Research Article

Discrete Mixed Petrov-Galerkin Finite Element Method for a Fourth-Order Two-Point Boundary Value Problem

L. Jones Tarcus Doss¹ and A. P. Nandini²

¹ Department of Mathematics, Anna University Chennai, CEG Campus, Chennai 600 025, India

² Department of Mathematics, M.N.M Jain Engineering College, Thoraipakkam, Chennai 600097, India

Correspondence should be addressed to L. Jones Tarcus Doss, jones@annauniv.edu

Received 20 July 2011; Revised 24 November 2011; Accepted 25 November 2011

Academic Editor: Attila Gilányi

Copyright © 2012 L. J. T. Doss and A. P. Nandini. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

A quadrature-based mixed Petrov-Galerkin finite element method is applied to a fourth-order linear ordinary differential equation. After employing a splitting technique, a cubic spline trial space and a piecewise linear test space are considered in the method. The integrals are then replaced by the Gauss quadrature rule in the formulation itself. Optimal order *a priori* error estimates are obtained without any restriction on the mesh.

1. Introduction

In this paper, we develop a quadrature-based Petrov-Galerkin mixed finite element method for the following fourth-order boundary value problem:

$$\frac{d^2}{dx^2} \left[a(x) \frac{d^2 u}{dx^2} \right] + b(x)u = f(x), \quad x \in I = (0, 1), \quad (1.1)$$

subject to the boundary conditions

$$u(0) = 0, \quad u(1) = 0; \quad u''(0) = 0, \quad u''(1) = 0, \quad (1.2)$$

where $a(x) \neq 0$, $x \in I$. Let $\alpha(x) = 1/a(x)$. We, hereafter, suppress the dependency of the independent variable x on the functions $\alpha(x)$, $b(x)$, and $f(x)$. Therefore, we write α , b , and f instead of these functions.

Let us define the splitting of the above fourth-order equation as follows.

Set

$$u'' = \alpha v, \quad x \in I. \quad (1.3)$$

Then the differential equation (1.1) with the boundary conditions (1.2) can be written as a coupled system of equations as follows:

$$u'' = \alpha v, \quad x \in I, \text{ with } u(0) = u(1) = 0, \quad (1.4)$$

$$v'' + bu = f, \quad x \in I, \text{ with } v(0) = v(1) = 0. \quad (1.5)$$

In this paper, the error analysis will take place in the usual Sobolev space $W_p^m(I)$ defined on the domain $I = (0, 1)$ with $H^m(I)$ denoting $W_2^m(I)$. The Sobolev norms are given below. For an open interval E and a non negative integer m ,

$$\begin{aligned} \|v\|_{W_p^m(E)} &= \left(\sum_{i=0}^m \|v^{(i)}\|_{L_p(E)}^p \right)^{1/p}, \quad \text{if } 1 \leq p < \infty, \\ &= \max_{1 \leq i \leq m} \|v^{(i)}\|_{L_\infty(E)}, \quad \text{if } p = \infty. \end{aligned} \quad (1.6)$$

We suppress the dependence of the norms on I when $E = I$. Further, $H_0^m(I)$ denotes the function space

$$\{\phi \in H^m(I) : \phi(0) = \phi(1) = 0\}. \quad (1.7)$$

2. Continuous and Discrete H^1 -Galerkin Formulation

Given $n > 1$, let

$$\Pi_n : 0 = x_0 < x_1 < \cdots < x_n = 1 \quad (2.1)$$

be an arbitrary partition of $[0, 1]$ with the property that $h \rightarrow 0$ as $n \rightarrow \infty$, where $h = \max_{1 \leq k \leq n} h_k$ and $h_k = x_k - x_{k-1}$, $k = 1, \dots, n$. Let (u, v) represent the L_2 inner product, and let $\langle u, v \rangle_h$ represent the discrete inner product of any two functions $u, v \in L_2(I)$ and be defined as follows:

$$(u, v) = \int uv \, dx, \quad \langle u, v \rangle_h = Q_h(uv), \quad (2.2)$$

where Q_h is the fourth-order Gaussian quadrature rule:

$$Q_h(g) := \frac{1}{2} \sum_{i=1}^n h_k [g(x_{k,1}) + g(x_{k,2})]. \quad (2.3)$$

Here, $x_{k,i} = x_{k-1} + \xi_i h_k$, $i = 1, 2$, are the two Gaussian points in the subinterval $[x_{k-1}, x_k]$ with $\xi_1 = (1/2)(1 - 1/\sqrt{3})$, $\xi_2 = 1 - \xi_1$.

Let us now consider the following cubic spline space as trial space:

$$S_{h,3} = \left\{ \varphi \in C^2(I) : \varphi|_{I_k} \in P_3(I_k), k = 1, 2, \dots, n \right\}, \quad (2.4)$$

where $P_r(I_k)$ is the space of polynomials of degree r defined over the k th subinterval $I_k = [x_{k-1}, x_k]$.

The corresponding space with zero Dirichlet boundary condition is denoted by

$$S_{h,3}^0 = \left\{ \varphi \in S_{h,3} : \varphi(0) = \varphi(1) = 0 \right\}. \quad (2.5)$$

Further, let us consider the following piecewise linear space

$$S_{h,1} = \left\{ \varphi \in C(I) : \varphi|_{I_k} \in P_1(I_k), k = 1, 2, \dots, n \right\} \quad (2.6)$$

as the test space.

2.1. Weak Formulation

The weak formulation corresponding to the split equations (1.4) and (1.5) is defined, respectively, as follows.

Find $\{u, v\} \in H_0^2(I)$ such that

$$(u'', \phi) = (\alpha v, \phi), \quad \phi \in H^2(0, 1), \quad (2.7)$$

$$(v'' + bu, \phi) = (f, \phi), \quad \phi \in H^2(0, 1). \quad (2.8)$$

2.2. The Petrov-Galerkin Formulation

The Petrov-Galerkin formulation corresponding to the above weak formulation (2.7) and (2.8) is defined, respectively, as follows.

