Research Article
On Subspaces of an Almost $\varphi$-Lagrange Space

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We discuss the subspaces of an almost $\varphi$-Lagrange space (APL space in short). We obtain the induced nonlinear connection, coefficients of coupling, coefficients of induced tangent and induced normal connections, the Gauss-Weingarten formulae, and the Gauss-Codazzi equations for a subspace of an APL-space. Some consequences of the Gauss-Weingarten formulae have also been discussed.

1. Introduction
The credit for introducing the geometry of Lagrange spaces and their subspaces goes to the famous Romanian geometer Miron [1]. He developed the theory of subspaces of a Lagrange space together with Bejancu [2]. Miron and Anastasiei [3] and Sakaguchi [4] studied the subspaces of generalized Lagrange spaces (GL spaces in short). Antonelli and Hrimiuc [5, 6] introduced the concept of $\varphi$-Lagrangians and studied $\varphi$-Lagrange manifolds. Generalizing the notion of a $\varphi$-Lagrange manifold, the present authors recently studied the geometry of an almost $\varphi$-Lagrange space (APL space briefly) and obtained the fundamental entities related to such space [7]. This paper is devoted to the subspaces of an APL space.

Let $F^n = (M, F(x, y))$ be an $n$-dimensional Finsler space and $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}$ a smooth function. If the function $\varphi$ has the following properties:

(a) $\varphi'(t) \neq 0,$
(b) $\varphi'(t) + \varphi''(t) \neq 0,$ for every $t \in \text{Im}(F^2),$

then the Lagrangian given by

$$L(x, y) = \varphi(F^2) + A_i(x)y^i + U(x),$$  \hspace{1cm} (1.1)
where $A_i(x)$ is a covector and $U(x)$ is a smooth function, is a regular Lagrangian [7]. The space $L^n = (M, L(x, y))$ is a Lagrange space. The present authors [7] called such space as an almost $\varphi$-Lagrange space (shortly APL space) associated to the Finsler space $F^n$. An APL space reduces to a $\varphi$-Lagrange space if and only if $A_i(x) = 0$ and $U(x) = 0$. We take

$$g_{ij} = \frac{1}{2} \partial_i \partial_j F^2, \quad a_{ij} = \frac{1}{2} \partial_i \partial_j L,$$

where $\partial_i \equiv \frac{\partial}{\partial y^i}$. (1.2)

We indicate all the geometrical objects related to $F^n$ by putting a small circle “$\circ$” over them. Equations (1.2), in view of (1.1), provide the following expressions for $a_{ij}$ and its inverse (cf. [7]):

$$a_{ij} = \varphi' \left( g_{ij} + \frac{2q''}{q'} y^i y^j \right), \quad a^{ij} = \frac{1}{q'} \left( g^{ij} - \frac{2q''}{q' + 2F^2 q'} y^i y^j \right),$$

(1.3)

where $g_{ij} y'^i = y^i$. Let $\tilde{M}$ be a smooth manifold of dimension $m$, $1 < m < n$, immersed in $M$ by immersion $i : \tilde{M} \rightarrow M$. The immersion $i$ induces an immersion $T_i : T\tilde{M} \rightarrow TM$ making the following diagram commutative:

$$\begin{array}{ccc}
T\tilde{M} & \xrightarrow{T_i} & TM \\
\pi \downarrow & & \downarrow \pi \\
\tilde{M} & \xrightarrow{i} & M.
\end{array}$$

(1.4)

Let $(u^\alpha, v^\alpha)$ (throughout the paper, the Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to $m$) be local coordinates on $T\tilde{M}$. The restriction of the Lagrangian $L$ on $T\tilde{M}$ is $L(u, v) = L(x(u), y(u, v))$. Let $a_{\alpha \beta} = (1/2)(\partial^2 L/\partial u^\alpha \partial u^\beta)$. Then, we have (cf. [8]) $a_{\alpha \beta} = B^i_{\alpha} B^j_{\beta} a_{ij}$ where $B^i_{\alpha}(u) = \partial x^i/\partial u^\alpha$ are the projection factors. The pair $L^n = (\tilde{M}, \tilde{L}(u, v))$ is also a Lagrange space, called the subspace of $L^n$. For the natural bases $(\partial/\partial x^i, \partial/\partial y^i)$ on $TM$ and $(\partial/\partial u^\alpha, \partial/\partial v^\alpha)$ on $T\tilde{M}$, we have [8]

$$\frac{\partial}{\partial u^\alpha} = B^i_{\alpha} \frac{\partial}{\partial x^i} + B^i_{\alpha \beta} \frac{\partial}{\partial y^i}, \quad \frac{\partial}{\partial v^\alpha} = B^i_{\alpha} \frac{\partial}{\partial y^i},$$

(1.5)

where $B^i_{\alpha} = B^{ij}_{\alpha \beta} v^\beta$, $B^i_{\alpha \beta} = \partial^2 x^i/\partial u^\alpha \partial v^\beta$.

