x_n \rangle_{n=1}^\infty) \cup PP(\langle x_n \rangle_{n=1}^\infty)$ is contained in one cell of the partition $\mathcal{R}$.*

Proof. See [4, Theorem 2.11]. □

A similar result in this direction in the case of dyadic rational numbers has been proved by V. Bergelson et al..

Theorem 1.7. *There exists a finite partition $\mathbb{D} \setminus \{0\} = \bigcup_{i=1}^r A_i$ such that there do not exist $i \in \{1, 2, \dots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ with*

$$FS(\langle x_n \rangle_{n=1}^\infty) \cup PP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i. \quad (1.7)$$

Proof. See [5, Theorem 5.9]. □

In [5], the authors also presented the following conjecture and question.

Conjecture 1.8. *There exists a finite partition $\mathbb{Q} \setminus \{0\} = \bigcup_{i=1}^r A_i$ such that there do not exist $i \in \{1, 2, \dots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ with*

$$FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i. \quad (1.8)$$

Question 1. Does there exist a finite partition $\mathbb{R} \setminus \{0\} = \bigcup_{i=1}^r A_i$ such that there do not exist $i \in \{1, 2, \dots, r\}$ and a sequence $\langle x_n \rangle_{n=1}^\infty$ with

$$FS(\langle x_n \rangle_{n=1}^\infty) \cup FP(\langle x_n \rangle_{n=1}^\infty) \subseteq A_i? \quad (1.9)$$

In the present paper our aim is to extend Theorems 1.2 and 1.5 for dense subsemigroups of $((0, \infty), +)$.

Definition 1.9. If S is a dense subsemigroup of $((0, \infty), +)$ one defines $0^+(S) = \{p \in \beta S_d : \text{for all } \epsilon > 0, (0, \epsilon) \in p\}$.

It is proved in [6], that $0^+(S)$ is a compact right topological subsemigroup of $(\beta S_d, +)$ which is disjoint from $K(\beta S_d)$ and hence gives some new information which are not available from $K(\beta S_d)$. Being compact right topological semigroup $0^+(S)$ contains minimal idempotents of $0^+(S)$. A subset A of S is said to be IP^* -set near 0 if it belongs to every idempotent of $0^+(S)$ and a subset C of S is said to be central* set near 0 if it belongs to every minimal idempotent of $0^+(S)$. In [7] the authors applied the algebraic structure of $0^+(S)$ on their investigation of image partition regularity near 0 of finite and infinite matrices. Article [8] used algebraic structure of $0^+(\mathbb{R})$ to investigate image partition regularity of matrices with real entries from $\mathbb{R}$.

2. IP^* and Central* Set Near 0

In the following discussion, we will extend Theorem 1.2 for a dense subsemigroup of $((0, \infty), +)$ in the appropriate context.

Definition 2.1. Let S be a dense subsemigroup of $((0, \infty), +)$. A subset A of S is said to be an *IP set near 0* if there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ such that $\sum_{n=1}^\infty x_n$ converges and such that $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq A$. One calls a subset D of S an *IP^* set near 0* if for every subset C of S which is *IP set near 0*, $C \cap D$ is *IP set near 0*.

From [6, Theorem 3.1] it follows that for a dense subsemigroup S of $((0, \infty), +)$ a subset A of S is an *IP set near 0* if and only if there exists some idempotent $p \in 0^+(S)$ with $A \in p$. Further it can be easily observed that a subset D of S is an IP^* set near 0 if and only if it belongs to every idempotent of $0^+(S)$.

Given $c \in \mathbb{R} \setminus \{0\}$ and $p \in \beta \mathbb{R}_d \setminus \{0\}$, the product $c \cdot p$ is defined in $(\beta \mathbb{R}_d, \cdot)$. One has $A \subseteq \mathbb{R}$ is a member of $c \cdot p$ if and only if $c^{-1}A = \{x \in \mathbb{R} : c \cdot x \in A\}$ is a member of p .

