Research Article

Fredholm Weighted Composition Operators on Dirichlet Space

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This paper characterizes bounded Fredholm, bounded invertible, and unitary weighted composition operators on Dirichlet space.

1. Introduction

Let $H$ be a Hilbert space of analytic functions on the unit disk $D$. For an analytic function $\varphi$ on $D$, we can define the multiplication operator $M_\varphi : f \mapsto \varphi f$, $f \in H$. For an analytic self-mapping $\varphi$ of $D$, the composition operator $C_\varphi$ defined on $H$ as $C_\varphi f = f \circ \varphi$, $f \in H$. These operators are two classes of important operators in the study of operator theory in function spaces [1–3]. Furthermore, for $\varphi$ and $\psi$, we define the weighted composition operator $C_{\psi,\varphi}$ on $H$ as

$$C_{\psi,\varphi} : f \mapsto \psi(f \circ \varphi), \quad f \in H.$$  \hfill (1.1)

Recently, the boundedness, compactness, norm, and essential norm of weighted composition operators on various spaces of analytic functions have been studied intensively, see [4–9] and so on. In this paper, we characterize bounded Fredholm weighted composition operators on Dirichlet space of the unit disk.
Recall the Dirichlet space $\mathcal{D}$ that consists of analytic function $f$ on $D$ with finite Dirichlet integral:

$$D(f) = \int_D |f'|^2 dA < \infty,$$

where $dA$ is the normalized Lebesgue area measure on $D$. It is well known that $\mathcal{D}$ is the only mobius invariant Hilbert space up to an isomorphism [10]. Endow $\mathcal{D}$ with norm

$$\|f\| = \left( |f(0)|^2 + D(f) \right)^{1/2}, \quad f \in \mathcal{D}.$$  

$\mathcal{D}$ is a Hilbert space with inner product

$$\langle f, g \rangle = f(0)g(0) + \int_D f'(z)g'(z)dA(z), \quad f, g \in \mathcal{D}.  \tag{1.4}$$

Furthermore $\mathcal{D}$ is a reproducing function space with reproducing kernel

$$K_\lambda(z) = 1 + \log \frac{1}{1 - \lambda z}, \quad \lambda, z \in D. \tag{1.5}$$

Denote $\mathcal{M} = \{ \psi : \psi$ is analytic on $D, \psi f \in \mathcal{D}$ for $f \in \mathcal{D} \}$. $\mathcal{M}$ is called the multiplier space of $\mathcal{D}$. By the closed graph theorem, the multiplication operator $M_\psi$, defined by $\psi \in \mathcal{M}$ is bounded on $\mathcal{D}$. For the characterization of the element in $\mathcal{M}$, see [11].

For analytic function $\psi$ on $D$ and analytic self-mapping $\varphi$ of $D$, the weighted composition operator $C_{\psi, \varphi}$ on $\mathcal{D}$ is not necessarily bounded. Even the composition operator $C_\varphi$ is not necessarily bounded on $\mathcal{D}$, which is different from the cases in Hardy space and Bergman space. See [12] for more information about the properties of composition operators acting on the Dirichlet space.

The main result of the paper reads as the following.

**Theorem 1.1.** Let $\psi$ and $\varphi$ be analytic functions on $D$ with $\varphi(D) \subset D$. Then $C_{\psi, \varphi}$ is a bounded Fredholm operator on $\mathcal{D}$ if and only if $\psi \in \mathcal{M}$, bounded away from zero near the unit circle, and $\varphi$ is an automorphism of $D$.

If $\varphi(z) = 1$, then the result above gives the characterization of bounded Fredholm composition operator $C_{\psi, \varphi}$ on $\mathcal{D}$, which was obtained in [12].

As corollaries, in the end of this paper one gives the characterization of bounded invertible and unitary weighted composition operator on $\mathcal{D}$, respectively. Some idea of this paper is derived from [4, 13], which characterize normal and bounded invertible weighted composition operator on the Hardy space, respectively.

**2. Proof of the Main Result**

In the following, $\psi$ and $\varphi$ denote analytic functions on $D$ with $\varphi(D) \subset D$. It is easy to verify that $\psi \in \mathcal{D}$ if $C_{\psi, \varphi}$ is defined on $\mathcal{D}$.
Proposition 2.1. Let $C_{ψ,ϕ}$ be a bounded Fredholm operator on $\mathfrak{D}$. Then $ψ$ has at most finite zeroes in $D$ and $ϕ$ is an inner function.