For the bases $(dx^i, dy^j)$ and $(du^\alpha, dv^\alpha)$, we have

$$dx^i = B^i_{\alpha} du^\alpha, \quad dy^j = B^i_{\alpha} dv^\alpha + B^i_{\alpha \beta} du^\alpha.$$ 

(1.6)

Since $(B^i_{\alpha})$ are $m$ linearly independent vector fields tangent to $\tilde{M}$, a vector field $\xi^i(x, y)$ is normal to $\tilde{M}$ along $T\tilde{M}$ if on $T\tilde{M}$, we have

$$a_{ij}(x(u), y(u, v)) B^{ij}_{\alpha \beta} = 0, \quad \forall \alpha = 1, 2, \ldots, m.$$ 

(1.7)
There are, at least locally, \((n - m)\) unit vector fields \(B^i_a(u, v)\) \((a = m + 1, m + 2, \ldots, n)\) normal to \(\mathcal{M}\) and mutually orthonormal, that is,

\[
a_{ij}B^i_aB^j_b = 0, \quad a_{ij}B^i_aB^j_b = \delta_{ab}, \quad (a, b = m + 1, m + 2, \ldots, n). \tag{1.8}
\]

Thus, at every point \((u, v) \in T\mathcal{M}\), we have a moving frame \(\mathcal{R} = ((u, v), B^i_a(u, v), B^i_a(u, v))\). Using (1.3) in the first expression of (1.8) and keeping \(\dot{y}_iB^i_a = 0\) (this fact is clear from \(g_{ij}y^iB^j_a = 0\)) in view, we observe that \(B^i_a's\) are normal to \(\mathcal{M}\) with respect to \(L^n\) if and only if they are so with respect to \(F^n\). The dual frame of \(\mathcal{R}\) is \(\mathcal{R}^n = ((u, v), B^i_a(u, v), B^i_a(u, v))\) with the following duality conditions:

\[
B^i_iB^i_a = \delta^i_a, \quad B^i_aB^i_j = 0, \quad B^i_iB^i_b = \delta^i_b, \quad B^i_iB^a + B^i_aB^a = \delta^i_j. \tag{1.9}
\]

We will make use of the following results due to the present authors [7], during further discussion.

**Theorem 1.1** (cf. [7]). *The canonical nonlinear connection of an APL space \(L^n\) has the local coefficients given by*

\[
N^i_j = \ddot{N}^i_j - V^i_j, \tag{1.10}
\]

where \(V^i_j = (1/2)F^i_j - S^i_j \left(2F_{jk}y^k + \partial_rU\right)\),

\[
S^i_j = \frac{1}{2\varphi'}C^i_j\varphi'' + \frac{1}{2\varphi'^2}\varphi'^r\varphi'^j_y + \frac{\varphi''\left(\delta^i_jy^j + \delta^i_jy^j\right)}{2\varphi' + 2F^2\varphi''} + \frac{\varphi^2\varphi''' - 2\varphi'^{r3}F^2 - 4\varphi'\varphi'^2}{2\varphi'^2(\varphi' + 2F^2\varphi'')}y^iy^j\varphi', \tag{1.11}
\]

\[
F_{jk}(x) = \frac{1}{2}(\partial_rA_k - \partial_kA_r), \quad F^i_j = \delta^i_jF_{kj}.
\]

**Theorem 1.2** (cf. [7]). *The coefficients of the canonical metrical \(d\)-connection \(C(N)\) of an APL space \(L^n\) are given by*

\[
C^i_{jk} = C^i_{jk} + \frac{\varphi''}{\varphi'}\left(\delta^i_jy^k + \delta^i_ky^j\right) + \frac{\varphi''}{\varphi' + 2F^2\varphi''}\delta_{ik}y^j + \frac{2(\varphi''\varphi' - 2\varphi'^2)}{\varphi' + 2F^2\varphi''}y^iy^j\varphi', \tag{1.12}
\]

\[
L^i_{jk} = L^i_{jk} + V^r_kC^i_{jr} + V^r_jC^i_{kr} + V^r_p\delta^p\varphi'C_{rkj}. \tag{1.13}
\]

For basic notations related to a Finsler space, a Lagrange space, and their subspaces, we refer to the books [8, 9].


\section{Induced Nonlinear Connection}

Let \( \tilde{N} = (\tilde{N}^\alpha_\beta(u,v)) \) be a nonlinear connection for \( \tilde{L}^m = (\tilde{M}, \tilde{L}(u,v)) \). The adapted basis of \( T_{(u,v)}T\tilde{M} \) induced by \( \tilde{N} \) is \( (\delta/\delta u^\alpha = \delta_\alpha, \partial/\partial v^\alpha = \dot{\delta}_\alpha) \), where

\[ \dot{\delta}_\alpha = \delta_\alpha - \tilde{N}_{\dot{\alpha}}^\beta \dot{\delta}_\beta. \] \hspace{1cm} (2.1)

The dual basis (cobasis) of the adapted basis \( (\delta_\alpha, \dot{\delta}_\alpha) \) is \( (\delta \dot{u}^\alpha, \delta \dot{v}^\alpha) \).

