Research Article

Convergence Theorem for a Family of Generalized Asymptotically Nonexpansive Semigroup in Banach Spaces

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Let E be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let $\mathfrak{J} = \{T(t) : t \geq 0\}$ be a family of uniformly asymptotically regular generalized asymptotically nonexpansive semigroup of E , with functions $u, v : [0, \infty) \rightarrow [0, \infty)$. Let $F := F(\mathfrak{J}) = \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$ and $f : K \rightarrow K$ be a weakly contractive map. For some positive real numbers λ and δ satisfying $\delta + \lambda > 1$, let $G : E \rightarrow E$ be a δ -strongly accretive and λ -strictly pseudocontractive map. Let $\{t_n\}$ be an increasing sequence in $[0, \infty)$ with $\lim_{n \rightarrow \infty} t_n = \infty$, and let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1]$ satisfying some conditions. Strong convergence of a viscosity iterative sequence to common fixed points of the family $\mathfrak{J}$ of uniformly asymptotically regular asymptotically nonexpansive semigroup, which also solves the variational inequality $\langle (G - \gamma f)p, j(p - x) \rangle \leq 0$, for all $x \in F$, is proved in a framework of a real Banach space.

1. Introduction

Let E be a real Banach space. We denote by J the normalized duality map from E to 2^{E^*} (E^* is the dual space of E), and it is defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}. \quad (1.1)$$

A mapping $T : E \rightarrow E$ is said to be contractive if $\|Tx - Ty\| \leq \alpha \|x - y\|$, for $x, y \in E$, and some constant $\alpha \in [0, 1)$. It is said to be weakly contractive if there exists a nondecreasing function $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\psi(t) = 0$ if and only if $t = 0$ and $\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|)$, for all $x, y \in E$. It is known that the class of weakly contractive maps

contain properly the class of contractive ones, see [1, 2]. A mapping $T : E \rightarrow E$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in E$ and asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ as $n \rightarrow \infty$ and $\|T^n x - T^n y\| \leq k_n \|x - y\|$, for all $x, y \in E$. We denote by $F(T) = \{x \in E : Tx = x\}$ the set of fixed points of a map T .

A mapping $T : E \rightarrow E$ is said to be total asymptotically nonexpansive (see [3]) if there exist nonnegative real sequences $\{u_n\}$ and $\{v_n\}$, with $u_n \rightarrow 0$ and $v_n \rightarrow 0$ as $n \rightarrow \infty$ and strictly increasing and continuous functions $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with $\psi(0) = 0$ such that

$$\|T^n x - T^n y\| \leq \|x - y\| + u_n \psi(\|x - y\|) + v_n, \quad \forall x, y \in E. \quad (1.2)$$

Remark 1.1. If $\psi(\lambda) = \lambda$, the total asymptotically nonexpansive mapping coincides with generalized asymptotically nonexpansive mapping. In addition, for all $n \in \mathbb{N}$, if $v_n = 0$, then generalized asymptotically nonexpansive mapping coincides with asymptotically nonexpansive mapping; if $u_n = 0$, $v_n = \max\{0, p_n\}$ where $p_n := \sup_{x, y \in E} (\|T_n x - T_n y\| - \|x - y\|)$, then generalized asymptotically nonexpansive mapping coincide with asymptotically nonexpansive mapping in the intermediate sense; if $u_n = 0$, and $v_n = 0$ then we obtain from (1.2) the class of nonexpansive mapping.

A one-parameter family of generalized asymptotically nonexpansive semigroup is a family $\mathfrak{J} = \{T(t) : t \geq 0\}$ of self-mapping of E such that

- (i) $T(0)x = x$ for $x \in E$,
- (ii) $T(s+t)x = T(s)T(t)x$ for all $t, s \geq 0$ and $x \in E$,
- (iii) $\lim_{t \rightarrow 0} T(t)x = x$ for $x \in E$,
- (iv) there exist functions $u, v : [0, \infty) \rightarrow [0, \infty)$ such that $u(t) \rightarrow 0, v(t) \rightarrow 0$ as $t \rightarrow \infty$, and

$$\|T(t)x - T(t)y\| \leq (1 + u(t))\|x - y\| + v(t) \quad \forall x, y \in E. \quad (1.3)$$

We will denote by F the common fixed-point set of $\mathfrak{J}$, that is,

$$F := \text{Fix}(\mathfrak{J}) = \{x \in E : T(t)x = x, t \geq 0\} = \bigcap_{t \geq 0} \text{Fix}(T(t)). \quad (1.4)$$