Find $\{u_h, v_h\} \in S_{h,3}^0$ such that

$$\begin{aligned} (u_h'', \phi_h) &= (\alpha v_h, \phi_h), \quad \phi_h \in S_{h,1}, \\ (v_h'' + bu_h, \phi_h) &= (f, \phi_h), \quad \phi_h \in S_{h,1}. \end{aligned} \quad (2.9)$$

The integrals in the above Petrov-Galerkin formulation are not evaluated exactly at the implementation level. We, therefore, define the following discrete Petrov-Galerkin procedure in which the integrals are replaced by the Gaussian quadrature in the scheme as follows.

2.3. Discrete Petrov-Galerkin Formulation

The discrete Petrov-Galerkin formulation corresponding to (2.7) and (2.8) is defined, respectively, as follows.

Find $\{u_h, v_h\} \in S_{h,3}^0$ such that

$$\langle u_h'', \phi_h \rangle_h = \langle \alpha v_h, \phi_h \rangle_h, \quad \phi_h \in S_{h,1}, \quad (2.10)$$

$$\langle v_h'' + bu_h, \phi_h \rangle_h = \langle f, \phi_h \rangle_h, \quad \phi_h \in S_{h,1}. \quad (2.11)$$

The approximate solutions u_h and v_h without any conditions on boundary points are expressed as a linear combination of the B-splines as follows:

$$u_h(x) = \sum_{j=-1}^{n+1} \gamma_j B_j(x), \quad v_h(x) = \sum_{j=-1}^{n+1} \delta_j B_j(x), \quad (2.12)$$

where the j th basis $B_j(x)$ of the cubic B-splines space $S_{h,3}$ for $j = -1, 0, 1, 2, \dots, n, n+1$ is given below:

$$B_j(x) = \begin{cases} 0, & \text{if } x \leq x_{j-2}, \\ \frac{1}{6h^3}(x - x_{j-2})^3, & \text{if } x_{j-2} \leq x \leq x_{j-1}, \\ \frac{1}{6h^3}(h^3 + 3h^2(x - x_{j-1}) + 3h(x - x_{j-1})^2 - 3(x - x_{j-1})^3), & \text{if } x_{j-1} \leq x \leq x_j, \\ \frac{1}{6h^3}(h^3 + 3h^2(x_{j+1} - x) + 3h(x_{j+1} - x)^2 - 3(x_{j+1} - x)^3), & \text{if } x_j \leq x \leq x_{j+1}, \\ \frac{1}{6h^3}(x_{j+2} - x)^3, & \text{if } x_{j+1} \leq x \leq x_{j+2}, \\ 0, & \text{if } x \geq x_{j+2}. \end{cases} \quad (2.13)$$

For $j = -1, 0$ and $j = n, n+1$, the basis functions are defined as in the above form, after extending the partition by introducing fictitious nodal points x_{-3}, x_{-2}, x_{-1} on the left-hand side and $x_{n+1}, x_{n+2}, x_{n+3}$ on the right-hand side, respectively. Further, the i th basis $\phi_i(x)$ of the piecewise linear “hat” splines space $S_{h,1}$ for $i = 0, 1, 2, \dots, n$ is given below:

$$\phi_i(x) = \begin{cases} 0, & \text{if } x \leq x_{i-1}, \\ \frac{1}{h}(x - x_{i-1}), & \text{if } x_{i-1} \leq x \leq x_i, \\ \frac{1}{h}(x_{i+1} - x), & \text{if } x_i \leq x \leq x_{i+1}, \\ 0, & \text{if } x \geq x_{i+1}. \end{cases} \quad (2.14)$$

In a similar manner, for $i = 0$ and $i = n$, the basis functions are defined as in the above form, after extending the partition by introducing fictitious nodal point x_{-1} on the left-hand side and x_{n+1} on the right-hand side, respectively. The mixed discrete Petrov-Galerkin method for (2.10) and (2.11) without assuming boundary conditions in the trial space is given as follows:

$$\begin{aligned} \sum_{j=-1}^{n+1} \gamma_j \langle B_j'', \phi_i \rangle_h - \sum_{j=-1}^{n+1} \delta_j \langle \alpha B_j, \phi_i \rangle_h &= 0, \quad i = 0, 1, 2, \dots, n, \\ \sum_{j=-1}^{n+1} \gamma_j \langle b B_j, \phi_i \rangle_h + \sum_{j=-1}^{n+1} \delta_j \langle B_j'', \phi_i \rangle_h &= \langle f, \phi_i \rangle_h, \quad i = 0, 1, 2, \dots, n, \end{aligned} \quad (2.15)$$

with the corresponding equations:

$$\begin{aligned} \sum_{j=-1}^{n+1} \gamma_j B_j(0) &= 0, & \sum_{j=-1}^{n+1} \gamma_j B_j(1) &= 0, \\ \sum_{j=-1}^{n+1} \delta_j B_j(0) &= 0, & \sum_{j=-1}^{n+1} \delta_j B_j(1) &= 0, \end{aligned} \quad (2.16)$$

referring to the zero-boundary conditions:

$$u_h(0) = 0, \quad u_h(1) = 0, \quad v_h(0) = 0, \quad v_h(1) = 0. \quad (2.17)$$

The above set of equations (2.15)–(2.16) can be written as a set of $2n + 6$ equations in $2n + 6$ unknowns. Here, we study the effect of quadrature rule in the error analysis. Since we compute the approximations for the solution $u(x)$ as well as for its second derivative $v(x)$ with integrals replaced by Gaussian quadrature rule in the formulation, this work may be considered as a quadrature-based mixed Petrov-Galerkin method.

3. Overview of Discrete Petrov-Galerkin Method

Here, the integrals are replaced by composite two-point Gauss rule. Therefore, the resulting method may be described as a “qualocation” approximation, that is, a quadrature-based modification of the collocation method. Further, it may be considered as a Petrov-Galerkin method with a quadrature rule because the test space and trial space are different. Hence, it may be referred to as discrete Petrov-Galerkin method. One practical advantage of this procedure over the orthogonal spline collocation method described in Douglas Jr. and Dupont [1, 2] is that for a given partition there are only half the number of unknowns, and therefore it reduces the size of the matrix.