\textbf{Definition 2.1 (cf. [8])}. A nonlinear connection \( \tilde{N} = (\tilde{N}^\alpha_\beta(u,v)) \) of \( L^m \) is said to be induced by the canonical nonlinear connection \( N \) if the following equation holds good:

\[ \delta v^\alpha = B^i_i \delta y^i. \] \hspace{1cm} (2.2)

The local coefficients of the induced nonlinear connection \( \tilde{N} = (\tilde{N}^\alpha_\beta(u,v)) \) for the subspace \( \tilde{L}^m = (\tilde{M}, \tilde{L}(u,v)) \) of a Lagrange space \( L^n = (M, L(x,y)) \) are given by (cf. [8])

\[ \tilde{N}^\alpha_\beta = B^i_i \left( N^i_j B^j_\beta + B^j_0 \right), \] \hspace{1cm} (2.3)

\( N^i_j \) being the local coefficients of canonical nonlinear connection \( N \) of the Lagrange space \( L^n = (M, L(x,y)). \) Now using (1.10) in (2.3), we get

\[ \tilde{N}^\alpha_\beta = B^i_i \left( \overset{\circ}{N}^i_j B^j_\beta + B^j_0 \right) - B^a_j V^j_i B^i_\beta. \] \hspace{1cm} (2.4)

If we take \( \overset{\circ}{N}^\alpha_\beta = B^i_i \left( \overset{\circ}{N}^i_j B^j_\beta + B^j_0 \right) \), it follows from (2.4) that

\[ \overset{\circ}{N}^\alpha_\beta = \tilde{N}^\alpha_\beta - B^a_j V^j_i B^i_\beta. \] \hspace{1cm} (2.5)

Thus, we have the following.

\textbf{Theorem 2.2}. The local coefficients of the induced nonlinear connection \( \tilde{N} \) of the subspace \( L^m \) of an APL space \( L^n \) are given by (2.5).

In view of (2.5), (2.1) takes the following form, for the subspace \( L^m \) of an APL space \( L^n \):

\[ \delta \dot{\beta} = \overset{\circ}{\delta}_\beta + B^a_j V^j_i B^i_\beta \dot{\delta}_\alpha, \] \hspace{1cm} (2.6)

where \( \overset{\circ}{\delta}_\beta = \delta \dot{\beta} - \tilde{N}^\alpha_\beta \dot{\delta}_\alpha. \)
We can put \( dx^i, \delta y^i \) as (cf. [8])

\[
dx^i = B^i_a du^a, \quad \delta y^i = B^i_a \delta y^a + B^i_a H^a \delta u^a,
\]

where

\[
H^a = B^a_i \left( N^i_j B^j_a + B^i_{0a} \right).
\]

Using (1.10) in (2.8) and simplifying, we get

\[
H^a = B^a_i \left( N^i_j B^j_a + B^i_{0a} \right) - B^a_i V^j B^j_a.
\]

Taking \( H^a = B^a_i (N^i_j B^j_a + B^i_{0a}) \), in (2.9), it follows that

\[
H^a = \tH^a - B^a_i V^j B^j_a.
\]

Now, \( dx^i = B^i_a du^a, \delta y^i = B^i_a \delta y^a \) if and only if \( H^a = 0 \), that is, if and only if \( \tH^a = B^a_i V^j B^j_a \).

Thus, we have the following.

**Theorem 2.3.** The adapted cobasis \( (dx^i, \delta y^i) \) of the basis \( (\partial/\partial x^i, \partial/\partial y^i) \) induced by the nonlinear connection \( N \) of an APL space \( L^n \) is of the form \( dx^i = B^i_a du^a, \delta y^i = B^i_a \delta y^a \) if and only if \( H^a = B^a_i V^j B^j_a \).

**Definition 2.4** (cf. [8]). Let \( D = D \Gamma(N) \) be the canonical metrical \( d \)-connection of \( L^n \). An operator \( \tilde{D} \) is said to be a coupling of \( D \) with \( \tilde{N} \) if

\[
\tilde{D} X^i = X^i_a du^a + X^i|_a \delta v^a,
\]

where \( X^i_a = \delta_a X^i + X^i L^i_{ja} \), \( X^i|_a = \delta_a X^i + X^i \tilde{C}^i_{ja} \).

The coefficients \( (L^i_{ja}, \tilde{C}^i_{ja}) \) of coupling \( \tilde{D} \) of \( D \) with \( \tilde{N} \) are given by

\[
L^i_{ja} = L^i_{jk} B^k_a + C^i_{jk} B^k_a H^a,
\]

\[
\tilde{C}^i_{ja} = C^i_{jk} B^k_a.
\]
Using (1.12) and (1.13) in (2.12), we have

\[
\tilde{L}^i_{j\beta} = \left( L^i_{jk} + V^i_k C^i_{jr} + V^i_j C^i_{kr} + V^i_p a^p C_{rkj} \right) B^k_\beta \\
+ \left[ \tilde{C}^i_{jk} + \frac{\varphi''}{\varphi'} (\tilde{\delta}^i_j y_k + \tilde{\delta}^i_k y_j) + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} S_{jk} y^i \\
+ 2 \left( \varphi'' \varphi' - 2 \varphi''^2 \right) y^i y_j y_k \right] B^k_\beta \\
\begin{equation}
(2.14)
\end{equation}
\]

