Research Article

On Almost Orthogonal Frames

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Almost orthogonal frames have been introduced and studied. It has been proved that a bounded almost orthogonal frame satisfies Feichtinger conjecture. Also, we prove that a bounded almost orthogonal frame contains a Riesz basis.

1. Introduction

Frames were formally introduced in 1952 by Duffin and Schaeffer [1]. In 1985, frames were resurfaced in the book by Young [2]. The theory of frames began to be more widely studied only after the landmark paper of Daubechies et al. [3] in 1986. For an introduction to frames, one may refer to [4–6].

Feichtinger in his work on time frequency analysis noted that all Gabor frames (which he was using for his work) had the property that they could be divided into a finite number of subsets which were Riesz basis sequences. This observation led to the following conjecture, called the Feichtinger conjecture “Every bounded frame can be written as a finite union of Riesz basic sequences.”

Feichtinger conjecture is connected to the famous Kadison-Singer conjecture. It was shown in [7] that Kadison-Singer conjecture implies Feichtinger conjecture. For literature related to Feichtinger conjecture, one may refer to [7, 8].

In the present paper, we introduce and study almost orthogonal frames in Hilbert spaces and prove that a bounded almost orthogonal frame satisfies Feichtinger conjecture. Also, we prove that a bounded almost orthogonal frame contains a Riesz basis.


2. Preliminaries

Throughout the paper, $H$ will denote an infinite-dimensional Hilbert space, $\{n_k\}$ an infinite-increasing sequence in $\mathbb{N}$, $[x_n]$ the closed linear span of $\{x_n\}$, and for any set $D$, $|D|$ will denote cardinality of $D$.

**Definition 2.1.** A sequence $\{x_n\}$ in a Hilbert space $H$ is said to be a frame for $H$ if there exist constants $A$ and $B$ with $0 < A \leq B < \infty$ such that
\[
A\|x\|^2 \leq \sum_n |\langle x, x_n \rangle|^2 \leq B\|x\|^2, \quad x \in H.
\] (2.1)

The positive constants $A$ and $B$, respectively, are called lower and upper frame bounds for the frame $\{x_n\}$. The inequality (2.1) is called the frame inequality for the frame $\{x_n\}$.

A frame $\{x_n\}$ in $H$ is called tight if it is possible to choose $A, B$ satisfying inequality (2.1) with $A = B$ as frame bounds and is called normalized tight if $A = B = 1$. A frame $\{x_n\}$ in $H$ is called exact if removal of any $x_n$ renders the collection $\{x_n\}$ no longer a frame for $H$. A sequence $\{x_n\} \in H$ is called a Bessel sequence if it satisfies upper frame inequality in (2.1).

**Definition 2.2.** A sequence $\{x_n\}$ in $H$ is called a Riesz basic sequence if there exist positive constants $A$ and $B$ such that for all finite sequence of scalars $\{\alpha_k\}$, we have
\[
A\sum_k |\alpha_k|^2 \leq \left\| \sum_k \alpha_k x_k \right\|^2 \leq B\sum_k |\alpha_k|^2.
\] (2.2)

In case, the Riesz basic sequence $\{x_n\}$ is complete in $H$, it is called a Riesz basis for $H$.

**Definition 2.3.** A sequence $\{y_n\}$ in a Hilbert space $H$ is said to be a block sequence with respect to a given sequence $\{x_n\}$ in $H$, if it is of the form
\[
y_n = \sum_{i \in D_n} \alpha_i x_i \neq 0, \quad n \in \mathbb{N},
\] (2.3)

where $D_n$’s are finite subsets of $\mathbb{N}$ with $D_n \cap D_m = \emptyset, n \neq m$, $\bigcup_{n \in \mathbb{N}} D_n = \mathbb{N}$ and $\alpha_i$’s are any scalars.

It has been observed in [9] that a block sequence with respect to a frame in a Hilbert space may not be a frame for $H$. Also, a block sequence with respect to a sequence in $H$ which is not even a frame for $H$ may be a frame for $H$.

3. Main Results

We begin with a sufficient condition for a bounded frame to satisfy the Feichtinger conjecture.

**Theorem 3.1.** Let $\{x_n\}$ be a bounded frame for $H$. If there exists a sequence of finite subsets $\{D_n\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ with $D_i \cap D_j = \emptyset, \forall i \neq j,$ $\bigcup_{i=1}^{\infty} D_i = \mathbb{N}$ and $\sup_n |\{D_n\}| < \infty$ such that $H = \bigoplus_{n \in \mathbb{N}} V_n$, where $V_n = [x_n]_{n \in D_n}$, then $\{x_n\}$ can be decomposed into a finite union of a Riesz basic sequences.
Proof. Suppose the problem has an affirmative answer. Let \( \{D_n\}_{n \in \mathbb{N}} \) be sequence of finite subsets of \( \mathbb{N} \) with \( D_n \cap D_m = \emptyset, \ n \neq m \) and \( \bigcup_{n \in \mathbb{N}} D_n = \mathbb{N} \) such that \( H = \bigoplus_{n \in \mathbb{N}} V_n \), where \( V_n = \{x_i\}_{i \in D_n} \) and \( \{x_n\} \) is a bounded frame for \( H \). Let \( \{G_n\} \) be a sequence of sets given by

\[
G_n = \{x_i\}_{i \in D_n}, \quad \forall n \in \mathbb{N}.
\] (3.1)