The qualocation method was first introduced and analysed by Sloan [3] for boundary integral equation on smooth curves. Later on Sloan et al. [4] extended this method to a class of linear second-order two-point boundary value problems and derived optimal error estimates without quasi-uniformity assumption on the finite element mesh. Then, Jones Doss and Pani [5] discussed the qualocation method for a second-order semilinear two-point boundary

value problem. Further, Pani [6] expanded its scope by adapting the analysis to a semilinear parabolic initial and boundary value problem in a single space variable. Jones Doss and Pani [7] extended this method to the free boundary problem, that is, one-dimensional single-phase Stefan problem for which part of the boundary has to be found out along with the solution process. A quadrature-based Petrov-Galerkin method applied to higher dimensional boundary value problems is studied in Bialecki et al. [8, 9] and Ganesh and Mustapha [10].

The main idea of this paper is that a quadrature based approximation for a fourth order problem is analyzed in mixed Galerkin setting. The organization of this paper is as follows. In previous Sections 1 and 2, the problem is introduced; the weak and the Galerkin formulations are defined. Overview of discrete Petrov-Galerkin method is discussed in Section 3. Preliminaries required for our analysis are mentioned in Section 4. Error analysis is carried over in Section 5. Throughout this paper C is a generic positive constant, whose dependence on the smoothness of the exact solution can be easily determined from the proofs.

4. Preliminaries

We assume that α and b are such that

$$\alpha, b \in C^4(\bar{I}), \quad (4.1)$$

where $\bar{I} = [0, 1]$. We assume that the problem consisting of the coupled equations (1.4) and (1.5) is uniquely solvable for a given sufficiently smooth function $f(x)$. It can be proved that the quadrature rule in (2.3) has an error bound of the form

$$E_h(g) = \left| Q_h(g) - \int g \right| \leq C \sum_{i=1}^n h_k^4 \|g^{(4)}\|_{L_1(I_k)}. \quad (4.2)$$

This follows from Peano's kernel theorem (see [11]).

The following inequality is frequently used in our analysis. If $v \in W_p^m(E)$ with $p \in [1, \infty]$, then there exists a positive constant C depending only on m such that, for any δ satisfying $0 < \delta \leq |E| \leq 1$,

$$\|v\|_{W_p^i(E)} \leq C \left[\delta^{m-i} \|v\|_{W_p^m(E)} + \delta^{-i} \|v\|_{L_p(E)} \right], \quad 0 \leq i \leq m-1, \quad (4.3)$$

where $|E|$ denotes the length of E . For a detailed proof, one may refer to appendix of Sloan et al. [4] or Chapter 4 of Adams [12]. Let us use the following notation:

$$Lv := v''. \quad (4.4)$$

The adjoint operator L^* with corresponding adjoint boundary condition is defined as follows:

$$\begin{aligned} L^*\phi &= \phi'', \\ \phi(0) &= \phi(1) = 0. \end{aligned} \quad (4.5)$$

Since L is a self-adjoint operator, we mention below the regularity of L^* (equal to L) in the q norm. We make a stronger assumption as in Sloan et al. [4] that for arbitrary $q \in [1, \infty]$, there exists a positive constant C such that

$$\|L^*u\|_{L_q(I)} \geq C\|u\|_{W_q^1(I)}. \quad (4.6)$$

We have the following inequality due to the Sobolev embedding theorem; the proof of which can be found in page 97, Adams [12],

$$\|\phi\|_{L_\infty(I_k)} \leq \|\phi\|_{W_p^1(I_k)}; \quad 1 \leq p \leq \infty, \quad \phi \in W_p^1(I_k). \quad (4.7)$$

5. Convergence Analysis

Hereafter throughout this section, for p and q with $1 \leq p, q \leq \infty$, s and $p^{-1} + q^{-1} = 1$, we use the following notations:

$$\|v\|_{0,p} = \|v\|_{L_p}, \quad \|v\|_{s,p} = \|v\|_{W_p^s}, \quad \|v\|_{s,p,k} = \|v\|_{W_p^s(I_k)}. \quad (5.1)$$

Let us denote the error between u and u_h by ε_h and the error between v and v_h by e_h , respectively, that is, $\varepsilon_h = u - u_h$ and $e_h = v - v_h$. Using (2.11) and (1.5), we obtain the following error equations:

$$\langle e_h'', \phi_h \rangle_h = \langle v'' - v_h'', \phi_h \rangle_h = \langle v'', \phi_h \rangle_h - \langle f - bu_h, \phi_h \rangle_h = -\langle b(u - u_h), \phi_h \rangle_h = -\langle b\varepsilon_h, \phi_h \rangle_h, \quad (5.2)$$

and therefore we get

$$\langle e_h'', \phi_h \rangle_h = -\langle b\varepsilon_h, \phi_h \rangle_h, \quad \phi_h \in S_{h,1}. \quad (5.3)$$

Further, using (2.10) and (1.4),

$$\langle \varepsilon_h'', \phi_h \rangle_h = \langle u'' - u_h'', \phi_h \rangle_h = \langle \alpha(v - v_h), \phi_h \rangle_h = \langle \alpha e_h, \phi_h \rangle_h, \quad (5.4)$$

and therefore we have

$$\langle \varepsilon_h'', \phi_h \rangle_h = \langle \alpha e_h, \phi_h \rangle_h, \quad \phi_h \in S_{h,1}. \quad (5.5)$$

The following lemma gives estimates for the error in the quadrature rule for the term $(e_h''\chi_h)$ and $(\varepsilon_h''\chi_h)$ for $\chi_h \in S_{h,1}$. These estimates are required for our error analysis later. The proof of the lemma is similar to the proof of Lemma 4.2 of Sloan et al. [4].

Lemma 5.1. For all $\chi_h \in S_{h,1}$ and h sufficiently small,

$$(a) E_h(e_h'' \chi_h) \leq Ch^4 \|v\|_{6,p} \|\chi_h\|_{1,q},$$

$$(b) E_h(e_h'' \chi_h) \leq Ch^3 \|v\|_{6,p} \|\chi_h\|_{0,q},$$

$$(c) E_h(\varepsilon_h'' \chi_h) \leq Ch^4 \|u\|_{6,p} \|\chi_h\|_{1,q},$$

$$(d) E_h(\varepsilon_h'' \chi_h) \leq Ch^3 \|u\|_{6,p} \|\chi_h\|_{0,q}.$$

The following result gives estimate for $\varepsilon_h(\bar{x})$, where $\bar{x}$ is any arbitrary point in I . This estimate is crucial for our error analysis.