The family $\mathfrak{J} = \{T(t) : t \geq 0\}$ is said to be asymptotically regular if

$$\lim_{s \rightarrow \infty} \|T(s+t)x - T(s)x\| = 0, \quad (1.5)$$

for all $t \in [0, \infty)$ and $x \in E$. It is said to be uniformly asymptotically regular if, for any $t \geq 0$ and for any bounded subset C of E ,

$$\lim_{s \rightarrow \infty} \sup_{x \in C} \|T(s+t)x - T(s)x\| = 0. \quad (1.6)$$

For some positive real numbers δ and λ , a mapping $G : E \rightarrow E$ is said to be δ -strongly accretive if for any $x, y \in E$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Gx - Gy, j(x - y) \rangle \geq \delta \|x - y\|^2, \quad (1.7)$$

and it is called λ -strictly pseudocontractive if

$$\langle Gx - Gy, j(x - y) \rangle \leq \|x - y\|^2 - \lambda \|(I - G)x - (I - G)y\|^2. \quad (1.8)$$

Let E be a real Banach space, and let δ, λ , and τ be positive real numbers satisfying $\delta + \lambda > 1$ and $\tau \in (0, 1)$. Let $G : E \rightarrow E$ be a δ -strongly accretive and λ -strictly pseudocontractive, then the following holds, see [4], for $x, y \in E$:

$$\begin{aligned} \|(I - G)x - (I - G)y\| &\leq \left(\sqrt{\frac{1 - \delta}{\lambda}} \right) \|x - y\|, \\ \|(I - \tau G)x - (I - \tau G)y\| &\leq 1 - \tau \left(1 - \sqrt{\frac{1 - \delta}{\lambda}} \right) \|x - y\|, \end{aligned} \quad (1.9)$$

that is, $(I - G)$ and $(I - \tau G)$ are contractive mappings.

Let C be a nonempty closed-convex subset of E and $T : E \rightarrow E$ a map. Then, a variational inequality problem with respect to C and T is found to be $x^* \in C$ such that

$$\langle Tx^*, j(y - x^*) \rangle \geq 0, \quad \forall y \in C, \quad j(y - x^*) \in J(y - x^*). \quad (1.10)$$

Recently, convergence theorems for fixed points of nonexpansive mappings, common fixed points of family of nonexpansive mappings, nonexpansive semigroup, and their generalisation have been studied by numerous authors (see, e.g., [5–21]).

Acedo and Suzuki [22], recently, proved the strong convergence of the Browder's implicit scheme, $x_0, u \in C$,

$$x_n = \alpha_n u + (1 - \alpha_n)T(t_n)x_n, \quad n \geq 0, \quad (1.11)$$

to a common fixed point of a uniformly asymptotically regular family $\{T(t) : t \geq 0\}$ of nonexpansive semigroup in the framework of a real Hilbert space.

Li et al. [23] proved strong convergence theorems for implicit viscosity schemes for common fixed points of family of generalized asymptotically nonexpansive semigroups in Banach spaces.

Let S be a semigroup and $B(S)$ the subspace of all bounded real-valued functions defined on S with supremum norm. For each $s \in S$, the left translator operator $l(s)$ on $B(S)$ is defined by $(l(s)f)(t) = f(st)$ for each $t \in S$ and $f \in B(S)$. Let X be a subspace of $B(S)$ containing 1, and let X^* be its topological dual. An element μ of X^* is said to be a mean on X if $\|\mu\| = \mu(1) = 1$. Let X be l_s invariant, that is, $l_s(X) \subset X$ for each $s \in S$. A mean μ on X is said to be left invariant if $\mu(l_s f) = \mu(f)$ for each $s \in S$ and $f \in X$.

Recently, Saeidi and Naseri [24] studied the problem of approximating common fixed point of a family of nonexpansive semigroup and solution of some variational inequality problem in a real Hilbert space. They proved the following theorem.

