Research Article

On BRK-Algebras

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The notion of BRK-algebra is introduced which is a generalization of BCK/BCI/BCH/Q/QS/BM-algebras. The concepts of $G$-part, $p$-radical, medial of a BRK-algebra are introduced and studied their properties. We proved that the variety of all medial BRK-algebras is congruence permutable and showed that every associative BRK-algebra is a group.

1. Introduction

In 1996, Imai and Iséki [1] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. These algebras have been extensively studied since their introduction. In 1983, Hu and Li [2] introduced the notion of a BCH-algebra which is a generalization of the notion of BCK and BCI-algebras and studied a few properties of these algebras. In 2001, Neggers et al. [3] introduced a new notion, called a Q-algebra and generalized some theorems discussed in BCI/BCK-algebras. In 2002, Neggers and Kim [4] introduced a new notion, called a B-algebra, and obtained several results. In 2007, Walendziak [5] introduced a new notion, called a BF-algebra, which is a generalization of B-algebra. In [6], C. B. Kim and H. S. Kim introduced BG-algebra as a generalization of B-algebra. We introduce a new notion, called a BRK-algebra, which is a generalization of BCK/BCI/BCH/Q/QS/BM-algebras. The concept of $G$-part, $p$-radical, and medial of a BRK-algebra are introduced and studied their properties.

2. Preliminaries

First, we recall certain definitions from [2–5, 7, 8] that are required in the paper.
Definition 2.1. A BCI-algebra is an algebra \((X, \ast, 0)\) of type \((2, 0)\) satisfying the following conditions:

\[
\begin{align*}
(B_1) \ (x \ast y) \ast (x \ast z) & \leq (z \ast y), \\
(B_2) \ x \ast (x \ast y) & \leq y, \\
(B_3) \ x & \leq x, \\
(B_4) \ x \leq y \text{ and } y \leq x \text{ imply } x = y, \\
(B_5) \ x \leq 0 \text{ implies } x = 0, \text{ where } x \leq y \text{ is defined by } x \ast y = 0, \text{ for all } x, y, z \in X.
\end{align*}
\]

If \((B_5)\) is replaced by \((B_6)\): \(0 \leq x\), then the algebra is called a BCK-algebra. It is known that every BCK-algebra is a BCI-algebra but not conversely.

Definition 2.2. A BCH-algebra is an algebra \((X, \ast, 0)\) of type \((2,0)\) satisfying \((B_3)\), \((B_4)\), and \((B_7)\): \((x \ast y) \ast z = (x \ast z) \ast y\).

It is shown that every BCI-algebra is a BCH-algebra but not conversely.

Definition 2.3. A Q-algebra is an algebra \((X, \ast, 0)\) of type \((2,0)\) satisfying \((B_3)\), \((B_7)\), and \((B_8)\): \(x \ast 0 = x\).

A Q-algebra is said to be a QS-algebra if it satisfies the additional relation:

\[(B_9) \ (x \ast y) \ast (x \ast z) = z \ast y,
\]

for any \(x, y, z \in X\). It is shown that every BCH-algebra is a Q-algebra but not conversely.

Definition 2.4. A B-algebra is an algebra \((X, \ast, 0)\) of type \((2,0)\) satisfying \((B_3)\), \((B_8)\), and \((B_{10})\):

\[(x \ast y) \ast z = x \ast (z \ast (0 \ast y)).
\]

A B-algebra is said to be 0-commutative if \(a \ast (0 \ast b) = b \ast (0 \ast a)\) for any \(a, b \in X\). In [3], it is shown that Q-algebras and B-algebras are different notions.

Definition 2.5. A BF-algebra is an algebra \((X, \ast, 0)\) of type \((2,0)\) satisfying \((B_3)\), \((B_8)\), and \((B_{11})\):

\[0 \ast (x \ast y) = (y \ast x).
\]

It is shown that every B-algebra is BF-algebra but not conversely.

Definition 2.6. A BM-algebra is an algebra \((X, \ast, 0)\) of type \((2,0)\) satisfying \((B_8)\) and \((B_{12})\):

\[(x \ast y) \ast (x \ast z) = z \ast y.
\]

Definition 2.7. A BH-algebra is an algebra \((X, \ast, 0)\) of type \((2,0)\) satisfying \((B_3)\), \((B_4)\), and \((B_8)\).