In view of (2.10) and \( \tilde{y}_i B^i_a = 0 \), (2.14) becomes

\[
\tilde{L}^i_{j\beta} = \left( \tilde{L}^i_{jk} B^k_\beta + \tilde{C}^i_{jk} B^k_\beta \tilde{H}^a_\beta \right) + \left( V^i_k C^i_{jr} + V^i_j C^i_{kr} + V^i_p a^p C_{rkj} - \tilde{C}^i_{jk} B^k_\beta B^p_\beta \tilde{V}^p_k \right) B^k_\beta \\
+ \left( \frac{\varphi''}{\varphi'} \tilde{y}^i_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} S_{jk} y^i \right) B^k_\beta, \\
\begin{equation}
(2.15)
\end{equation}
\]

that is,

\[
L^i_{j\beta} = \tilde{L}^i_{j\beta} + \left( V^i_k C^i_{jr} + V^i_j C^i_{kr} + V^i_p a^p C_{rkj} - \tilde{C}^i_{jk} B^k_\beta B^p_\beta \tilde{V}^p_k \right) B^k_\beta \\
+ \left( \frac{\varphi''}{\varphi'} \tilde{y}^i_j \delta^i_k + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} S_{jk} y^i \right) B^k_\beta, \\
\begin{equation}
(2.16)
\end{equation}
\]

where \( L^i_{j\beta} = \tilde{L}^i_{j\beta} B^k_\beta + \tilde{C}^i_{jk} B^k_\beta \tilde{H}^a_\beta \).

Using (1.12) in (2.13), we find that

\[
\tilde{C}^i_{j\beta} = \tilde{C}^i_{j\beta} + \left( \frac{\varphi''}{\varphi'} (\tilde{\delta}^i_j y_k + \tilde{\delta}^i_k y_j) + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} S_{jk} y^i \\
+ 2 \left( \varphi'' \varphi' - 2 \varphi''^2 \right) y^i y_j y_k \right) B^k_\beta, \\
\begin{equation}
(2.17)
\end{equation}
\]

that is,

\[
\tilde{C}^i_{j\beta} = \tilde{C}^i_{j\beta} + \left( \frac{\varphi''}{\varphi'} (\tilde{\delta}^i_j y_k + \tilde{\delta}^i_k y_j) + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} S_{jk} y^i \\
+ 2 \left( \varphi'' \varphi' - 2 \varphi''^2 \right) y^i y_j y_k \right) B^k_\beta, \\
\begin{equation}
(2.18)
\end{equation}
\]
where \( \mathcal{C}_{ij}^{\beta} = \delta_{\beta}^{j} B_{\beta}^{k} \). Thus, we have the following.

**Theorem 2.5.** The coefficients of coupling for the subspace \( L^m \) of an APL space \( L^n \) are given by (2.16) and (2.18).

**Definition 2.6** (cf. [8]). An operator \( D^{T} \) given by

\[
D^{T}X^{a} = X_{\beta}^{a}|u^{\beta} + X^{a}|_{\beta} \delta_{\beta}^{\gamma},
\]

is called the induced tangent connection by \( D \). This defines an \( N \)-linear connection for \( \check{L}^m \).

Using (2.16) in (2.20), we get

\[
L_{\beta}^{a} = B_{i}^{a} \left( \delta_{\beta}^{j} + B_{\beta}^{j} + \delta_{\beta}^{j} \right) + B_{i}^{a} \left[ \left( V_{k}^{\beta} C_{jr}^{i} + \delta_{\beta}^{j} C_{kr}^{i} - C_{jp} B_{p}^{k} V_{k}^{i} \right) B_{i}^{k} \right. \\
+ \left( \frac{\phi_{\gamma}^{\gamma} \delta_{k}^{i} \delta_{\beta}^{j} + \frac{\phi_{\gamma}^{\gamma}}{\phi_{\gamma}^{\gamma} + 2 F^{2} \phi_{\gamma}^{\gamma} \delta_{k}^{i} \delta_{\beta}^{j}} \right) B_{i}^{a} H_{i}^{a} \right],
\]

that is,

\[
L_{\beta}^{a} = B_{i}^{a} \left( \delta_{\beta}^{j} + B_{\beta}^{j} \right) + B_{i}^{a} \left[ \left( V_{k}^{\beta} C_{jr}^{i} + \delta_{\beta}^{j} C_{kr}^{i} - C_{jp} B_{p}^{k} V_{k}^{i} \right) B_{i}^{k} \right. \\
+ \left( \frac{\phi_{\gamma}^{\gamma} \delta_{k}^{i} \delta_{\beta}^{j} + \frac{\phi_{\gamma}^{\gamma}}{\phi_{\gamma}^{\gamma} + 2 F^{2} \phi_{\gamma}^{\gamma} \delta_{k}^{i} \delta_{\beta}^{j}} \right) B_{i}^{a} H_{i}^{a} \right].
\]