Lemma 2.2. *Let S be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$. If A is an *IP set near 0* in S then sA is also an *IP set near 0* for every $s \in S \cap (0, 1)$. Further if A is an IP^* set near 0 in $(S, +)$ then $s^{-1}A$ is also an IP^* set near 0 for every $s \in S \cap (0, 1)$.*

Proof. Since A is an *IP set near 0* then by [6, Theorem 3.1] there exists a sequence $\langle x_n \rangle_{n=1}^\infty$ in S with the property that $\sum_{n=1}^\infty x_n$ converges and $FS(\langle x_n \rangle_{n=1}^\infty) \subseteq A$. This implies that $\sum_{n=1}^\infty (s \cdot x_n)$ is also convergent and $FS(\langle sx_n \rangle_{n=1}^\infty) \subseteq sA$. This proves that sA is also an *IP set near 0*.

For the second let A be an IP^* set near 0 and $s \in S \cap (0, 1)$. To prove that $s^{-1}A$ is an IP^* set near 0 it is sufficient to show that if B is any *IP set near 0* then $B \cap s^{-1}A \neq \emptyset$. Since B

is an IP set near 0, sB is also an IP set near 0 by the first part of the proof, so that $A \cap sB \neq \emptyset$. Choose $t \in sB \cap A$ and $k \in B$ such that $t = sk$. Therefore $k \in s^{-1}A$ so that $B \cap s^{-1}A \neq \emptyset$. $\square$

Given $A \subseteq S$ and $s \in S$, $s^{-1}A = \{t \in S: st \in A\}$, and $-s + A = \{t \in S: s + t \in A\}$.

Theorem 2.3. Let S be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$. Also let $\langle x_n \rangle_{n=1}^\infty$ be a sequence in S such that $\sum_{n=1}^\infty x_n$ converges and let A be a IP^* set near 0 in S . Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that

$$FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A. \quad (2.1)$$

Proof. Since $\sum_{n=1}^\infty x_n$ converges, from [6, Theorem 3.1] it follows that we can find some idempotent $p \in 0^+(S)$ for which $FS(\langle x_n \rangle_{n=1}^\infty) \in p$. In fact $T = \bigcap_{m=1}^\infty c\ell_{\beta S_d} FS(\langle y_n \rangle_{n=m}^\infty) \subseteq 0^+(S)$ and $p \in T$. Again, since A is a IP^* set near 0 in S , by Lemma 2.2 for every $s \in S \cap (0, 1)$, $s^{-1}A \in p$. Let $A^* = \{s \in A: -s + A \in p\}$. Then by [1, Lemma 4.14] $A^* \in p$. We can choose $y_1 \in A^* \cap FS(\langle x_n \rangle_{n=1}^\infty)$. Inductively let $m \in \mathbb{N}$ and $\langle y_i \rangle_{i=1}^m, \langle H_i \rangle_{i=1}^m$ in $\mathcal{P}_f(\mathbb{N})$ be chosen with the following properties:

- (1) $i \in \{1, 2, \dots, m-1\} \max H_i < \min H_{i+1}$;
- (2) if $y_i = \sum_{t \in H_i} x_t$ then $\sum_{t \in H_m} x_t \in A^*$ and $FP(\langle y_i \rangle_{i=1}^m) \subseteq A$.

We observe that $\{\sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}), \min H > \max H_m\} \in p$. Let $B = \{\sum_{t \in H} x_t : H \in \mathcal{P}_f(\mathbb{N}), \min H > \max H_m\}$, let $E_1 = FS(\langle y_i \rangle_{i=1}^m)$ and $E_2 = FP(\langle y_i \rangle_{i=1}^m)$. Now consider

$$D = B \cap A^* \cap \bigcap_{s \in E_1} (-s + A^*) \cap \bigcap_{s \in E_2} (s^{-1}A^*). \quad (2.2)$$

Then $D \in p$. Now choose $y_{m+1} \in D$ and $H_{m+1} \in \mathcal{P}_f(\mathbb{N})$ such that $\min H_{m+1} > \max H_m$. Putting $y_{m+1} = \sum_{t \in H_{m+1}} x_t$ shows that the induction can be continued and proves the theorem. $\square$

If we turn our attention to central* sets then the above result holds for a restricted class of sequences which we call minimal sequence near 0.