Definition 2.8. A BG-algebra is an algebra \((X, \ast, 0)\) of type \((2,0)\) satisfying \((B_3)\), \((B_8)\), and \((BG)\):

\[(x \ast y) \ast (0 \ast y) = x.
\]

3. BRK-Algebras

In this section, we define the notion of BRK-algebra and observe that the axioms in the definition are independent.
Definition 3.1. A BRK-algebra is a nonempty set $A$ with a constant 0 and a binary operation $*$ satisfying axioms:

(B$_8$) $x * 0 = x$,
(B$_{13}$) $(x * y) * x = 0 * y$ for any $x, y \in A$.

For brevity, we also call $A$ a BRK-algebra. In $A$, we can define a binary relation “$\leq$” by $x \leq y$ if and only if $x * y = 0$.

Example 3.2. Let $A := \mathbb{R} - \{-n\}, 0 \neq n \in \mathbb{Z}^+$ where $\mathbb{R}$ is the set of all real numbers and $\mathbb{Z}^+$ is the set of all positive integers. If we define a binary operation $*$ on $A$ by

$$x * y = \frac{n(x - y)}{n + y}, \quad (3.1)$$

then $(A, *, 0)$ is an BRK-algebra.

Example 3.3. Let $A = \{0, 1, 2\}$ in which $*$ is defined by

$$
\begin{array}{c|ccc}
* & 0 & 1 & 2 \\
\hline
0 & 0 & 2 & 2 \\
1 & 1 & 0 & 0 \\
2 & 2 & 0 & 0 \\
\end{array}
\quad (3.2)
$$

Then $(A, *, 0)$ is a BRK-algebra.

We know that every BCK-algebra is a BCI-algebra and every BCI-algebra is a BCH-algebra and every BCH-algebra is a Q-algebra. We can observe that every Q-algebra is a BRK-algebra but converse needs not be true.

Example 3.4. Let $A = \{0, 1, 2, 3\}$ in which $*$ is defined by

$$
\begin{array}{c|cccc}
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 \\
2 & 2 & 1 & 0 & 1 \\
3 & 3 & 2 & 3 & 0 \\
\end{array}
\quad (3.3)
$$

Then $(A, *, 0)$ is a BRK-algebra, which is not a BCK/BCI/BCH/Q-algebra.

We know that every QS-algebra is a BM-algebra and we can observe that every BM-algebra is a BRK-algebra but converses need not be true.
Example 3.5. Let $A = \{0, 1, 2, 3\}$ in which $\ast$ is defined by

$$
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 2 & 2 & 0 \\
1 & 1 & 0 & 0 & 2 \\
2 & 2 & 0 & 0 & 2 \\
3 & 3 & 1 & 1 & 0 \\
\end{array}
$$

(3.4)

Then $(A, \ast, 0)$ is a BRK-algebra, which is not a QS/BM-algebra.

It is easy to see that B/BG/BF/BH-algebra and BRK-algebras are different notions. For example, Example 3.3 is a BRK-algebra which is not a BH-algebra and Example 3.4 is an BRK-algebra which is not B/BG/BF-algebra. Consider the following example. Let $A = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table:

$$
\begin{array}{c|ccc|ccc}
\ast & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
0 & 0 & 2 & 1 & 3 & 4 & 5 \\
1 & 1 & 0 & 2 & 4 & 5 & 3 \\
2 & 2 & 1 & 0 & 5 & 3 & 4 \\
3 & 3 & 4 & 5 & 0 & 2 & 1 \\
4 & 4 & 5 & 3 & 1 & 0 & 2 \\
5 & 5 & 3 & 4 & 2 & 1 & 0 \\
\end{array}
$$

(3.5)

Then $(A, \ast, 0)$ is a B/BF/BG/BH-algebra which is not an BRK-algebra.