If we take \( \check{L}_{\beta}^{a} = B_{i}^{a} \left( \delta_{\beta}^{j} + B_{\beta}^{j} \right) \), the last expression gives

\[
L_{\beta}^{a} = \check{L}_{\beta}^{a} + B_{i}^{a} \left[ \left( V_{k}^{\beta} C_{jr}^{i} + \delta_{\beta}^{j} C_{kr}^{i} - C_{jp} B_{p}^{k} V_{k}^{i} \right) B_{i}^{k} \right. \\
+ \left( \frac{\phi_{\gamma}^{\gamma} \check{L}_{\beta}^{a} \delta_{k}^{i} \delta_{\beta}^{j} + \frac{\phi_{\gamma}^{\gamma}}{\phi_{\gamma}^{\gamma} + 2 F^{2} \phi_{\gamma}^{\gamma} \delta_{k}^{i} \delta_{\beta}^{j}} \right) B_{i}^{a} H_{i}^{a} \right].
\]
Next, using (2.18) in (2.21), we obtain
\[
C^a_{\beta i} = B^a_i B^j_{\beta \rho} \tilde{C}^i_{j\rho} + \left( \frac{\varphi''}{\varphi'} \left( \delta^i_j y_k + \delta^i_k y_j \right) + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} \delta^i_j y_k y_j + \frac{2(\varphi'' \varphi' - 2\varphi''^2)}{\varphi' (\varphi' + 2F^2 \varphi'')} y^i_j y_j y_k \right) B^k_i B^a_{\beta j}.
\] (2.25)

If we take \( C^a_{\beta i} = B^a_i B^j_{\beta \rho} \tilde{C}^i_{j\rho} \), (2.25) becomes
\[
C^a_{\beta i} = C^a_{\beta i} + \left( \frac{\varphi''}{\varphi'} \left( \delta^i_j y_k + \delta^i_k y_j \right) + \frac{\varphi''}{\varphi' + 2F^2 \varphi''} \delta^i_j y_k y_j + \frac{2(\varphi'' \varphi' - 2\varphi''^2)}{\varphi' (\varphi' + 2F^2 \varphi'')} y^i_j y_j y_k \right) B^k_i B^a_{\beta j}.
\] (2.26)

Thus, we have the following.

**Theorem 2.7.** The coefficients of the induced tangent connection \( D^i \) for the subspace \( L^m \) of an APL space are given by (2.24) and (2.26).

**Remarks.** The torsion \( T^a_{\beta i} = L^a_{\beta i} - L^a_{i \beta} \) does not vanish, in general, while \( S^a_{\beta i} = C^a_{\beta i} - C^a_{i \beta} = 0 \). These facts may be observed from (2.24) and (2.26).

**Definition 2.8** (cf. [8]). An operator \( D^i \) given by
\[
D^i X^a = X^a_{\beta i} du^a + X^a_{\beta i} \delta \sigma a^\nu,
\] (2.27)

where \( X^a_{\beta i} = \delta a X^a + X^b L^a_{\beta i} \), \( X^a_{\beta i} = \delta a X^a + X^b C^a_{\beta i} \), is called the induced normal connection by \( D \).

The coefficients \((L^a_{\beta i}, C^a_{\beta i})\) of \( D^i \) are given by
\[
L^a_{\beta i} = B^a_i \left( \delta^i_j B^j_{\beta j} + B^j_{\beta j} \tilde{L}_{i j} \right),
\] (2.28)
\[
C^a_{\beta i} = B^a_i \left( \delta^i_j B^j_{\beta j} + B^j_{\beta j} \tilde{C}_{i j} \right).
\] (2.29)

Using (2.6) and (2.16) in (2.28), we find
\[
L^a_{\beta i} = B^a_i \delta^i_{\gamma} B^j_{\beta} + B^a_i B^p_{\beta j} V^p_j B^i_{\gamma} \delta a B^i_{\beta j}
\]
\[+ B^a_i B^p_{\beta j} \left[ \tilde{L}_{i j} + \left( V^p_{\gamma} C^i_{j \gamma} + V^p_{\gamma} C^i_{\gamma j} + V^p_{\gamma} a^\mu c_{\gamma j} - C^a_{\mu i} B^a_{\beta j} V^p_{\gamma k} \right) \right] B^k_i.
\] (2.30)
Taking $L_{by}^a = B_i^a (\delta_1 B_b^i + B_b^i L_{j^1})$ and using $y_j B_b^j = 0$, (2.30) reduces to

$$L_{by}^a = L_{by}^a + B_i^a B_b^b V_j^b B_i^i \delta_2 B_b^i + \left( V_k^i C_j^i + V_j^i C_k^i + V_j^i a^p C_{r k j} - C_j^i B_b^b B_p^j V_k^p \right) B_i^a B_b^b B_i^k$$
$$+ \frac{q''}{q' + 2F^2 q''} g_{jk} y^j B_c^b H_c^c B_i^a B_b^i.$$  