Proof. If $C_{ψ,ϕ}$ is a bounded Fredholm operator, then there exist a bounded operator $T$ and a compact operator $S$ on $\mathfrak{D}$ such that

$$T(C_{ψ,ϕ})^* = I + S,$$  \hspace{1cm} (2.1)

where $I$ is the identity operator.

Since

$$(C_{ψ,ϕ})^* K_w(z) = \left\langle C_{ψ,ϕ} K_w, K_z \right\rangle = (K_w, C_{ψ,ϕ} K_z)\right.$$

$$= \left\langle K_w, ϕ K_z \circ ϕ \right\rangle = ϕ(w) K_z(ϕ(w))$$

$$= ϕ(w) K_{ϕ(w)}(z),$$  \hspace{1cm} (2.2)

we have

$$\|T\|\|ϕ(w)\| \|K_{ϕ(w)}\| \geq \|T(C_{ψ,ϕ})^* K_w\|$$

$$\geq \|k_w\| - \|Sk_w\|$$

$$= 1 - \|Sk_w\|,$$  \hspace{1cm} (2.3)

where $k_w = K_w/\|K_w\|$ is the normalization of reproducing kernel function $K_w$.

Since $S$ is compact and $k_w$ weakly converges to 0 as $|w| \to 1$, $\|Sk_w\| \to 0$ as $|w| \to 1$. It follows that there exists constant $r$, $0 < r < 1$, such that $\|Sk_w\| < 1/2$ for all $w$ with $r < |w| < 1$. Inequality (2.3) shows that

$$\left| \frac{ϕ(w)}{\|K_w\|} \right| \geq \frac{1}{2\|T\|\|K_{ϕ(w)}\|}, \quad r < |w| < 1,$$  \hspace{1cm} (2.4)

which implies that $ψ$ has no zeroes in $\{w ∈ D, \ r < |w| < 1\}$, and, hence, $ψ$ has at most finite zeroes in $\{w ∈ D, \ |w| ≤ r\}$.

Since $k_w$ weakly converges to 0 as $|w| \to 1$, $⟨ϕ, k_w⟩ \to 0$ as $|w| \to 1$, that is,

$$\frac{ϕ(w)}{\|K_w\|} \to 0, \quad |w| \to 1.$$  \hspace{1cm} (2.5)

It follows from (2.4) that $\|K_{ϕ(w)}⟩ = (1 + \log(1/(1 - |ϕ(w)|^2)))^{1/2} \to \infty$ and hence $|ϕ(w)| \to 1$ as $|w| \to 1$, that is, $ϕ$ is an inner function. \hfill $\square$

For the proof of the following lemma, we cite Carleson’s formula for the Dirichlet integral [14].
Lemma 2.2. Let \( f \in \mathcal{D} \) be the canonical factorization of \( f \) as a function in the Hardy space, where \( B = \prod_{j=1}^{\infty} (\bar{a}_j/|a_j|)((a_j - z)/(1 - \bar{a}_j z)) \), is a Blaschke product, \( S \) is the singular part of \( f \) and \( F \) is the outer part of \( f \). Then

\[
D(f) = \int_{\mathbb{T}} \sum_{n=1}^{\infty} P_n(\xi) |f(\xi)|^2 \frac{|d\xi|}{2\pi} + \int_{\mathbb{T}} \int \frac{2}{|\xi - \zeta|^2} \left| f(\xi) \right|^2 |d\mu(\xi)\frac{|d\xi|}{2\pi} + \int_{\mathbb{T}} \frac{e^{2i\xi} - e^{2i\zeta}}{|\xi - \zeta|^2} (u(\xi) - u(\zeta)) \frac{|d\xi|}{2\pi} \frac{|d\zeta|}{2\pi},
\]

where \( \mathbb{T} \) is the unit circle, \( u(\xi) = \log |f(\xi)| \), \( P_n(\xi) \) is the Poisson kernel, and \( \mu \) is the singular measure corresponding to \( S \).

**Lemma 2.2.** Let \( C_{\psi, \phi} \) be a bounded operator on \( \mathcal{D} \), \( \psi = BF \) with \( B \) a finite Blaschke product. Then \( C_{F, \phi} \) is bounded.