Definition 2.4. Let S be a dense subsemigroup of $((0, \infty), +)$. A sequence $\langle x_n \rangle_{n=1}^\infty$ in S is said to be a *minimal sequence near 0* if

$$\bigcap_{m=1}^\infty \overline{FS(\langle x_n \rangle_{n=m}^\infty)} \cap K(0^+(S)) \neq \emptyset. \quad (2.3)$$

The notion of piecewise syndetic set near 0 was first introduced in [6].

Definition 2.5. For a dense subsemigroup S of $((0, \infty), +)$, a subset A of S is *piecewise syndetic near 0* if and only if $c\ell_{\beta S_d} A \cap K(0^+(S)) \neq \emptyset$.

The following theorem characterizes minimal sequences near 0 in terms of piecewise syndetic set near 0.

Theorem 2.6. Let S be a dense subsemigroup of $((0, \infty), +)$. Then the following conditions are equivalent:

- (a) $\langle x_n \rangle_{n=1}^\infty$ is a minimal sequence near 0.
- (b) $FS(\langle x_n \rangle_{n=1}^\infty)$ is piecewise syndetic near 0.
- (c) There is an idempotent in $\bigcap_{m=1}^\infty \overline{FS(\langle x_n \rangle_{n=m}^\infty)} \cap K(0^+(S)) \neq \emptyset$.

Proof. (a) $\Rightarrow$ (b) follows from (see [6, Theorem 3.5]).

To prove that (b) implies (a) let us consider that $FS(\langle x_n \rangle_{n=1}^\infty)$ be a piecewise syndetic near 0. Then there exists a minimal left ideal L of $0^+(S)$ such that $L \cap \overline{FS(\langle x_n \rangle_{n=1}^\infty)} \neq \emptyset$. We choose $q \in L \cap \overline{FS(\langle x_n \rangle_{n=1}^\infty)}$. By [6, Theorem 3.1], $\bigcap_{m=1}^\infty c\ell_{\beta S_d} FS(\langle x_n \rangle_{n=m}^\infty)$ is a subsemigroup of $0^+(S)$, so it suffices to show that for each $m \in \mathbb{N}$, $L \cap \overline{FS(\langle x_n \rangle_{n=m}^\infty)} \neq \emptyset$. In fact minimal left ideals being closed, we can conclude that $L \cap \bigcap_{n=m}^\infty \overline{FS(\langle x_n \rangle_{n=m}^\infty)} \neq \emptyset$ and so $L \cap \bigcap_{n=m}^\infty \overline{FS(\langle x_n \rangle_{n=m}^\infty)}$ is a compact right topological semigroup so that it contains idempotents. To this end, let $m \in \mathbb{N}$ with $m > 1$. Then $FS(\langle x_n \rangle_{n=1}^\infty) = FS(\langle x_n \rangle_{n=m}^\infty) \cup FS(\langle x_n \rangle_{n=1}^{m-1}) \cup \bigcup \{t + FS(\langle x_n \rangle_{n=m}^\infty) : t \in FS(\langle x_n \rangle_{n=1}^{m-1})\}$. So we must have one of the following:

- (i) $FS(\langle x_n \rangle_{n=m}^\infty) \in q$,
- (ii) $FS(\langle x_n \rangle_{n=1}^{m-1}) \in q$,
- (iii) $t + FS(\langle x_n \rangle_{n=m}^\infty) \in q$ for some $t \in FS(\langle x_n \rangle_{n=1}^{m-1})$.

Clearly (ii) does not hold, because in that case q becomes a member of S while it is a member of minimal left ideal. If (iii) holds then we have $t + FS(\langle x_n \rangle_{n=m}^\infty) \in q$ for some $t \in FS(\langle x_n \rangle_{n=1}^{m-1})$. Since $q \in 0^+(S)$, we have $(0, t) \cap S \in q$. But $(0, t) \cap (t + FS(\langle x_n \rangle_{n=m}^\infty)) = \emptyset$, a contradiction. Hence (i) must hold so that $q \in L \cap \overline{FS(\langle x_n \rangle_{n=m}^\infty)}$.