(2.31)

Next, using (2.18) in (2.29), we have

$$C_{by}^a = B_i^a \left( \delta_1 B_b^i + B_b^i C_{j^1} \right) + \frac{q''}{q'} \left( \delta_2 y_k + \delta_2^i y_j \right) + \frac{q''}{q' + 2F^2 q''} g_{jk} y^j$$
$$+ \frac{2(q'' q' - 2q''^2)}{q' (q' + 2F^2 q'')} \left( y_j y_k \right) B_i^k B^a B_b^i.$$  

(2.32)

Taking $C_{by}^a = B_i^a (\delta_1 B_b^i + B_b^i C_{j^1})$ and using (1.9) and $y_j B_b^j = 0$, the last equation yields

$$C_{by}^a = C_{by}^a + \frac{q''}{q'} \delta_2 y_k B_b^i + \frac{q''}{q' + 2F^2 q''} g_{jk} y^j B_i^k B_b^i.$$  

(2.33)

Thus, we have the following.

**Theorem 2.9.** The coefficients of induced normal connection $D^2$ for the subspace $L_m$ of an APL space $L_n$ are given by (2.31) and (2.33).

**Definition 2.10 (cf. [8]).** The (mixed) derivative of a mixed d-tensor field $T_{j^i - \beta^a - \rho^a - b}$ is given by

$$\nabla T_{j^i - \beta^a - \rho^a - b} = \left( \delta_1 T_{j^i - \beta^a - \rho^a} + T_{j^i - \beta^a - \rho^a} \cdot \delta_1 \cdot \delta_1 \right) + \ldots + T_{j^i - \beta^a - \rho^a} \cdot \delta_1 \cdot \delta_1.$$  

(2.34)

The connection 1-forms,

$$\omega_j^i = L_{ja}^i \delta u^a + \dot{C}_{ja}^i \delta v^a,$$  

(2.35)

$$\omega^a = L_{\rho j}^a \delta u^j + \dot{C}_{\rho j}^a \delta v^j,$$  

(2.36)

$$\omega^a = L_{\rho j}^a \delta u^j + \dot{C}_{\rho j}^a \delta v^j.$$  

(2.37)
are called the connection 1-forms of $\nabla$. We have the following structure equations of $\nabla$.

**Theorem 2.11 (cf. [8]).** The structure equations of $\nabla$ are as follows:

\[
\begin{align*}
    d(\delta u^a) - d\delta^a \wedge \omega^i_a &= -\Omega^a, \\
    d(\delta u^a) - \delta u^a \wedge \omega^i_a &= -\bar{\Omega}^a, \\
    d\bar{\omega}^i_j - \bar{\omega}^h_j \wedge \bar{\omega}^i_h &= -\bar{\Omega}^i_j, \\
    d\omega^a_i - \omega^a_i \wedge \omega^b_i &= -\Omega^a_i, \\
    d\bar{\omega}^i_a - \omega^c_i \wedge \omega^a_c &= -\Omega^a_c.
\end{align*}
\]

(2.38)

where the 2-forms of torsions $\Omega^a, \bar{\Omega}^a$ are given by

\[
\begin{align*}
    \Omega^a &= \frac{1}{2} T^a_{\beta\gamma} du^\beta \wedge du^\gamma + C^a_{\beta\gamma} du^\beta \wedge \delta v^\gamma, \\
    \bar{\Omega}^a &= \frac{1}{2} R^a_{\beta\gamma} du^\beta \wedge du^\gamma + \bar{P}^a_{\beta\gamma} du^\beta \wedge \delta v^\gamma,
\end{align*}
\]

(2.39)

with $P^a_{\beta\gamma} = \partial_\gamma \bar{N}^a_{\beta} - L^a_{\beta\gamma}$, and the 2-forms of curvature $\bar{\Omega}^i_j, \Omega^a_i$ and $\Omega^a_c$, are given by

\[
\begin{align*}
    \bar{\Omega}^i_j &= \frac{1}{2} R^i_{\gamma\delta} du^\gamma \wedge du^\delta + \bar{P}^i_{\gamma\delta} du^\gamma \wedge \delta v^\delta + \frac{1}{2} \bar{S}^i_{\gamma\delta} \delta v^\gamma \wedge \delta v^\delta, \\
    \Omega^a_i &= \frac{1}{2} R^a_{\beta\gamma} du^\beta \wedge du^\gamma + P^a_{\beta\gamma} du^\beta \wedge \delta v^\gamma \wedge \delta v^\delta + \frac{1}{2} S^a_{\beta\gamma} \delta v^\beta \wedge \delta v^\gamma \wedge \delta v^\delta, \\
    \Omega^a_c &= \frac{1}{2} R^a_{\beta\gamma} du^\beta \wedge du^\gamma + P^a_{\beta\gamma} du^\beta \wedge \delta v^\gamma \wedge \delta v^\delta + \frac{1}{2} S^a_{\beta\gamma} \delta v^\beta \wedge \delta v^\gamma \wedge \delta v^\delta.
\end{align*}
\]

(2.40)

We will use the following notations in Section 4:

(a) $\bar{\Omega}_{ij} = \bar{\Omega}^h_i a_{hj}$,  \hspace{1cm} (b) $\Omega_{ij} = \Omega^a_i a_{j\beta}$,  \hspace{1cm} (c) $\Omega_{ab} = \Omega^c_{b \alpha} \delta_{ac}$.