Theorem 1.2 (Saeidi and Naseri [24]). *Let $\mathfrak{J} = \{T(t) : t \in S\}$ be a nonexpansive semigroup in a real Hilbert space H such that $F(\mathfrak{J}) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \rightarrow \langle T(t)x, y \rangle$ is an element of X for each $x, y \in H$. Let $f : E \rightarrow E$ be a contraction with constant α , and let $G : H \rightarrow H$ be strongly positive map with constant $\bar{\gamma} > 0$. Let $\{\mu_n\}$ be a left regular sequence of means on X , and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\gamma \in (0, \bar{\gamma}/\alpha)$, and let $\{x_n\}$ be a sequence generated by $x_0 \in H$,*

$$x_{n+1} = (I - \alpha_n G)T(\mu_n)x_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \quad (1.12)$$

Then, $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which is the unique solution of the variational inequality $\langle (G - \gamma f)x^, j(x - x^*) \rangle \geq 0$ for all $x \in F(\mathfrak{J})$. Equivalently one has $P_{F(\mathfrak{J})}(I - G + \gamma f)x^* = x^*$.*

More recently, as commented by Golkarmanesh and Naseri [25], Piri and Vaezi [4] gave a minor variation of Theorem 1.2 as follows.

Theorem 1.3 (Piri and Vaezi [4]). *Let $\mathfrak{J} = \{T(t) : t \in S\}$ be a nonexpansive semigroup on a real Hilbert space H such that $F(\mathfrak{J}) \neq \emptyset$. Let X be a left invariant subspace of $B(S)$ such that $1 \in X$, and the function $t \rightarrow \langle T(t)x, y \rangle$ is an element of X for each $x, y \in H$. Let $f : E \rightarrow E$ be a contraction with constant α , and let $G : H \rightarrow H$ be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$. Let $\{\mu_n\}$ be a left regular sequence of means on X , and let $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\lim_{n \rightarrow \infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{x_n\}$ be a sequence generated by $x_0 \in H$,*

$$x_{n+1} = (I - \alpha_n G)T(\mu_n)x_n + \alpha_n \gamma f(x_n), \quad n \geq 0, \quad (1.13)$$

where $0 < \gamma < (1 - \sqrt{(1 - \delta/\lambda)})/\alpha$, then, $\{x_n\}$ converges strongly to a common fixed point of the family $F(\mathfrak{J})$ which is the unique solution of the variational inequality $\langle (G - \gamma f)x^, j(x - x^*) \rangle \geq 0$ for all $x \in F(\mathfrak{J})$. Equivalently one has $P_{F(\mathfrak{J})}(I - G + \gamma f)x^* = x^*$.*

Very recently, Ali [26] continued the study of the problem in [4, 24] and proved a strong convergence theorem in a Banach space setting much more general than Hilbert space. He actually proved the following theorem.

Theorem 1.4 (Ali [26]). *Let E be a real Banach space with local uniform Opial's property whose duality mapping is sequentially continuous. Let $\mathfrak{J} = \{T(t) : t \geq 0\}$ be a uniformly asymptotically regular family of asymptotically nonexpansive semigroup of E with function $k : [0, \infty) \rightarrow [0, \infty)$ and $F := F(\mathfrak{J}) = \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$. Let $f : E \rightarrow E$ be weakly contractive, and let $G : E \rightarrow E$ be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$. Let $\eta := (1 - \sqrt{(1 - \delta)/\lambda})$ and $\gamma \in (0, \min\{\eta, \delta/2\})$. Let $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1)$, and let $\{t_n\}$ be an increasing sequence in $[0, \infty)$ satisfying the following conditions:*

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \lim_{n \rightarrow \infty} \frac{k_n}{\alpha_n} = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \quad 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1. \quad (1.14)$$

Define a sequence $\{x_n\}$ by $x_0 \in E$,

$$\begin{aligned} x_{n+1} &= \beta_n x_n + (1 - \beta_n) y_n, \\ y_n &= (I - \alpha_n G)T(t_n)x_n + \alpha_n \gamma_n f(x_n), \quad n \geq 0. \end{aligned} \quad (1.15)$$

Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which solves the variational inequality

$$\langle (G - \gamma f)q, j(x - q) \rangle \geq 0, \quad \forall x \in F. \quad (1.16)$$

Remark 1.5. It is well known that all l^p ($1 < p < \infty$) spaces satisfy Opial's condition and possess a weakly sequentially continuous duality mapping. However, L^p ($1 < p < \infty$) spaces and consequently all Sobolev spaces do not satisfy either of the properties.