We observe that the two axioms $B_8$ and $B_{13}$ are independent. Let $A = \{0, 1, 2\}$ be a set with the following left table:

$$
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 2 \\
1 & 1 & 1 & 2 \\
2 & 2 & 1 & 2 \\
\end{array}
\quad
\begin{array}{c|cccc}
\ast & 0 & 1 & 2 \\
\hline
0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
2 & 2 & 0 & 1 \\
\end{array}
$$

(3.6)

Then the axiom $B_8$ holds but not $B_{13}$, since $(1 \ast 2) \ast 1 = 2 \ast 1 = 1 \neq 2 = 0 \ast 2$. Similarly, the set $A = \{0, 1, 2\}$ with the above right table satisfies the axiom $B_{13}$ but not $B_8$, since $2 \ast 0 = 0 \neq 2$.

Proposition 3.6. If $(A, \ast, 0)$ is a BRK-algebra, then, for any $x, y \in A$, the following conditions hold:

1. $x \ast x = 0$,
2. $x \ast y = 0 \Rightarrow 0 \ast x = 0 \ast y$. 
Proof. Let $(A, *, 0)$ be a BRK-algebra and $x, y \in A$. Then

1. $x * x = (x * 0) * x = 0 * 0 = 0$ (by B8 and B13),
2. $x * y = 0 \Rightarrow (x * y) * x = 0 * x \Rightarrow 0 * y = 0 * x$.

$\square$

**Proposition 3.7.** Every BRK-algebra $A$ satisfies the following property:

$$0 * (x * y) = (0 * x) * (0 * y),$$

for any $x, y \in A$.

Proof. Let $x, y \in A$. Then

$$0 * (x * y) = ((0 * y) * (x * y)) * (0 * y) \quad \text{(by B13)}$$
$$= [(x * y) * x] * (x * y) * (0 * y) \quad \text{(by B13)}$$
$$= (0 * x) * (0 * y).$$

$\square$

**Theorem 3.8.** Every BRK-algebra $A$ satisfying $x * (x * y) = x * y$ for all $x, y \in A$ is a trivial algebra.

Proof. Putting $x = y$ in the equation $x * (x * y) = x * y$, we obtain $x * 0 = 0 \Rightarrow x = 0$. Hence, $A$ is a trivial algebra. $\square$

**Theorem 3.9.** Every BRK-algebra $A$ satisfying $(x * y) * (x * z) = z * y$ for all $x, y, z \in A$ is a BCI-algebra.

Proof. Let $(A, *, 0)$ be a BRK-algebra and $(x * y) * (x * z) = z * y$ for all $x, y, z \in A$. Then

1. $(x * y) * (x * z) * (z * y) = (z * y) * (z * y) = 0,$
2. $x * y * x = 0,$
3. $x * y = 0 = y * x.$

Let $x * y = 0 = y * x$. Then $x = x * 0 = x * (x * y) = (x * 0) * (x * y) = y * 0 = y, y * 0 = 0 \Rightarrow x = 0.$

$\square$

**Theorem 3.10.** Every 0-commutative B-algebra is a BRK-algebra.

Proof. Let $A$ be a 0-commutative B-algebra. Then $x * (x * y) = y$ for all $x, y \in A$. Hence, $(x * y) * x = x * (x * (0 * y)) = 0 * y.$

The following theorem can be proved easily.

**Theorem 3.11.** Let $(A, *, 0)$ be a BRK-algebra. Then, for any $x, y \in A$, the following conditions hold.

1. If $(x * y) * (0 * (0 * y)) = (x * y) * y$, then $0 * (0 * (0 * y)) = 0 * y.$
2. If $(x * y) * (0 * y) = (x * y) * y$, then $0 * (0 * y) = 0 * y.$
3. If $x * (y * x) = x * (0 * (x * y))$, then $0 * (y * x) = 0 * (0 * (x * y)).$
4. G-Part of BRK-Algebras

In this section, we define \( G \)-part, \( p \)-radical and medial of a BRK-algebra. We give a necessary and sufficient condition for a BRK-algebra to become a medial BRK-algebra and investigate the properties of \( G \)-part in BRK-algebras.

Definition 4.1. A nonempty subset \( I \) of a BRK-algebra \( A \) is called a subalgebra of \( A \) if \( \star x \in I \) whenever \( x, y \in I \).