**Proof.** Let \( M_B \) be the multiplication operator on \( \mathcal{D} \). Then \( C_{\psi, \phi} = M_B C_{F, \phi} \). Since \( B \) is a finite Blaschke product, by the Carleson’s formula, we have

\[
D(\psi(f \circ \phi)) = D(BF(f \circ \phi)) \geq D(F(f \circ \phi)), \quad f \in \mathcal{D}.
\]

Since \( \|f\|^2 = |f(0)|^2 + D(f), \) \( f \in \mathcal{D} \), by the inequality above it is easy to verify that \( C_{F, \phi} \) is bounded if \( C_{\psi, \phi} \) is bounded. \( \square \)

**Lemma 2.3.** Let \( F \) be an analytic function on \( D \) with zero-free. If \( C_{F, \phi} \) is a bounded Fredholm operator on \( \mathcal{D} \), then \( \psi \) is univalent.

**Proof.** If \( \psi(a) = \psi(b) \) for \( a, b \in D \) with \( a \neq b \), by a similar reasoning as [1, Lemma 3.26], there exist infinite sets \( \{a_n\} \) and \( \{b_n\} \) in \( D \) which is disjoint such that \( \psi(a_n) = \psi(b_n) \). Hence,

\[
(C_{F, \phi})^* \left( \frac{K_{a_n}}{F(a_n)} - \frac{K_{b_n}}{F(b_n)} \right) = 0,
\]

which contradicts to that kernel of \( (C_{F, \phi})^* \) is finite dimensional. \( \square \)

**Corollary 2.4.** If \( C_{\psi, \phi} \) is a bounded Fredholm operator on \( \mathcal{D} \), then \( \psi \) is an automorphism of \( D \) and \( \phi \in \mathcal{M} \).

**Proof.** By Proposition 2.1, \( \psi \) has the factorization of \( BF \) with \( B \) a finite Blaschke product and \( F \) zero free in \( D \). By Lemma 2.2, \( C_{\psi, \phi} \) is a bounded operator on \( \mathcal{D} \). Since \( C_{\psi, \phi} = M_B C_{F, \phi} \) and \( M_B \) is a Fredholm operator, \( C_{F, \phi} \) is a Fredholm operator also. By Proposition 2.1 and Lemma 2.3, \( \psi \) is an univalent inner function, it follows from [1, Corollary 3.28] that \( \psi \) is an automorphism of \( D \).

Since \( C_{\psi, \phi} C_{\psi^{-1}} = M_{\psi}, \) \( M_{\psi} \) is a bounded multiplication operator on \( \mathcal{D} \), which implies that \( \psi \in \mathcal{M} \). \( \square \)

The following lemmas is well-known. It is easy to verify by the fact \( M_{\psi}^* K_{\phi} = \overline{\psi(\phi)} K_{\phi} \) also.
Lemma 2.5. Let $\psi \in \mathcal{M}$. Then $M_\psi$ is an invertible operator on $\mathfrak{D}$ if and only if $\psi$ is invertible in $\mathcal{M}$.

Lemma 2.6. Let $\psi \in \mathcal{M}$. Then $M_\psi$ is a Fredholm operator on $\mathfrak{D}$ if and only if $\psi$ is bounded away from the unit circle.

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. If $C_{\psi,\phi}$ is a bounded Fredholm operator on $\mathfrak{D}$, by Corollary 2.4, $\psi \in \mathcal{M}$ and $\phi$ is an automorphism of $D$. Since $C_{\psi}$ is invertible, $M_\psi$ is a Fredholm operator. So $\psi$ is bounded away from the unit circle follows from Lemma 2.6.

On the other hand, if $\psi \in \mathcal{M}$ and bounded away from the unit circle, then $M_\psi$ is a bounded Fredholm operator on $\mathfrak{D}$. If $\phi$ is an automorphism of $D$, then $C_{\phi}$ is invertible. Hence $C_{\psi,\phi} = M_\psi C_{\phi}$ is a bounded Fredholm operator on $\mathfrak{D}$.

As corollaries, in the following, we characterize bounded invertible and unitary weighted composition operators on $\mathfrak{D}$.

Corollary 2.7. Let $\psi$ and $\phi$ be analytic functions on $D$ with $\phi(D) \subset D$. Then $C_{\psi,\phi}$ is a bounded invertible operator on $\mathfrak{D}$ if and only if $\psi \in \mathcal{M}$, invertible in $\mathcal{M}$, and $\phi$ is an automorphism of $D$.

Proof. Since a bounded invertible operator is a bounded Fredholm operator, the proof is similar to the proof of Theorem 1.1.

Corollary 2.8. Let $\psi$ and $\phi$ be analytic functions on $D$ with $\phi(D) \subset D$. $C_{\psi,\phi}$ is a bounded operator on $\mathfrak{D}$. Then $C_{\psi,\phi}$ is a unitary operator if and only if $\psi$ is a constant with $|\psi| = 1$ and $\phi$ is a rotation of $D$.