Lemma 5.2. Let u be the weak solution of (1.4) defined through (2.7). Further, let u_h be the corresponding discrete Petrov-Galerkin solution defined through (2.10). Then, the error $\varepsilon_h = u - u_h$ satisfies

$$|\varepsilon_h(\bar{x})| \leq C \left[h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p} \right], \quad (5.6)$$

where $\bar{x}$ is an arbitrary point in $[0, 1]$.

Proof. For a given $\bar{x} \in [0, 1]$, let Φ be an element of $L_p(I) \cap C(I)$ satisfying the following auxiliary problem:

$$\begin{aligned} \Phi'' &= 0, & x \in I - \{\bar{x}\}, \\ \Phi(0) &= \Phi(1) = 0, & \Phi'_-(\bar{x}) - \Phi'_+(\bar{x}) = -1. \end{aligned} \quad (5.7)$$

The above problem has a solution. For example,

$$\Phi(x) = \begin{cases} (\bar{x} - 1)x, & 0 \leq x \leq \bar{x}, \\ \bar{x}(x - 1), & \bar{x} \leq x \leq 1 \end{cases} \quad (5.8)$$

satisfies the above differential equation, the boundary conditions, and the jump condition.

Let us define Ψ as follows:

$$\Psi(x) = \begin{cases} \Phi'', & x \in I - \{\bar{x}\}, \\ 0, & \text{at } x = \bar{x}. \end{cases} \quad (5.9)$$

Then, $\Psi = 0$ a.e. on I . We first multiply ε_h with Ψ and then integrate over I . On applying integration by parts, using the fact that $\varepsilon_h(0) = \varepsilon_h(1) = 0$ and the jump condition for Φ' , we obtain

$$\begin{aligned} 0 &= (\varepsilon_h, \Psi) = \int_0^{\bar{x}} \varepsilon_h \Psi + \int_{\bar{x}}^1 \varepsilon_h \Psi = \int_0^{\bar{x}} \varepsilon_h \Phi'' + \int_{\bar{x}}^1 \varepsilon_h \Phi'' \\ &= [\varepsilon_h \Phi']_0^{\bar{x}} - \int_0^{\bar{x}} \varepsilon_h' \Phi' + [\varepsilon_h \Phi']_{\bar{x}}^1 - \int_{\bar{x}}^1 \varepsilon_h' \Phi' = \varepsilon_h(\bar{x}) [\Phi'_-(\bar{x}) - \Phi'_+(\bar{x})] - \int_0^{\bar{x}} \varepsilon_h' \Phi' - \int_{\bar{x}}^1 \varepsilon_h' \Phi' \\ &= -\varepsilon_h(\bar{x}) - \int_0^{\bar{x}} \varepsilon_h' \Phi' - \int_{\bar{x}}^1 \varepsilon_h' \Phi'. \end{aligned} \quad (5.10)$$

Applying integration by parts once again, using boundary condition for Φ and the continuity of Φ , we obtain

$$0 = -\varepsilon_h(\bar{x}) - \left\{ [\varepsilon_h' \Phi]_0^{\bar{x}} - \int_0^{\bar{x}} \varepsilon_h'' \Phi + [\varepsilon_h' \Phi]_{\bar{x}}^1 - \int_{\bar{x}}^1 \varepsilon_h'' \Phi \right\} = -\varepsilon_h(\bar{x}) + (\varepsilon_h'', \Phi), \quad (5.11)$$

that is, $\varepsilon_h(\bar{x}) = (\varepsilon_h'', \Phi)$. Let Φ_h be the linear interpolant of Φ . Then, we have

$$\begin{aligned} \varepsilon_h(\bar{x}) &= (\varepsilon_h'', \Phi - \Phi_h) + (\varepsilon_h'', \Phi_h) - \langle \varepsilon_h'', \Phi_h \rangle_h + \langle \varepsilon_h'', \Phi_h \rangle_h \\ |\varepsilon_h(\bar{x})| &\leq \left| (\varepsilon_h'', \Phi - \Phi_h) \right| + \left| E_h(\varepsilon_h'' \Phi_h) \right| + \left| \langle \varepsilon_h'', \Phi_h \rangle_h \right| \leq T_1 + T_2 + T_3. \end{aligned} \quad (5.12)$$

We know that

$$\|\Phi_h\|_{1,q} \leq \|\Phi - \Phi_h\|_{1,q} + \|\Phi\|_{1,q} \leq Ch\|\Phi\|_{2,q} + \|\Phi\|_{2,q} \leq C\|\Phi\|_{2,q}. \quad (5.13)$$

We now compute the estimates for the terms T_1 , T_2 , and T_3 as follows:

$$T_1 = \left| (\varepsilon_h'', \Phi - \Phi_h) \right| \leq \left\| \varepsilon_h'' \right\|_{0,p} \|\Phi - \Phi_h\|_{0,q} \leq Ch^2 \|\varepsilon_h\|_{2,p} \|\Phi\|_{2,q}. \quad (5.14)$$

Using Lemma 5.1(c) and (5.13), we obtain

$$T_2 = \left| E_h(\varepsilon_h'' \Phi_h) \right| \leq Ch^4 \|u\|_{6,p} \|\Phi\|_{2,q}. \quad (5.15)$$

Using (5.5), (2.3), and the Sobolev embedding theorem (4.7) locally on I_k for both $\|e_h\|_{0,\infty,k}$ and $\|\Phi_h\|_{0,\infty,k}$, we have

$$T_3 = \left| \langle \varepsilon_h'', \Phi_h \rangle_h \right| = |\langle \alpha e_h, \Phi_h \rangle_h| \leq C \sum_{k=1}^n \frac{h_k}{2} \|e_h\|_{0,\infty,k} \|\Phi_h\|_{0,\infty,k} \leq C \sum_{k=1}^n \frac{h_k}{2} \|e_h\|_{1,p,k} \|\Phi_h\|_{1,q,k}. \quad (5.16)$$

Using Hölder's inequality for sums and (5.13), we have

$$T_3 \leq Ch \|e_h\|_{1,p} \|\Phi_h\|_{1,q} \leq Ch \|e_h\|_{1,p} \|\Phi\|_{2,q}. \quad (5.17)$$

For Φ satisfying the auxiliary problem, it is easy to verify that $\|\Phi\|_{2,q} \leq K$, where K is a constant not depending on h .