Now, for each \( j \in \mathbb{N} \), choose a sequence \( \{y_j^i\}_{i \in \mathbb{N}} \) such that

\[
y_j^i = \begin{cases} 
\text{ith element of } G_i, & \text{if } G_i \text{ contains } j \text{th element,} \\
\emptyset, & \text{otherwise.}
\end{cases}
\] (3.2)

Then, for each \( j \in \mathbb{N}, \) \( \{y_j^i\}_{i \in \mathbb{N}} \) is a sequence of orthogonal vectors which are norm bounded. So, \( \{y_j^i\}_{i \in \mathbb{N}} \) is a Riesz basic sequence for \( H \), for each \( j \in \mathbb{N} \). Also, note that

\[
\{x_n\} = \bigcup_j \{y_j^i\}.
\] (3.3)

Since \( D_n \)'s are finite, \( j \) varies on a finite set. Hence \( \{x_n\} \) is decomposed into finite number of Riesz basic sequences. \( \square \)

We will now introduce a concept which is more general than orthogonal frame and call it almost orthogonal frame. We give the following definition of almost orthogonal frame.

**Definition 3.2.** A frame \( \{x_n\} \) in a Hilbert space \( H \) is called an almost orthogonal frame of order \( K \) (\( K \in \mathbb{N} \)) if \( K \) is the smallest natural number for which there exists a permutation \( \{\sigma_n\} \) of \( \mathbb{N} \) such that

\[
\langle x_{\sigma_n}, x_{\sigma_m} \rangle = 0, \quad \forall \sigma_n, \sigma_m \text{ such that } |\sigma_n - \sigma_m| \geq K.
\] (3.4)

**Note 1.** We use \( \langle x_n \rangle \) instead of \( \langle x_{\sigma_n} \rangle \) for convenience.

**Example 3.3.** (I) An orthogonal basis is an almost orthogonal frame of order 1.

(II) \( \{e_1, e_2, e_3, e_4, \ldots, e_n, e_{n+1}, \ldots\} \) is an almost orthogonal frame of order 2.

(III) \( \{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, \ldots\} \) is not an almost orthogonal frame of any order.

(IV) \( \{e_1, e_2 + e_3, e_2 + e_3, e_3, e_3 + e_4, \ldots\} \) is an almost orthogonal frame of order 3.

(V) \( \{e_1, e_2 + e_1/4, e_3 + e_1/8, \ldots\} \) is not an almost orthogonal frame of any order.

(VI) \( \{e_i + (1/i)e_{i+1}\}_{i \geq 1} \) is an almost orthogonal frame of order 2.

(VII) \( \{e_1, (1/2)e_2, (1 - (1/2^2))^{1/2}e_2, (1/3)e_3, (1 - (1/3^2))^{1/2}e_3, \ldots\} = \{(1/n)e_n \} \cup \{(1 - (1/2^2))^{1/2}e_n\} \) is a tight frame with \( A = B = 1 \), which is almost orthogonal of order 2 and is not bounded below.

**Observations**

(I) A bounded frame may or may not be an almost orthogonal frame. (See Example I and Example V.)
(II) An almost orthogonal frame of some finite \((\neq 1)\) order may or may not be a Riesz basis. (See Example II and Example V.)

(III) A Riesz basis may or may not be an almost orthogonal frame. (See Example I and Example V.)

**Theorem 3.4.** A bounded almost orthogonal frame satisfies Feichtinger conjecture.

**Proof.** Let \(\{x_n\}\) be a bounded almost orthogonal frame of order \(K\). Define a sequence \(\{G_n\}\) of subspaces as follows:

\[
G_1 = [x_1, x_2, \ldots, x_K], \\
G_2 = [x_{K+1}, \ldots, x_{2K}], \\
\vdots \\
G_n = [x_{(n-1)K+1}, x_{(n-1)K+2}, \ldots, x_{nK}], \quad n \in \mathbb{N}. 
\] (3.5)

Now, since \(\{x_n\}\) is an almost orthogonal frame of degree \(K\). This gives

\[
\langle x_n, x_m \rangle = 0 \quad \forall n, m \in \mathbb{N} \text{ such that } |n - m| \geq K. 
\] (3.6)

Let \(x \in G_n\) and \(y \in G_{n+2}\), for any \(n \in \mathbb{N}\). Then

\[
x = \sum_{(n-1)K+1}^{nK} a_i x_i, \quad y = \sum_{(n+1)K+1}^{(n+2)K} \beta_j x_j. 
\] (3.7)