Definition 4.2. A nonempty subset \( I \) of a BRK-algebra \( A \) is called an ideal of \( A \) if for any \( x, y \in A \):

\( \begin{align*}
(\ i) \ 0 & \in I,
(\ ii) \ x \star y \in I \text{ and } y \in I \text{ imply } x \in I.
\end{align*} \)

Obviously, \( \{0\} \) and \( A \) are ideals of \( A \). We call \( \{0\} \) and \( A \) the zero ideal and the trivial ideal of \( A \), respectively. An ideal \( I \) is said to be proper if \( I \neq A \).

Definition 4.3. An ideal \( I \) of a BRK-algebra \( A \) is called a closed ideal of \( A \) if \( 0 \star x \in I \) for all \( x \in I \).

Example 4.4. Let \( A = \{0, 1, 2\} \) in which \( \star \) is defined by

\[
\begin{array}{c|ccc}
\star & 0 & 1 & 2 \\
\hline
0 & 0 & 2 & 2 \\
1 & 1 & 0 & 0 \\
2 & 2 & 0 & 0 \\
\end{array}
\]  \hspace{1cm} (4.1)

Then \( (A, \star, 0) \) is a BRK-algebra and the set \( I = \{0, 2\} \) is a subalgebra, an ideal, and a closed ideal of \( A \).

Definition 4.5. Let \( A \) be a BRK-algebra. For any subset \( S \) of \( A \), we define

\[
G(S) = \{x \in S \mid 0 \star x = x\}. \hspace{1cm} (4.2)
\]

In particular, if \( S = A \), then we say that \( G(A) \) is the \( G \)-part of a BRK-algebra.

For any BRK-algebra \( A \), the set:

\[
B(A) = \{x \in A \mid 0 \star x = 0\} \hspace{1cm} (4.3)
\]

is called a \( p \)-radical of \( A \). Clearly, \( B(A) \) is a subalgebra and an ideal of \( A \).

A BRK-algebra \( A \) is said to be \( p \)-semisimple if \( B(A) = \{0\} \).

The following property is obvious:

\[
G(A) \cap B(A) = \{0\}. \hspace{1cm} (4.4)
\]
Lemma 4.6. If $(A, \ast, 0)$ is a BRK-algebra and $a \ast b = a \ast c$ for $a, b, c \in A$, then $0 \ast b = 0 \ast c$.

Proof. Let $(A, \ast, 0)$ be a BRK-algebra and $a, b, c \in A$. Then by (B13), $a \ast b = a \ast c \Rightarrow (a \ast b) \ast a = (a \ast c) \ast a \Rightarrow 0 \ast b = 0 \ast c$. □

Theorem 4.7. Let $(A, \ast, 0)$ be a BRK-algebra. Then a left cancellation law holds in $G(A)$.

Proof. Let $a, b, c \in G(A)$ with $a \ast b = a \ast c$. Then, by Lemma 4.6, $0 \ast b = 0 \ast c$. Since $b, c \in G(A)$, we obtain $b = c$. □

Proposition 4.8. Let $(A, \ast, 0)$ be a BRK-algebra. If $x \in G(A)$, then $0 \ast x \in G(A)$.

Proof. Let $x \in G(A)$. Then $0 \ast x = x$ and hence $0 \ast (0 \ast x) = 0 \ast x$. Therefore, $0 \ast x \in G(A)$. □

Converse of the above proposition needs not be true. From Example 4.4, we can see that $0 \ast 1 = 2 \in \{0, 2\} = G(A)$ but $1 \notin G(A)$.

Theorem 4.9. If $x, y \in G(A)$, then $x \ast y \in G(A)$.

Proof. Let $x, y \in G(A)$. Then $0 \ast x = x$ and $0 \ast y = y$. Hence, $0 \ast (x \ast y) = (0 \ast x) \ast (0 \ast y) = x \ast y$. Therefore, $x \ast y \in G(A)$. □

Proposition 4.10. If $(A, \ast, 0)$ is a BRK-algebra and $x, y \in A$, then

$$y \in B(A) \iff (x \ast y) \ast x = 0.$$ (4.5)

Proof. Let $(A, \ast, 0)$ be a BRK-algebra and $x, y \in A$. Then, by (B13), $y \in B(A) \iff 0 \ast y = 0 \iff (x \ast y) \ast x = 0$. □

Theorem 4.11. If $S$ is a subalgebra of a BRK-algebra $(A, \ast, 0)$, then $G(A) \cap S = G(S)$.