(2.41)

### 3. The Gauss-Weingarten Formulae

The Gauss-Weingarten formulae for the subspace $L^m = (\tilde{M}, \tilde{L}(u, v))$ of a Lagrange space $L^n$ are given by (cf. [8])

\[
\nabla B^i_a = B^i_a \Pi^a_{\alpha} \quad \nabla B^i_a = -B^i_a \Pi^\beta_{\alpha}.
\]

(3.1)
where

\[
\Pi_a^a = H_{ab}^a du^b + K_{ab}^a \delta v^b, \\
\Pi_b^a = S^{ab} \delta \alpha \Pi_{\gamma}^\beta,
\]

(a) \( H_{ab}^a = B_i^a \left( \delta_i^j B_a^j + B_a^j \bar{L}_{i \beta}^j \right) \),

(b) \( K_{ab}^a = B_i^a B_a^j \bar{C}_{i \beta}^j \).

Using (2.6) and (2.16) in (3.3)(a), we have

\[
H_{ab}^a = B_i^a \left( \delta_i^j B_a^j + B_a^j \bar{L}_{i \beta}^j \right) + B_i^a B_j^a V_j^p B_p^j B_{ay}
\]

\[
+ \left( V_k^i C_j^i \right. + V_j^i C_k^i + V_p^i a^{ip} C_{krj} - C_{j \beta} B_a^p B_{p \beta} \right) B_i^a B_j^a B_k^a
\]

\[
+ \left( \frac{\phi''}{\phi'} y_j^i \delta_k^i + \frac{\phi''}{\phi' + 2F^a \phi''} \delta_j^i y_j^j \right) B_i^a H_k^a B_j^a B_a^i.
\]

If we take \( \delta_i^j B_a^j + B_a^j \bar{L}_{i \beta}^j \), the last expression provides

\[
H_{ab}^a = B_i^a \left( \delta_i^j B_a^j + B_a^j \bar{L}_{i \beta}^j \right) + B_i^a B_j^a V_j^p B_p^j B_{ay}
\]

\[
+ \left( V_k^i C_j^i \right. + V_j^i C_k^i + V_p^i a^{ip} C_{krj} - C_{j \beta} B_a^p B_{p \beta} \right) B_i^a B_j^a B_k^a
\]

\[
+ \left( \frac{\phi''}{\phi'} y_j^i \delta_k^i + \frac{\phi''}{\phi' + 2F^a \phi''} \delta_j^i y_j^j \right) B_i^a H_k^a B_j^a B_a^i.
\]

Next, using (2.18) in (3.3)(b) and keeping (1.9) in view, we find

\[
K_{ab}^a = K_{ab} + \left( \frac{\phi''}{\phi' + 2F^a \phi''} \delta_j^i y_j^j + \frac{2(\phi''' \phi' - \phi''^2)}{\phi' (\phi' + 2F^a \phi'')} y_j^i \delta_k^i \right) B_i^a B_j^a B_k^a,
\]

where \( K_{ab} = B_i^a B_a^j \bar{C}_{i \beta}^j \). Thus, we have the following.

**Theorem 3.1.** The following Gauss-Weingarten formulae for the subspace \( \bar{L}^m \) of an APL space hold:

\[
\nabla B_a^i = B_a^j \Pi_a^a \quad \nabla B_a^i = -B_p^i \Pi_a^b,
\]
where

\[ \Pi^a_{\alpha} = H^a_{\alpha \beta} du^\beta + K^a_{\alpha \beta} \sigma^\beta, \quad \Pi^\beta_{\alpha} = g^{\alpha \beta} \delta_{ab} \Pi^b_{\gamma}, \]

\[ H^a_{\alpha \beta} = \overset{o}{H}^a_{\alpha \beta} + B^a_{\beta} V^p_{\gamma} B^i_{\gamma} B^i_{\alpha \gamma} + \left( V^r_{\gamma} C^i_{\gamma j} + V^r_{\gamma} C^i_{\gamma kr} + V^p_{\gamma} a^{\gamma \nu} C_{r j} - C_{j r} B^p_{\gamma} B^p_{\nu} V^p_{\gamma} \right) B^a_{\beta} B^i_{\gamma} B^i_{\alpha}, \]

\[ + \left( \frac{g^{\alpha \beta}}{q'r' + 2F^2 q'' g'k'y'} \right) B^a_{\beta} H^b_{\gamma} B^i_{\alpha}, \]

\[ K^a_{\alpha \beta} = \overset{o}{K}^a_{\alpha \beta} + \left( \frac{g^{\alpha \beta}}{q'r' + 2F^2 q'' g'k'y'} \right) B^a_{\beta} H^b_{\gamma} B^i_{\alpha}, \]

\[ (3.8) \]

**Remark 3.2.** \( H^a_{\alpha \beta} \) and \( K^a_{\alpha \beta} \) given, respectively, by (3.5) and (3.6) are called the second fundamental \( d \)-tensor fields of immersion \( i \).