Therefore, we have

\[
\langle x, y \rangle = \left\langle \sum_{(n-1)K+1}^{nK} a_i x_i, \sum_{(n+1)K+1}^{(n+2)K} \beta_j x_j \right\rangle = \sum_{(n-1)K+1}^{nK} a_i \left\langle x_i, \sum_{(n+1)K+1}^{(n+2)K} \beta_j x_j \right\rangle \\
= \sum_{(n-1)K+1}^{nK} a_i \left( \sum_{(n+1)K+1}^{(n+2)K} \beta_j \langle x_i, x_j \rangle \right) = 0. 
\] (3.8)

\(\Rightarrow G_n \cap G_{n+2} = \phi, \quad \forall n \in \mathbb{N},\)

\(\Rightarrow \text{span} \{G_n, G_{n+2}\} = G_n \oplus G_{n+2}, \quad \forall n,\)

\(\Rightarrow \text{span} \{G_1, G_3, G_5, \ldots\} = G_1 \oplus G_3 \oplus G_5 \oplus \cdots = \bigoplus_{n \in \mathbb{N}} G_{2n-1} = H_1.\)

Also, we have

\[
\text{span} \{G_2, G_4, G_6, \ldots\} = \bigoplus_{n \in \mathbb{N}} G_{2n} = H_2. 
\] (3.9)
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So, by Theorem 3.1

\[ \{ x_1, x_2, \ldots, x_K, x_{2K+1}, x_{2K+2}, \ldots, x_{3K}, \ldots \} \] (3.10)

can be written as finite union of Riesz basic sequences.

Similarly, using Theorem 3.1,

\[ \{ x_K, x_{K+2}, \ldots, x_{2K+1}, x_{3K+1}, \ldots, x_{4K}, \ldots \} \] (3.11)

can be written as finite union of Riesz basic sequences.

Hence, \( \{ x_n \} \) can be written as finite union of Riesz basic sequences.

**Remark 3.5.** Almost orthogonal frames produce fusion frames (nonorthogonal) and fusion frame systems. Indeed, let \( \{ x_n \} \) be an almost orthogonal frame of order \( K \). Proceeding as in Theorem 3.4, we get a sequence of subspaces \( \{ G_n \} \) satisfying

\[
\begin{align*}
\text{span}[G_1, G_3, G_5, \ldots] &= G_1 \oplus G_3 \oplus G_5 \oplus \cdots = H_1, \\
\text{span}[G_2, G_4, G_6, \ldots] &= G_2 \oplus G_4 \oplus G_6 \oplus \cdots = H_2.
\end{align*}
\] (3.12)

Now, define a sequence of projections \( \{ v_i \} \) \( (v_i : H \to G_i) \). Then, we can easily verify that \( \{ v_{2i-1}, G_{2i-1} \}_{i \in \mathbb{N}} \) is a fusion frame for \( H_1 \) and \( \{ v_{2i}, G_{2i} \}_{i \in \mathbb{N}} \) is a fusion frame for \( H_2 \). So, \( \{ v_i, G_i \}_{i \in \mathbb{N}} \) is a fusion frame for \( H \).

Finally, we prove that for any bounded almost orthogonal frame, there exists a block sequence with respect to the almost orthogonal frame such that the block sequence is a Riesz basis. More precisely, we have the following.

**Theorem 3.6.** A bounded almost orthogonal frame contains a Riesz basis.

**Proof.** Let \( \{ x_n \} \) be an almost orthogonal frame of order \( K \). Consider \( \{ x_1, x_2, \ldots, x_K, x_{K+1}, \ldots, x_{2K}, x_{2K+1}, \ldots \} \). Then, following the steps in Theorem 3.4, we get a sequence of subspaces \( \{ G_n \} \) which are finite dimensional. So, we can extract a Riesz basis for \( G_n \) out of \( \{ x_{(n-1)K+1}, x_{(n-1)K+2}, \ldots, x_{nK} \} \) and let it be \( \{ x_n^i \} \). Then \( \bigcup_{n \in \mathbb{N}} \{ 2^{n-1} \} \) is a Riesz basis for \( H_1 \) and \( \bigcup_{n \in \mathbb{N}} \{ 2^n \} \) is a Riesz basis for \( H_2 \), where \( H_1 \) and \( H_2 \) are as in Theorem 3.4. Write \( F_n = G_n \cap G_{n+1} \) for all \( n \in \mathbb{N} \), then, for each \( n \in \mathbb{N} \), \( F_n \) is a finite-dimensional subspace of \( G_n \). Let \( \{ x_i^n \} \) be an extracted Riesz basis for \( F_n \), which is extracted from \( \{ x_{(n-1)K+1}, x_{(n-1)K+2}, \ldots, x_{nK} \} \) or \( \{ x_{nK+1}, \ldots, x_{(n+1)K} \} \). Then, \( \bigcup_n \{ x_i^n \} \sim \bigcup_n \{ x_i^n \} \) is the desired Riesz basis for \( H \).

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