Proof. Clearly, $G(A) \cap S \subseteq G(S)$. If $x \in G(S)$, then $0 \ast x = x$ and $x \in S \subseteq A$. Hence, $x \in G(A)$. Therefore, $x \in G(A) \cap S$. Thus, $G(A) \cap S = G(S)$. □

Theorem 4.12. Let $(A, \ast, 0)$ be a BRK-algebra. If $G(A) = A$, then $A$ is p-semisimple.

Proof. Assume that $G(A) = A$. Then $\{0\} = G(A) \cap B(A) = A \cap B(A) = B(A)$. Hence, $A$ is p-semisimple. □

Theorem 4.13. Every closed ideal of a BRK-algebra is a subalgebra.

Proof. Let $I$ be a closed ideal of a BRK-algebra $(A, \ast, 0)$ and $x, y \in I$. Then $0 \ast y \in I$. By (B13), $(x \ast y) \ast x = 0 \ast y \in I$. Since $I$ is an ideal and $x \in I$, we have $x \ast y \in I$. So $I$ is a subalgebra of $A$. □

Note that the converse of the above theorem is not true. In Example 3.4, the set $\{0, 1, 2\}$ is a subalgebra but not a closed ideal.

Theorem 4.14. Let $I$ be a subset of a BRK-algebra $A$. Then $I$ is a closed ideal of $A$ if and only if it satisfies (i) $0 \in I$ (ii) $x \ast z \in I$, $y \ast z \in I$ and $z \in I$ imply $x \ast y \in I$, for all $x, y, z \in A$. Theorem 4.14 is not true. In Example 3.4, the set $\{0, 1, 2\}$ is a subalgebra but not a closed ideal.
Proof. Let $I$ be a closed ideal of $A$. Clearly, $0 \in I$. Assume that $x \ast z, y \ast z, z \in I$. Since $I$ is an ideal, we have $x, y \in I$ which implies that $x \ast y \in I$ because $I$ is a closed ideal and hence a subalgebra of $A$. Conversely, assume that $I$ satisfies (i) and (ii). Let $x \ast y, y \in I$. Since $0 \ast 0, y \ast 0, 0 \in I$, by (ii) we have $0 \ast y \in I$. From (ii), again it follows that $x = x \ast 0 \in I$ so that $I$ is an ideal of $A$. Now suppose that $x \in I$. Since $0 \ast 0, x \ast 0, 0 \in I$, we obtain $0 \ast x \in I$ by (ii). This completes the proof.

Definition 4.15. A BRK-algebra $(A, \ast, 0)$ is said to be positive implicative if

$$((x \ast y) \ast y) \ast (0 \ast y) = x \ast y$$

for all $x, y \in A$.

The BRK-algebra in Example 3.3 is positive implicative.

Definition 4.16. Let $(A, \ast, 0)$ be a BRK-algebra. For a fixed $a \in A$. The map $R_a : A \rightarrow A$ given by $R_a(y) = y \ast a$ for all $y \in A$ is called right translation of $A$. Similarly, the map $L_a : A \rightarrow A$ given by $L_a(y) = a \ast y$ for all $y \in A$ is called a left translation of $A$.

Definition 4.17. Let $(A, \ast, 0)$ be a BRK-algebra. For a fixed $a \in A$. The map $T_a : A \rightarrow A$ given by $T_a(y) = (y \ast a) \ast (0 \ast a)$ for all $y \in A$ is called a weak right translation of $A$. Similarly, the map $M_a : A \rightarrow A$ given by $M_a(y) = (a \ast y) \ast (0 \ast y)$ for all $y \in A$ is called a weak left translation of $A$.

Theorem 4.18. A BRK-algebra $(A, \ast, 0)$ is positive implicative if and only if $R_z = T_z \circ R_z$ for all $z \in A$.