It is our purpose in this paper to prove a strong convergence theorem for approximating common fixed points of family of uniformly asymptotically regular generalized asymptotically nonexpansive semigroup in a real reflexive and strictly convex Banach space E with a uniformly Gâteaux differentiable norm. Our theorem is applicable in $L_p(\ell_p)$ spaces, $1 < p < \infty$ (and consequently in Sobolev spaces). Our theorem extends and improves some recent important results. For instance, our theorem presents a convergence of an explicit scheme that extends Theorem 1.4 to a more general setting of Banach spaces that includes L^p ($1 < p < \infty$) spaces on one hand and for more general class of maps on the other hand.

2. Preliminaries

Let $S := \{x \in E : \|x\| = 1\}$ denote the unit sphere of a real Banach space E . E is said to have a *Gâteaux differentiable* norm if the limit

$$\lim_{t \rightarrow 0} \frac{\|x + ty\| - \|x\|}{t} \quad (2.1)$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$, the limit is attained uniformly for $x \in S$. A Banach space E is said to be strictly convex if $\|x + y\|/2 < 1$ for $x \neq y$ and $\|x\| = \|y\| = 1$.

Let K be a nonempty, closed, convex, and bounded subset of a real Banach space E , and let the diameter of K be defined by $d(K) := \sup\{\|x - y\| : x, y \in K\}$. For each $x \in K$, let $r(x, K) := \sup\{\|x - y\| : y \in K\}$ and $r(K) := \inf\{r(x, K) : x \in K\}$ denote the Chebyshev radius of K relative to itself. The *normal structure coefficient* $N(E)$ of E (introduced in 1980 by Bynum [27], see also Lim [28] and the references contained therein) is defined by $N(E) := \inf\{(d(K)/r(K)) : K \text{ is a closed convex and bounded subset of } E \text{ with } d(K) > 0\}$. A space E such that $N(E) > 1$ is said to have *uniform normal structures*. It is known that every space with a uniform normal structure is reflexive, and that all uniformly convex and uniformly smooth Banach spaces have uniform normal structure (see, e.g., [29]).

Let E be a real Banach space with uniformly Gâteaux differentiable norm, then the normalized duality mapping $J : E \rightarrow 2^{E^*}$, defined by (1.1), is singled valued and uniformly continuous from the norm topology of E to the weak* topology of E^* on each bounded subset of E , see, for example [30].

Definition 2.1. Let μ be a continuous linear functional on l^∞ , and let $(a_0, a_1, \dots) \in l^\infty$. We write $\mu_n(a_n)$ instead of $\mu(a_0, a_1, \dots)$. The function μ is called a Banach limit when μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $(a_0, a_1, \dots) \in l^\infty$.

For a Banach limit μ , it is known that $\liminf_{n \rightarrow \infty} a_n \leq \mu_n(a_n) \leq \limsup_{n \rightarrow \infty} a_n$ for every $a = (a_0, a_1, \dots) \in l^\infty$. So if $a = (a_0, a_1, \dots) \in l^\infty$ and $a_n - b_n \rightarrow 0$ as $n \rightarrow \infty$, we have $\mu_n(a_n) = \mu_n(b_n)$.

We will make use of the following well-known result.

Lemma 2.2. Let E be a real-normed linear space. Then, the following inequality holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad \forall x, y \in E, j(x + y) \in J(x + y). \quad (2.2)$$

In the sequel, we shall also make use of the following lemmas.

Lemma 2.3 (Suzuki [31]). Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a real Banach space E , and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf \beta_n \leq \limsup \beta_n < 1$. Suppose that $x_{n+1} = \beta_n y_n + (1 - \beta_n)x_n$ for all integer $n \geq 1$ and $\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0$.

Lemma 2.4 (Shioji and Takahashi [32]). Let $(a_0, a_1, a_2, \dots) \in l^\infty$ be such that $\mu_n a_n \leq 0$ for all Banach limits μ . If $\limsup_{n \rightarrow \infty} (a_{n+1} - a_n) \leq 0$, then $\limsup_{n \rightarrow \infty} a_n \leq 0$.

Lemma 2.5 (Xu [33]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 0, \quad (2.3)$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ (ii) $\limsup_{n \rightarrow \infty} \sigma_n \leq 0$ (iii) $\gamma_n \geq 0$ and $(n \geq 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then, $a_n \rightarrow 0$ as $n \rightarrow \infty$.