Using T_1 , T_2 , and T_3 in (5.12), we have

$$|\varepsilon_h(\bar{x})| \leq C \left[h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p} \right]. \quad (5.18)$$

This completes the proof. $\square$

In the following lemma, we initially compute the error $(v - v_h)$ in terms of $(u - u_h)$, and then later on we establish an optimal estimate of error $(v - v_h)$ independent of $(u - u_h)$.

Lemma 5.3. *Let u and v be the weak solutions of the coupled equations (1.4) and (1.5) defined through (2.7) and (2.8), respectively. Further, let u_h and v_h be the corresponding discrete Petrov-Galerkin solutions defined through (2.10) and (2.11), respectively. Then the estimates of the errors $e_h = v - v_h$ in L_p , W_p^1 , and W_p^2 norms are given as follows:*

$$\begin{aligned} \|e_h\|_{0,p} &\leq C \left[h^4 \|v\|_{6,p} + h^5 \|u\|_{6,p} + h^3 \|\varepsilon_h\|_{2,p} \right], \\ \|e_h\|_{1,p} &\leq C \left[h^3 \|v\|_{6,p} + h^4 \|u\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} \right], \\ \|e_h\|_{2,p} &\leq C \left[h^2 \|v\|_{6,p} + h^4 \|u\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} \right]. \end{aligned} \quad (5.19)$$

Proof. Let η be an arbitrary element of L_q , and let $\phi \in W_q^2$ be the solution of the auxiliary problem

$$\begin{aligned} L^* \phi &= \eta, \\ \phi(0) &= \phi(1) = 0. \end{aligned} \quad (5.20)$$

We now have

$$\begin{aligned} (e_h, \eta) &= (e_h, L^* \phi) = (Le_h, \phi) = (e_h'', \phi - \phi_h) + (e_h'', \phi_h) \\ &= (e_h'', \phi - \phi_h) + (e_h'', \phi_h) - \langle e_h'', \phi_h \rangle_h + \langle e_h'', \phi_h \rangle_h \\ &= (e_h'', \phi - \phi_h) + E_h(e_h'' \phi_h) + \langle e_h'', \phi_h \rangle_h, \\ |(e_h, \eta)| &\leq \left| (e_h'', \phi - \phi_h) \right| + \left| E_h(e_h'' \phi_h) \right| + \left| \langle e_h'', \phi_h \rangle_h \right| \\ &\leq T_4 + T_5 + T_6, \end{aligned} \quad (5.21)$$

where $\phi_h \in S_{h,1}$ is the linear interpolant of ϕ .

We know that

$$\|\phi_h\|_{1,q} \leq \|\phi - \phi_h\|_{1,q} + \|\phi\|_{1,q} \leq Ch\|\phi\|_{2,q} + \|\phi\|_{2,q} \leq C\|\phi\|_{2,q}. \quad (5.22)$$

We shall compute the estimates for the terms T_4 , T_5 , and T_6 as follows:

$$\begin{aligned} T_4 &= \left| \left(e_h'', \phi - \phi_h \right) \right| \leq \|e_h''\|_{0,p} \|\phi - \phi_h\|_{0,q} \leq Ch^2 \|e_h\|_{2,p} \|\phi\|_{2,q}, \\ T_5 &= \left| E_h(e_h'' \phi_h) \right| \leq Ch^4 \|v\|_{6,p} \|\phi_h\|_{1,q} \leq Ch^4 \|v\|_{6,p} \|\phi\|_{2,q} \text{ by Lemma 5.1(a), } (5.22). \end{aligned} \quad (5.23)$$

Using (5.3), (2.3), and the Sobolev embedding theorem (4.7) locally on I_k for $\|\phi_h\|_{0,\infty,k}$, we have

$$T_6 = \left| \left\langle e_h'', \phi_h \right\rangle_h \right| = \left| -\langle b\varepsilon_h, \phi_h \rangle_h \right| \leq C \sum_{k=1}^n \frac{h_k}{2} \|\varepsilon_h\|_{0,\infty,k} \|\phi_h\|_{0,\infty,k} \leq C \sum_{k=1}^n \frac{h_k}{2} \|\varepsilon_h\|_{0,\infty,k} \|\phi_h\|_{1,q,k}. \quad (5.24)$$

Using Hölder's inequality for sums, Lemma 5.2, and (5.22), we obtain

$$T_6 \leq Ch \left[h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p} \right] \|\phi_h\|_{1,q} \leq C \left[h^3 \|\varepsilon_h\|_{2,p} + h^5 \|u\|_{6,p} + h^2 \|e_h\|_{1,p} \right] \|\phi\|_{2,q}. \quad (5.25)$$

Substituting T_4 , T_5 , and T_6 in (5.21), we have

$$\left| (e_h, \eta) \right| \leq C \left[h^2 \|e_h\|_{2,p} + h^4 \|v\|_{6,p} + h^3 \|\varepsilon_h\|_{2,p} + h^5 \|u\|_{6,p} + h^2 \|e_h\|_{1,p} \right] \|\phi\|_{2,q}. \quad (5.26)$$

Using (4.6) and the regularity of the auxiliary problem, we have $\|\phi\|_{2,q} \leq C\|\eta\|_{0,q}$. Since $\eta \in L_q$ is arbitrary, we have

$$\|e_h\|_{0,p} \leq C \left(h^2 \|e_h\|_{2,p} + h^3 \|\varepsilon_h\|_{2,p} + h^4 \|v\|_{6,p} + h^5 \|u\|_{6,p} \right). \quad (5.27)$$

We now estimate $\|e_h''\|$ via a projection argument. Let P_h be the orthogonal projection onto $S_{h,1}$ with respect to L_2 inner product defined by

$$\left(v'' - P_h v'', \varphi_h \right) = 0, \quad \varphi_h \in S_{h,1}. \quad (5.28)$$

The domain of P_h may be taken to be L_1 . From Crouzeix and Thomée [13] and de Boor [14], it is seen that the L_2 projection is stable. Thus,

$$\|P_h v\|_{0,p} \leq C \|v\|_{0,p}. \quad (5.29)$$

Then the error e_h'' can be interpreted in terms of the error of the above projection:

$$\|e_h''\|_{0,p} = \|v'' - v_h''\|_{0,p} \leq \|v'' - P_h v''\|_{0,p} + \|P_h v'' - v_h''\|_{0,p}. \quad (5.30)$$

From the stability property (5.29), the error in the projection follows as in de Boor [14], that is,

$$\|v'' - P_h v''\|_{0,p} \leq Ch^2 \|v''\|_{2,p} \leq Ch^2 \|v\|_{4,p}. \quad (5.31)$$

Then the remaining task is to compute the estimate of $\|P_h v'' - v_h''\|_{0,p}$.