Proof. Let $A$ be a BRK-algebra and $R_z = T_z \circ R_z$ for $z \in A$. Then

$$y \ast z = R_z(y) = (T_z \circ R_z)(y) = T_z(R_z(y)) = T_z(y \ast z) = ((y \ast z) \ast z) \ast (0 \ast z), \quad \forall y, z \in A.$$  \hfill (4.7)

Hence, $A$ is positive implicative BRK-algebra. Conversely, assume that $A$ is positive implicative BRK-algebra. Let $x, y \in A$. Then

$$R_x(y) = y \ast x = ((y \ast x) \ast x) \ast (0 \ast x) = (R_x(y) \ast x) \ast (0 \ast x)$$

$$= T_x(R_x(y)) = (T_z \circ R_z)(y).$$  \hfill (4.8)

Hence, $R_z = T_z \circ R_z$.

Definition 4.19. A BRK-algebra $(A, \ast, 0)$ satisfying

$$(x \ast y) \ast (z \ast u) = (x \ast z) \ast (y \ast u)$$  \hfill (4.9)

for any $x, y, z$ and $u \in A$, is called a medial BRK-algebra.
Example 4.20. Let $A := \mathbb{R} - \{-n\}, \; 0 \neq n \in \mathbb{Z}^+$ where $\mathbb{R}$ is the set of all real numbers and $\mathbb{Z}^+$ is the set of all positive integers. If we define a binary operation $*$ on $A$ by

$$x * y = \frac{n(x - y)}{n + y},$$

then $(A, *, 0)$ is a medial BRK-algebra.

**Theorem 4.21.** If $A$ is a medial BRK-algebra, then, for any $x, y, z \in A$, the following hold:

(i) $x * (y * z) = (x * y) * (0 * z),$

(ii) $(x * y) * z = (x * z) * y.$

**Proof.** Let $A$ be a medial BRK-algebra and $x, y, z \in A$. Then

(i) $(x * y) * (0 * z) = (x * 0) * (y * z) = x * (y * z),$

(ii) $(x * y) * z = (x * y) * (z * 0) = (x * z) * (y * 0) = (x * z) * y.$

By the above theorem, the following corollary follows.

**Corollary 4.22.** Every medial BRK-algebra is a $Q$-algebra.

**Theorem 4.23.** Let $A$ be a medial BRK-algebra. Then the right cancellation law holds in $G(A)$.

**Proof.** Let $a, b, x \in G(A)$ with $a * x = b * x$. Then, for any $y \in G(A), \; x * y = (0 * x) * y = (0 * y) * x = y * x$. Therefore,

$$a = 0 * a = (x * a) * x = (a * x) * x = (b * x) * x = (x * b) * x = 0 * b = b. \quad (4.11)$$

Now, we give a necessary and sufficient condition for a BRK-algebra to become a medial BRK-algebra.

**Theorem 4.24.** A BRK-algebra $A$ is medial if and only if it satisfies:

(i) $x * y = 0 * (y * x)$ for all $x, y \in A,$

(ii) $(x * y) * z = (x * z) * y$ for all $x, y, z \in A.$

**Proof.** Suppose $(A, *, 0)$ is medial and $x, y, z \in A$. Then

(i) $0 * (y * x) = (x * x) * (y * x) = (x * y) * (x * x) = (x * y) * 0 = x * y,$

(ii) $(x * y) * z = (x * y) * (z * 0) = (x * z) * (y * 0) = (x * z) * y.$
Conversely, assume that the conditions hold. Then

\[(x * y) * (z * u) = 0 * ((z * u) * (x * y)) \quad \text{(by (i))}\]
\[= 0 * ((z * (x * y)) * u) \quad \text{(by (ii))}\]
\[= (0 * (z * (x * y))) * (0 * u) \quad \text{(by Proposition 3.7)}\]
\[= ((x * y) * z) * (0 * u) \quad \text{(by (i))}\]
\[= ((x * z) * y) * (0 * u) \quad \text{(by (ii))}\]
\[= ((x * z) * (0 * u)) * y \quad \text{(by (ii))}\]
\[= (0 * ((0 * u) * (x * z))) * y \quad \text{(by (i))}\]
\[= (0 * ((z * x) * u)) * y \quad \text{(by (ii) \& (i))}\]
\[= (u * (z * x)) * y \quad \text{(by (i))}\]
\[= (u * y) * (z * x) \quad \text{(by (ii))}\]
\[= 0 * ((z * x) * (u * y)) \quad \text{(by (i))}\]
\[= (x * z) * (y * u) \quad \text{(by Proposition 3.7 and (i))}\]  

Therefore, \( A \) is medial. \( \square \)

**Corollary 4.25.** A BRK-algebra \( A \) is medial if and only if it is a medial QS-algebra.

The following theorem can be proved easily.