3. Main Results

Theorem 3.1. Let E be a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm, and let $\mathfrak{J} = \{T(t) : t \geq 0\}$ be uniformly asymptotically regular family of generalized asymptotically nonexpansive semigroup of E , with functions $u, v : [0, \infty) \rightarrow [0, \infty)$ and $F := F(\mathfrak{J}) = \bigcap_{t \geq 0} F(T(t)) \neq \emptyset$. Let $f : E \rightarrow E$ be weakly contractive, and let $G : E \rightarrow E$ be δ -strongly accretive and λ -strictly pseudocontractive with $\delta + \lambda > 1$. Let $\eta := (1 - \sqrt{(1 - \delta)/\lambda})$ and $\gamma \in (0, \min\{\delta, \eta/2\})$. Let $\{\beta_n\}$ and $\{\alpha_n\}$ be sequences in $(0, 1]$ and $\{t_n\}$ an increasing sequence in $[0, \infty)$ satisfying the following conditions:

$$\begin{aligned} \lim_{n \rightarrow \infty} \alpha_n = 0, \quad \lim_{n \rightarrow \infty} \frac{u(t_n)}{\alpha_n} = 0, \quad \lim_{n \rightarrow \infty} \frac{v(t_n)}{\alpha_n} = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty, \\ 0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1, \quad \lim_{n \rightarrow \infty} t_n = \infty. \end{aligned} \quad (3.1)$$

Define a sequence $\{x_n\}$ by $x_0 \in E$,

$$\begin{aligned} x_{n+1} &= \beta_n x_n + (1 - \beta_n) y_n, \\ y_n &= (I - \alpha_n G)T(t_n)x_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \end{aligned} \quad (3.2)$$

Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which solves the variational inequality

$$\langle (G - \gamma f)q, j(x - q) \rangle \geq 0, \quad \forall x \in F. \quad (3.3)$$

Proof. We start by showing that solution of the variational inequality (3.3) in F is at most one. Assume that $q, p \in F$ are solutions of the variational inequality (3.3), then

$$\langle (G - \gamma f)p, j(q - p) \rangle \geq 0, \quad \langle (G - \gamma f)q, j(p - q) \rangle \geq 0. \quad (3.4)$$

Adding these two inequalities, we get

$$\langle (G - \gamma f)p - (G - \gamma f)q, j(p - q) \rangle \leq 0. \quad (3.5)$$

Therefore,

$$\begin{aligned} 0 &\geq \langle (G - \gamma f)p - (G - \gamma f)q, j(p - q) \rangle \\ &= \langle G(p) - G(q), j(p - q) \rangle - \gamma \langle f(p) - f(q), j(p - q) \rangle \\ &\geq \delta \|p - q\|^2 - \gamma \|f(p) - f(q)\| \|p - q\| \\ &\geq \delta \|p - q\|^2 + \gamma \psi(\|p - q\|) \|p - q\| - \gamma \|p - q\|^2 \\ &= (\delta - \gamma) \|p - q\|^2 + \gamma \psi(\|p - q\|) \|p - q\|. \end{aligned} \quad (3.6)$$

Since $\delta > \gamma$, we obtain that $p = q$, and so the solution is unique in F .

Now, let $p \in F$, since $(1 - \alpha_n \eta)(u(t_n)/\alpha_n) \rightarrow 0$ and $(1 - \alpha_n \eta)(v(t_n)/\alpha_n) \rightarrow 0$ as $n \rightarrow \infty$, then there exists $n_0 \in \mathbb{N}$ such that $(1 - \alpha_n \eta)(u(t_n)/\alpha_n) < (\eta - \gamma)/2$ and $(1 - \alpha_n \eta)(v(t_n)/\alpha_n) < (\eta - \gamma)/2$ for all $n \geq n_0$. Hence, for $n \geq n_0$, we have the following:

$$\begin{aligned} \|y_n - p\| &\leq \|(I - \alpha_n G)(T(t_n)x_n - p)\| + \alpha_n \|\gamma f(x_n) - G(p)\| \\ &\leq (1 - \alpha_n \eta) [(1 + u(t_n)) \|x_n - p\| + v(t_n)] + \alpha_n \gamma \|f(x_n) - f(p)\| + \alpha_n \|\gamma f(p) - G(p)\| \\ &\leq [1 - \alpha_n (\eta - \gamma) + (1 - \alpha_n \eta) u(t_n)] \|x_n - p\| + (1 - \alpha_n \eta) v(t_n) + \alpha_n \|\gamma f(p) - G(p)\|, \end{aligned} \quad (3.7)$$