The following consequences of Theorem 3.1 are straightforward.

**Corollary 3.3.** In a subspace \( \tilde{L}^m \) of an APL space, we have the following:

\[ (a) \quad \nabla a_{\alpha \beta} = 0, \]
\[ (b) \quad \nabla B^i_{\alpha} = 0, \]

if and only if

\[ \overset{o}{H}^a_{\alpha \beta} = - \left( B^a_{\beta} V^p_{\gamma} B^i_{\gamma} B^i_{\alpha \gamma} + \left( V^r_{\gamma} C^i_{\gamma j} + V^r_{\gamma} C^i_{\gamma kr} + V^p_{\gamma} a^{\gamma \nu} C_{r j} - C_{j r} B^p_{\gamma} B^p_{\nu} V^p_{\gamma} \right) B^a_{\beta} B^i_{\gamma} B^i_{\alpha}, \]

\[ + \left( \frac{g^{\alpha \beta}}{q'r' + 2F^2 q'' g'k'y'} \right) B^a_{\beta} H^b_{\gamma} B^i_{\alpha}, \]

\[ (3.10) \]

\[ \overset{o}{K}^a_{\alpha \beta} = - \left( \frac{g^{\alpha \beta}}{q'r' + 2F^2 q'' g'k'y'} \right) B^a_{\beta} H^b_{\gamma} B^i_{\alpha}, \]

**4. The Gauss-Codazzi Equations**

The Gauss-Codazzi Equations for the subspace \( \tilde{L}^m = (\tilde{M}, \tilde{L}(u, v)) \) of a Lagrange space \( L^n \) are given by (cf. [8])

\[ B^i_{\alpha} B^j_{\beta} \tilde{\Omega}_{ij} - \Omega_{\alpha \beta} = \Pi_{\beta \alpha} \wedge \Pi^\alpha_{\gamma}, \]

\[ (4.1) \]

\[ B^i_{\alpha} B^j_{\beta} \tilde{\Omega}_{ij} - \Omega_{\alpha \beta} = \Pi_{\beta \alpha} \wedge \Pi^\alpha_{\gamma}, \]

\[ (4.2) \]

\[ -B^i_{\alpha} B^j_{\beta} \tilde{\Omega}_{ij} = \delta_{ab} \left( d \Pi^b_{\alpha} + \Pi^b_{\beta} \wedge \omega^\beta_{a} - \Pi^b_{\alpha} \wedge \omega^\beta_{a} \right), \]

\[ (4.3) \]
where

\[(a) \; \Pi_{aa} = g_{\alpha \beta} \Pi_{\alpha \beta}, \quad (b) \; \Pi_{ab} = \delta_{bc} \Pi_{c}. \tag{4.4}\]

Using (1.3) in (2.41)(a), we find that

\[\Omega_{ij} = \psi^i \Omega^j_i g_{ij} + 2 \psi^i \omega^j_i y_i y_j. \tag{4.5}\]

Applying \(a_{i \beta} = B_i^j B_{\beta}^j a_{ij}\) in (2.41)(b), we have \(\Omega_{\alpha \beta} = B_i^j B_{\beta}^j \Omega_{\alpha ij} a_{ij}\), which in view of (1.3) becomes

\[\Omega_{\alpha \beta} = \psi^i g_{ij} B_i^j B_{\beta}^j \Omega_{\alpha ij} + 2 \psi^i y_i y_j B_i^j B_{\beta}^j \Omega_{\alpha ij}, \tag{4.6}\]

that is,

\[\Omega_{\alpha \beta} = \psi^i g_{ij} \Omega_{\alpha ij} + 2 \psi^i y_i y_j B_i^j B_{\beta}^j \Omega_{\alpha ij}. \tag{4.7}\]

For the subspace \(L^m\) of an APL space, (4.4)(a) is of the form \(\Pi_{aa} = a_{aa} \Pi_{aa}\), which in view of \(a_{a \beta} = B_i^j B_{\beta}^j a_{ij}\) and (1.3) becomes \(\Pi_{aa} = \psi^i B_i^j B_{\beta}^j a_{ij} \Pi_{\beta \alpha} \Pi_{a}, \Pi_{ab}, \Omega_{\alpha \beta}, \text{ and } \omega^b_c\), respectively, given by (4.8), (4.4)(b), (4.5), (4.7), and (2.37).

Thus, we have the following.

**Theorem 4.1.** The Gauss-Codazzi equations for a Lagrange subspace \(L^m\) of an APL space are given by (4.1)–(4.3) with \(\Pi_{aa}, \Pi_{ab}, \Omega_{ij}, \Omega_{\alpha \beta}, \text{ and } \omega^b_c\), respectively, given by (4.8), (4.4)(b), (4.5), (4.7), and (2.37).

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**References**