For $\psi_h \in S_{h,1}$,

$$\begin{aligned} (P_h v'' - v_h'', \psi_h) &= (P_h v'' - v'' + v'' - v_h'', \psi_h) \\ &= (P_h v'' - v'', \psi_h) + (v'' - v_h'', \psi_h) \\ &= (v'' - v_h'', \psi_h) \text{ using (5.28),} \\ (P_h v'' - v_h'', \psi_h) &= (e_h'', \psi_h) = (e_h'', \psi_h) - \langle e_h'', \psi_h \rangle_h + \langle e_h'', \psi_h \rangle_h \\ &= E_h(e_h'' \psi_h) + \langle e_h'', \psi_h \rangle_h, \\ |(P_h v'' - v_h'', \psi_h)| &\leq |E_h(e_h'' \psi_h)| + |\langle e_h'', \psi_h \rangle_h| \leq T_7 + T_8. \end{aligned} \quad (5.32)$$

We shall compute the estimates for the terms T_7 and T_8

$$T_7 = |E_h(e_h'' \psi_h)| \leq Ch^3 \|v\|_{6,p} \|\psi_h\|_{0,q} \quad (5.33)$$

by Lemma 5.1(b).

Following the steps of computation involved in the term T_6 , we obtain the estimate of T_8 as

$$T_8 = |\langle e_h'', \psi_h \rangle_h| \leq C [h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p}] \|\psi_h\|_{0,q}, \quad (5.34)$$

where we have used the inverse inequality $\|\psi_h\|_{1,q,k} \leq h_k^{-1} \|\psi_h\|_{0,q,k}$ locally. Using T_7 and T_8 in (5.32), we get

$$|(P_h v'' - v_h'', \psi_h)| \leq C [h^3 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p}] \|\psi_h\|_{0,q}. \quad (5.35)$$

We now show the above inequality for $\eta \in L_q$ to obtain $\|P_h v'' - v_h''\|_{0,p}$.

Now let η be an arbitrary element of L_q . Then since $v_h'' \in S_{h,1}$, it follows from the definition of $P_h \eta$, (5.35), and (5.29) with p replaced by q , that

$$\begin{aligned} 0 &= (P_h v'' - v_h'', \eta - P_h \eta), \\ |(P_h v'' - v_h'', \eta)| &= |(P_h v'' - v_h'', P_h \eta)| \leq C [h^3 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p}] \|P_h \eta\|_{0,q} \\ &\leq C [h^3 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p}] \|\eta\|_{0,q}, \\ \|P_h v'' - v_h''\|_{0,p} &\leq C [h^3 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p}]. \end{aligned} \quad (5.36)$$

Now, from (5.30), (5.31), and (5.36), we conclude that

$$\begin{aligned} \|e_h''\|_{0,p} &\leq Ch^2 \|v\|_{4,p} + C [h^3 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p}] \\ &\leq C [h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p}]. \end{aligned} \quad (5.37)$$

Now, using the fact $\|e_h\|_{2,p} \leq \|e_h\|_{1,p} + \|e_h''\|_{0,p}$ and the above estimate, we have

$$\begin{aligned} \|e_h\|_{2,p} &\leq \|e_h\|_{1,p} + C [h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p}] \\ &\leq C [\|e_h\|_{1,p} + h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p}] \\ &\leq C [\|e_h\|_{1,p} + h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p}]. \end{aligned} \quad (5.38)$$

Now using (4.3) with $m = 2$ and $i = 1$, we have

$$\|e_h\|_{1,p} \leq C (h^{-1} \|e_h\|_{0,p} + h \|e_h\|_{2,p}). \quad (5.39)$$

Substituting (5.39) in the above expression, we obtain

$$\|e_h\|_{2,p} \leq C [(h^{-1} \|e_h\|_{0,p} + h \|e_h\|_{2,p}) + h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p}]. \quad (5.40)$$

For sufficiently small h , we have

$$\|e_h\|_{2,p} \leq C [h^{-1} \|e_h\|_{0,p} + h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p}]. \quad (5.41)$$

Using (5.41) in (5.27),

$$\|e_h\|_{0,p} \leq C [h^2 (h^{-1} \|e_h\|_{0,p} + h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p}) + h^4 \|v\|_{6,p} + h^5 \|u\|_{6,p} + h^3 \|\varepsilon_h\|_{2,p}]. \quad (5.42)$$

For sufficiently small h , we get

$$\|e_h\|_{0,p} \leq C \left[h^4 \|v\|_{6,p} + h^5 \|u\|_{6,p} + h^3 \|\varepsilon_h\|_{2,p} \right]. \quad (5.43)$$

Using (5.43) in (5.41), we have

$$\begin{aligned} \|e_h\|_{2,p} &\leq C \left[h^{-1} \left(h^4 \|v\|_{6,p} + h^5 \|u\|_{6,p} + h^3 \|\varepsilon_h\|_{2,p} \right) + h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} \right] \\ &\leq C \left[h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} \right]. \end{aligned} \quad (5.44)$$

Using (5.43) and (5.44) in (5.39), we have

$$\begin{aligned} \|e_h\|_{1,p} &\leq C \left[h^{-1} \left(h^4 \|v\|_{6,p} + h^5 \|u\|_{6,p} + h^3 \|\varepsilon_h\|_{2,p} \right) + h \left(h^2 \|v\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} \right) \right] \\ &\leq C \left[h^3 \|v\|_{6,p} + h^4 \|u\|_{6,p} + h^2 \|\varepsilon_h\|_{2,p} \right]. \end{aligned} \quad (5.45)$$

Equations (5.43), (5.44), and (5.45) give the required result. $\square$

We now compute the error estimate of ε_h in L_p , W_p^1 , and W_p^2 norms as has been done in the previous case.