**Theorem 4.26.** An algebra \((A, *, 0)\) of type \((2, 0)\) is a medial BRK-algebra if and only if it satisfies:

(i) \( x * (y * z) = z * (y * x) \),

(ii) \( x * 0 = x \),

(iii) \( x * x = 0 \).

**Corollary 4.27.** If \( A \) is a medial BRK-algebra, then \( x * (x * y) = y \) for all \( x, y \in A \).

**Corollary 4.28.** The class of all of medial BRK-algebras forms a variety, written \( \nu(MR) \).

**Proposition 4.29.** A variety \( \nu \) is congruence-permutable if and only if there is a term \( p(x, y, z) \) such that

\[ \nu \models p(x, x, y) \Leftrightarrow y, \quad \nu \models p(x, y, y) \Leftrightarrow x. \]  

**Corollary 4.30.** The variety \( \nu(MR) \) is congruence permutative.

**Proof.** Let \( p(x, y, z) = x * (y * z) \). Then by Corollary 4.25 and \((B_6)\), we have \( p(x, x, y) = y \) and \( p(x, y, y) = x \), and so the variety \( \nu(MR) \) is congruence permutative. \( \square \)

The following example shows that a BRK-algebra may not satisfy the associative law.
Example 4.31. Let $A = \{0, 1, 2\}$ be a set with the following table:

<table>
<thead>
<tr>
<th>$*$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(4.14)

Then $(A, *, 0)$ is a BRK-algebra, but associativity does not hold since $(1*2)*1 = 0*1 = 2 \neq 1 = 1*0 = 1*(2*1)$.

Theorem 4.32. If $A$ is an associative BRK-algebra, then, for any $x \in B(A)$, $x = 0$.

Proof. Let $x \in B(A)$. Then $0 = 0 * x = (x * x) * x = x * (x * x) = x * 0 = x$. \qed

Theorem 4.33. If $A$ is an associative BRK-algebra, then $G(A) = A$.

Proof. Let $A$ be an associative BRK-algebra. Clearly, $G(A) \subseteq A$. Let $x \in A$. Then $0 * x = (x * x) * x = x * (x * x) = x * 0 = x$. Hence, $x \in G(A)$. Therefore, $G(A) = A$. \qed

Now, we prove that every associative BRK-algebra is a group.

Theorem 4.34. Every BRK-algebra $(A, *, 0)$ satisfying the associative law is a group under the operation “$*$”.

Proof. Putting $x = y = z$ in the associative law $(x * y) * z = x * (y * z)$ and using (B3) and (B8), we obtain $0 * x = x * 0 = x$. This means that 0 is the zero element of $A$. By (B3), every element $x$ of $A$ has as its inverse the element $x$ itself. Therefore, $(A, *)$ is a group. \qed

5. Conclusion and Future Research

In this paper, we have introduced the concept of BRK-algebra and studied their properties. In addition, we have defined $G$-part, $p$-radical, and medial of BRK-algebra and proved that the variety of medial algebras is congruence permutable. Finally, we proved that every associative BRK-algebra is a group.

In our future work, we introduce the concept of fuzzy BRK-algebra, interval-valued fuzzy BRK-algebra, intuitionistic fuzzy BRK-algebra, intuitionistic fuzzy structure of BRK-algebra, intuitionistic fuzzy ideals of BRK-algebra, and intuitionistic $(T,S)$-normed fuzzy subalgebras of BRK-algebras, intuitionistic $L$-fuzzy ideals of BRK-algebra.

I hope this work would serve as a foundation for further studies on the structure of BRK-algebras.

References