Proof. If $C_{\psi,\phi}$ is a unitary operator, then it must be an invertible operator. By Corollary 2.7, $\psi$ is an automorphism of $D$ and $\psi$ is invertible in $\mathcal{M}$.

Let $n$ be nonnegative integer, $e_n(z) = z^n$, $z \in D$. A unitary is also an isometry, so we have

$$\|\psi\| = \|C_{\psi,\phi}e_0\| = \|e_0\| = 1,$$

$$\|\psi \phi^n\| = \|C_{\psi,\phi}e_n\| = \|e_n\| = \sqrt{n}, \quad n \geq 1. \tag{2.9}$$

Let $a \in D$ such that $\phi(a) = 0$. Since $\phi$ is an automorphism of $D$, $\phi^n$ is a finite Blaschke product with zero $a$ of order $n$. By Carleson’s formula for Dirichlet integral, we have

$$D(\psi \phi^n) = n \int_{\Gamma} P_a(\xi)|\phi(\xi)|^2 \frac{|d\xi|}{2\pi} + D(\psi). \tag{2.10}$$

Hence,

$$n = \|\psi \phi^n\|^2 = |\psi(0)\phi(0)^n|^2 + D(\psi)$$

$$= |\psi(0)\phi(0)^n|^2 + n \int_{\Gamma} P_a(\xi)|\phi(\xi)|^2 \frac{|d\xi|}{2\pi} + D(\psi), \quad n \geq 1. \tag{2.11}$$

$$n = \|\psi \phi^n\|^2 = |\psi(0)\phi(0)^n|^2 + D(\psi)$$

$$= |\psi(0)\phi(0)^n|^2 + n \int_{\Gamma} P_a(\xi)|\phi(\xi)|^2 \frac{|d\xi|}{2\pi} + D(\psi), \quad n \geq 1. \tag{2.12}$$
That is,

\[
1 = \frac{|q(0)q'(0)|^2}{n} + \frac{\int_T P_a(\xi)|\psi(\xi)|^2|d\xi|}{2\pi} + \frac{D(q)}{n}, \quad n \geq 1. \tag{2.13}
\]

Let \( n \to \infty \), then \( 1 = \int_T P_a(\xi)|\psi(\xi)|^2(\|d\xi\|/2\pi) \).

By (2.12), we have \( D(q) = 0 \) and \( |q(0)q'(0)| = 0 \). By (2.9), we obtain \( q \) is a constant with \( |q| = 1 \), which implies that \( q(0) = 0 \), that is, \( q \) is a rotation of \( D \).

The sufficiency is easy to verify. \( \square \)

**Remark 2.9.** The key step in the proof of the main result is to analyze zeros of the symbol \( q \) and univalency of \( q \). The following result pointed out by the referee gives a simple characterization of the symbols \( q \) and \( \phi \) for the bounded Fredholm operator \( C_{q,\phi} \) on \( \mathcal{D} \).

**Proposition 2.10.** Let \( q \) and \( \phi \) be analytic functions on \( D \) with \( \phi(D) \subset D \). \( C_{q,\phi} \) is a bounded Fredholm operator on \( \mathcal{D} \). Then \( q \) has only finitely many zeros in \( D \) and \( \phi \) is univalent.

**Proof.** If \( q(a) = 0 \) for \( a \in D \), then \( C_{q,\phi}^*K_a = \overline{q(a)}K_q(a) = 0 \), which implies that \( K_a \) is in the kernel of \( C_{q,\phi}^* \). Thus if \( q \) had infinitely many zeros, the kernel of \( C_{q,\phi}^* \) would be infinite dimensional and hence this operator would not be Fredholm.

If \( q(a) = q(b) \) for \( a, b \in D \) with \( a \neq b \), by a similar reasoning as [1, Lemma 3.26], there exist infinite sets \( \{a_n\} \) and \( \{b_n\} \) in \( D \) which is disjoint such that \( q(a_n) = q(b_n) \). Hence, \( q(a_n) \neq q(b_n) \).

\[
(C_{q,\phi})^* \left( \frac{K_{a_n}}{q(a_n)} - \frac{K_{b_n}}{q(b_n)} \right) = 0. \tag{2.14}
\]

Since \( C_{q,\phi} \) is a Fredholm operator, \( q \) must be univalent. \( \square \)

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**References**


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