Lemma 5.4. *Let u and v be the weak solutions of the coupled equations (1.4) and (1.5) defined through (2.7) and (2.8), respectively. Further, let u_h and v_h be the corresponding discrete Petrov-Galerkin solutions defined through (2.10) and (2.11), respectively. Then the estimates of the errors $\varepsilon_h = u - u_h$ in L_p , W_p^1 and W_p^2 norms are given as follows:*

$$\begin{aligned} \|\varepsilon_h\|_{0,p} &\leq C \left[h^4 \|u\|_{6,p} + h \|e_h\|_{1,p} \right], \\ \|\varepsilon_h\|_{1,p} &\leq C \left[h^3 \|u\|_{6,p} + \|e_h\|_{1,p} \right], \\ \|\varepsilon_h\|_{2,p} &\leq C \left[h^2 \|u\|_{6,p} + \|e_h\|_{1,p} \right]. \end{aligned} \quad (5.46)$$

Proof. Let ρ be an arbitrary element of L_q , and let $\phi \in W_q^2$ be the unique solution of the auxiliary problem

$$\begin{aligned} L^* \phi &= \rho, \\ \phi(0) &= \phi(1) = 0. \end{aligned} \quad (5.47)$$

Then we have

$$(\varepsilon_h, \rho) = (\varepsilon_h, L^* \phi) = (L \varepsilon_h, \phi) = \left(\varepsilon_h'', \phi \right) = \left(\varepsilon_h'', \phi - \phi_h \right) + \left(\varepsilon_h'', \phi_h \right) - \left\langle \varepsilon_h'', \phi_h \right\rangle_h + \left\langle \varepsilon_h'', \phi_h \right\rangle_h, \quad (5.48)$$

where $\phi_h \in S_{h,1}$ is a linear interpolant of ϕ ,

$$|(\varepsilon_h, \rho)| \leq \left| \left(\varepsilon_h'', \phi - \phi_h \right) \right| + \left| E_h \left(\varepsilon_h'' \phi_h \right) \right| + \left| \left\langle \varepsilon_h'', \phi_h \right\rangle_h \right| \leq T_9 + T_{10} + T_{11}. \quad (5.49)$$

Following the steps involved in the computation of T_4 and T_5 , we obtain the estimates of T_9 and T_{10} as follows:

$$\begin{aligned} T_9 &\leq Ch^2 \|\varepsilon_h\|_{2,p} \|\phi\|_{2,q'}, \\ T_{10} &\leq Ch^4 \|u\|_{6,p} \|\phi\|_{2,q'}, \end{aligned} \quad (5.50)$$

by Lemma 5.1(c) and (5.22).

Using (5.5) and (2.3) first, then the Sobolev embedding theorem (4.7) locally on I_k for $\|\phi_h\|_{0,\infty,k}$ and $\|e_h\|_{0,\infty,k}$ to estimate T_{11} , we have

$$\begin{aligned} T_{11} &= \left| \left\langle \varepsilon_h'', \phi_h \right\rangle_h \right| = \left| \left\langle \alpha e_h, \phi_h \right\rangle_h \right| \leq C \sum_{k=1}^n \frac{h_k}{2} \|e_h\|_{0,\infty,k} \|\phi_h\|_{0,\infty,k} \leq C \sum_{k=1}^n \frac{h_k}{2} \|e_h\|_{0,\infty,k} \|\phi_h\|_{1,q,k} \\ &\leq C \sum_{k=1}^n \frac{h_k}{2} \|e_h\|_{1,p,k} \|\phi_h\|_{1,q,k}. \end{aligned} \quad (5.51)$$

Further, using Hölder's inequality for sums and (5.22), we obtain

$$T_{11} \leq Ch \|e_h\|_{1,p} \|\phi_h\|_{1,q} \leq Ch \|e_h\|_{1,p} \|\phi\|_{2,q}. \quad (5.52)$$

Substituting the estimates T_9 , T_{10} , and T_{11} in (5.49), we obtain

$$|(\varepsilon_h, \rho)| \leq C \left[h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p} \right] \|\phi\|_{2,q}. \quad (5.53)$$

Using (4.6) and regularity of the auxiliary problem, we have $\|\phi\|_{2,q} \leq C \|\rho\|_{0,q}$. Since $\rho \in L_q$ is arbitrary, we have

$$\|\varepsilon_h\|_{0,p} \leq C \left[h^2 \|\varepsilon_h\|_{2,p} + h^4 \|u\|_{6,p} + h \|e_h\|_{1,p} \right]. \quad (5.54)$$

The estimate of $\|\varepsilon_h''\|_{0,p}$ can be obtained through a projection argument as mentioned in Lemma 5.3 as

$$\|\varepsilon_h''\|_{0,p} \leq C \left[h^2 \|u\|_{6,p} + \|e_h\|_{1,p} \right], \quad (5.55)$$

where we have used Lemma 5.1(d). In a similar manner we can compute the estimates for $\|\varepsilon_h\|_{0,p}$, $\|\varepsilon_h\|_{1,p}$ and $\|\varepsilon_h\|_{2,p}$ as

$$\begin{aligned}\|\varepsilon_h\|_{0,p} &\leq C \left[h^4 \|u\|_{6,p} + h \|e_h\|_{1,p} \right], \\ \|\varepsilon_h\|_{1,p} &\leq C \left[h^3 \|u\|_{6,p} + \|e_h\|_{1,p} \right], \\ \|\varepsilon_h\|_{2,p} &\leq C \left[h^2 \|u\|_{6,p} + \|e_h\|_{1,p} \right].\end{aligned}\tag{5.56}$$

Using all the estimates from Lemmas 5.3 and 5.4, we have the following main error estimates. $\square$

Theorem 5.5. Assume that u and v satisfy (1.4) and (1.5), respectively, with (4.1). Assume also that $u \in W_p^6$ and $v \in W_p^6$, where $p \in [1, \infty]$. Then (2.10) and (2.11) have unique solutions $u_h \in \overset{0}{S}_{h,3}$ and $v_h \in \overset{0}{S}_{h,3}$, respectively, and for h sufficiently small, one has

$$\begin{aligned}\|u - u_h\|_{i,p} &\leq Ch^{4-i} \left[\|u\|_{6,p} + \|v\|_{6,p} \right], \\ \|v - v_h\|_{i,p} &\leq Ch^{4-i} \left[\|u\|_{6,p} + \|v\|_{6,p} \right], \quad i = 0, 1, 2.\end{aligned}\tag{5.57}$$