(a) $\Leftrightarrow$ (c) is obvious. □

Let us recall following lemma for our purpose.

Lemma 2.7. Let S be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$ and assume that for each $y \in S \cap (0, 1)$ and each $x \in S$, $x/y \in S$ and $yx \in S$. If $A \subseteq S$ and $y^{-1}A$ is a central set near 0, then A is also a central set near 0.

Proof. See [6, Lemma 4.8]. □

Lemma 2.8. Let S be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$ and assume that for each $s \in S \cap (0, 1)$ and each $t \in S$, $t/s \in S$ and $st \in S$. If A is central set near 0 in S then sA is also central set near 0.

Proof. Since $s^{-1}(sA) = A$ and A is central set near 0 then by Lemma 2.7, sA is central set near 0. □

Lemma 2.9. Let S be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$ and assume that for each $s \in S \cap (0, 1)$ and each $t \in S$, $t/s \in S$ and $st \in S$. If A is a central* set near 0 in $(S, +)$ then $s^{-1}A$ is also central* set near 0.

Proof. Let A be a central* set near 0 and $s \in S \cap (0, 1)$. To prove that $s^{-1}A$ is a central* set near 0 it is sufficient to show that for any central set near 0 C , $C \cap s^{-1}A \neq \emptyset$. Since C is central set

near 0, sC is also central set near 0 so that $A \cap sC \neq \emptyset$. Choose $t \in sC \cap A$ and $k \in C$ such that $t = sk$. Therefore $k \in s^{-1}A$ so that $C \cap s^{-1}A \neq \emptyset$. $\square$

We end this paper by following generalization of Theorem 2.3, whose proof is also straight forward generalization of Theorem 2.3 and hence omitted.

Theorem 2.10. *Let S be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$ and assume that for each $s \in S \cap (0, 1)$ and each $t \in S$, $t/s \in S$ and $st \in S$. Also let $\langle x_n \rangle_{n=1}^\infty$ be a minimal sequence near 0 and let A be a central* set near 0 in S . Then there exists a sum subsystem $\langle y_n \rangle_{n=1}^\infty$ of $\langle x_n \rangle_{n=1}^\infty$ such that*

$$FS(\langle y_n \rangle_{n=1}^\infty) \cup FP(\langle y_n \rangle_{n=1}^\infty) \subseteq A. \quad (2.4)$$

Acknowledgments

The authors would like to acknowledge the referee for her/his constructive report. The first named author is partially supported by DST-PURSE programme. The work of this paper was a part of second named authors Ph.D. dissertation which was supported by a CSIR Research Fellowship.

References

- [1] N. Hindman and D. Strauss, *Algebra in the Stone-Čech Compactification: Theory and Applications*, vol. 27, Walter de Gruyter, Berlin, Germany, 1998.
- [2] V. Bergelson and N. Hindman, "On IP^* sets and central sets," *Combinatorica*, vol. 14, no. 3, pp. 269–277, 1994.
- [3] D. De, "Combined algebraic properties of central* sets," *Integers*, vol. 7, p. A37, 2007.
- [4] N. Hindman, "Partitions and pairwise sums and products," *Journal of Combinatorial Theory A*, vol. 37, no. 1, pp. 46–60, 1984.
- [5] V. Bergelson, N. Hindman, and I. Leader, "Additive and multiplicative Ramsey theory in the reals and the rationals," *Journal of Combinatorial Theory A*, vol. 85, no. 1, pp. 41–68, 1999.
- [6] N. Hindman and I. Leader, "The semigroup of ultrafilters near 0," *Semigroup Forum*, vol. 59, no. 1, pp. 33–55, 1999.
- [7] D. De and N. Hindman, "Image partition regularity near zero," *Discrete Mathematics*, vol. 309, no. 10, pp. 3219–3232, 2009.
- [8] D. De and R. K. Paul, "Image partition regularity of matrices near 0 with real entries," *New York Journal of Mathematics*, vol. 17, pp. 149–161, 2011.