Proof. Assume temporarily that solutions u_h and v_h of (2.10) and (2.11), respectively, exist. Using (5.46) in (5.45), we obtain

$$\|e_h\|_{1,p} \leq C \left[h^3 \|v\|_{6,p} + h^4 \|u\|_{6,p} + h^2 \left(h^2 \|u\|_{6,p} + \|e_h\|_{1,p} \right) \right].\tag{5.58}$$

For sufficiently small h , we have

$$\|e_h\|_{1,p} \leq C \left(h^3 \|v\|_{6,p} + h^4 \|u\|_{6,p} \right).\tag{5.59}$$

An application of the above in (5.46), we get

$$\|\varepsilon_h\|_{2,p} \leq C \left[h^2 \|u\|_{6,p} + h^3 \|v\|_{6,p} \right].\tag{5.60}$$

Apply (5.59) in (5.56) to have

$$\|\varepsilon_h\|_{0,p} \leq C \left[h^4 \|u\|_{6,p} + h^4 \|v\|_{6,p} \right].\tag{5.61}$$

Use (5.60) in (5.43) to get

$$\|e_h\|_{0,p} \leq C \left[h^4 \|v\|_{6,p} + h^5 \|u\|_{6,p} \right].\tag{5.62}$$

Using (5.60) in (5.44), we obtain

$$\|e_h\|_{2,p} \leq C \left[h^2 \|v\|_{6,p} + h^4 \|u\|_{6,p} \right]. \quad (5.63)$$

Using (5.61) and (5.60) in (5.39) with e_h replaced by ε_h , we have

$$\|\varepsilon_h\|_{1,p} \leq C \left[h^3 \|u\|_{6,p} + h^3 \|v\|_{6,p} \right]. \quad (5.64)$$

The required result can be obtained from estimates (5.59) to (5.64). $\square$

So far we have assumed temporarily that solutions u_h and v_h exist. We now discuss the existence and uniqueness of discrete Petrov-Galerkin approximation. Since the matrix corresponding to (2.10) and (2.11) with zero boundary conditions for u_h and v_h is square, existence of $u_h \in \overset{0}{S}_{h,3}$ and $v_h \in \overset{0}{S}_{h,3}$ for any $f \in C^0(I)$ will follow from uniqueness, that is, from the property that the corresponding homogeneous equations have only trivial solutions.

Suppose that u_h and v_h corresponding to u and v satisfy

$$\begin{aligned} \langle u_h'' - \alpha v_h, \chi_h \rangle &= 0, \\ \langle v_h'' + b u_h, \chi_h \rangle &= 0, \quad \chi_h \in S_{h,1}. \end{aligned} \quad (5.65)$$

It follows from (5.61) and (5.62) (with u replaced by 0 and eventually $v \equiv 0$) that, for sufficiently small h ,

$$\|u_h\|_{0,p} \leq 0, \quad \|v_h\|_{0,p} \leq 0, \quad (5.66)$$

and hence $u_h \equiv 0$ and $v_h \equiv 0$. Thus, uniqueness is proved, and hence existence follows from uniqueness.

References

- [1] J. Douglas, Jr. and T. Dupont, "A finite element collocation method for quasilinear parabolic equations," *Mathematics of Computation*, vol. 27, pp. 17–28, 1973.
- [2] J. Douglas, Jr. and T. Dupont, *Collocation Methods for Parabolic Equations in a Single Space Variable*, vol. 385 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 1974.
- [3] I. H. Sloan, "A quadrature-based approach to improving the collocation method," *Numerische Mathematik*, vol. 54, no. 1, pp. 41–56, 1988.
- [4] I. H. Sloan, D. Tran, and G. Fairweather, "A fourth-order cubic spline method for linear second-order two-point boundary value problems," *IMA Journal of Numerical Analysis*, vol. 13, no. 4, pp. 591–607, 1993.
- [5] L. Jones Doss and A. K. Pani, "A qualocation method for a semilinear second-order two-point boundary value problem," in *Functional Analysis with Current Applications in Science, Technology and Industry*, M. Brokate and A. H. Siddiqi, Eds., vol. 377 of *Pitman Research Notes in Mathematics*, pp. 128–144, Addison Wesley Longman, Harlow, UK, 1998.
- [6] A. K. Pani, "A qualocation method for parabolic partial differential equations," *IMA Journal of Numerical Analysis*, vol. 19, no. 3, pp. 473–495, 1999.
- [7] L. Jones Doss and A. K. Pani, "A qualocation method for a unidimensional single phase semilinear Stefan problem," *IMA Journal of Numerical Analysis*, vol. 25, no. 1, pp. 139–159, 2005.

- [8] B. Bialecki, M. Ganesh, and K. Mustapha, "A Petrov-Galerkin method with quadrature for elliptic boundary value problems," *IMA Journal of Numerical Analysis*, vol. 24, no. 1, pp. 157–177, 2004.
- [9] B. Bialecki, M. Ganesh, and K. Mustapha, "An ADI Petrov-Galerkin method with quadrature for parabolic problems," *Numerical Methods for Partial Differential Equations*, vol. 25, no. 5, pp. 1129–1148, 2009.
- [10] M. Ganesh and K. Mustapha, "A Crank-Nicolson and ADI Galerkin method with quadrature for hyperbolic problems," *Numerical Methods for Partial Differential Equations*, vol. 21, no. 1, pp. 57–79, 2005.
- [11] P. J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, Academic Press, New York, NY, USA, 2nd edition, 1975.
- [12] R. A. Adams, *Sobolev Spaces*, vol. 65 of *Pure and Applied Mathematics*, Academic Press, New York, NY, USA, 1975.
- [13] M. Crouzeix and V. Thomée, "The stability in L_p and W_p^1 of the L_2 -projection onto finite element function spaces," *Mathematics of Computation*, vol. 48, no. 178, pp. 521–532, 1987.
- [14] C. de Boor, "A bound on the L^∞ -norm of L^2 -approximation by splines in terms of a global mesh ratio," *Mathematics of Computation*, vol. 30, no. 136, pp. 765–771, 1976.