so that

$$\begin{aligned}
 \|x_{n+1} - p\| &\leq \beta_n \|x_n - p\| + (1 - \beta_n) \|y_n - p\| \\
 &\leq [\beta_n + (1 - \beta_n) [1 - \alpha_n(\eta - \gamma) + (1 - \alpha_n\eta)u(t_n)]] \|x_n - p\| \\
 &\quad + (1 - \alpha_n\eta)(1 - \beta_n)v(t_n) + \alpha_n(1 - \beta_n) \|\gamma f(p) - G(p)\| \\
 &\leq \left[1 - \alpha_n(1 - \beta_n) \left((\eta - \gamma) - (1 - \alpha_n\eta) \frac{u(t_n)}{\alpha_n} \right) \right] \|x_n - p\| \\
 &\quad + \alpha_n(1 - \beta_n) \left[\|\gamma f(p) - G(p)\| + (1 - \alpha_n\eta) \frac{v(t_n)}{\alpha_n} \right] \\
 &\leq \left[1 - \alpha_n(1 - \beta_n) \left((\eta - \gamma) - (1 - \alpha_n\eta) \frac{u(t_n)}{\alpha_n} \right) \right] \|x_n - p\| \\
 &\quad + \alpha_n(1 - \beta_n) \left((\eta - \gamma) - (1 - \alpha_n\eta) \frac{u(t_n)}{\alpha_n} \right) \\
 &\quad \times \frac{2[\|\gamma f(p) - G(p)\| + (1 - \alpha_n\eta)(v(t_n)/\alpha_n)]}{\eta - \gamma} \\
 &\leq \max \left\{ \|x_n - p\|, \frac{2\|\gamma f(p) - G(p)\|}{\eta - \gamma} + 1 \right\}.
 \end{aligned} \tag{3.8}$$

By induction, we have

$$\|x_n - p\| \leq \max \left\{ \|x_{n_0} - p\|, \frac{2\|\gamma f(p) - G(p)\|}{\eta - \gamma} + 1 \right\}, \quad \forall n \geq 0. \tag{3.9}$$

Thus, $\{x_n\}$ is bounded and so are $\{T(t_n)x_n\}$, $\{GT(t_n)x_n\}$, $\{y_n\}$, and $\{f(x_n)\}$. Observe that

$$\begin{aligned}
 y_{n+1} - y_n &= ((I - \alpha_{n+1}G)T(t_{n+1})x_{n+1} - (I - \alpha_{n+1}G)T(t_{n+1})x_n) \\
 &\quad + ((I - \alpha_{n+1}G)T(t_{n+1})x_n - (I - \alpha_nG)T(t_{n+1})x_n) \\
 &\quad + ((I - \alpha_nG)T(t_{n+1})x_n - (I - \alpha_nG)T(t_n)x_n) \\
 &\quad + (\alpha_{n+1}\gamma f(x_{n+1}) - \alpha_{n+1}\gamma f(x_n)) + (\alpha_{n+1}\gamma f(x_n) - \alpha_n\gamma f(x_n)),
 \end{aligned} \tag{3.10}$$

so that

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq (1 - \alpha_{n+1}\eta)(1 + u(t_{n+1}))\|x_{n+1} - x_n\| + (1 - \alpha_{n+1}\eta)v(t_{n+1}) \\
&\quad + |\alpha_n - \alpha_{n+1}|\|GT(t_{n+1})x_n\| + (1 - \alpha_n\eta)\|T((t_{n+1} - t_n) + t_n)x_n - T(t_n)x_n\| \\
&\quad + \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\| \\
&\leq (1 - \alpha_{n+1}\eta)(1 + u(t_{n+1}))\|x_{n+1} - x_n\| + (1 - \alpha_{n+1}\eta)v(t_{n+1}) \\
&\quad + |\alpha_n - \alpha_{n+1}|\|GT(t_{n+1})x_n\| \\
&\quad + (1 - \alpha_n\eta) \sup_{z \in \{x_n\}, s \in \mathbb{R}^+} \|T(s + t_n)z - T(t_n)z\| \\
&\quad + \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\|.
\end{aligned} \tag{3.11}$$

From this, we obtain that

$$\begin{aligned}
\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| &\leq [(1 - \alpha_{n+1}\eta)(1 + u(t_{n+1})) - 1]\|x_{n+1} - x_n\| \\
&\quad + (1 - \alpha_{n+1}\eta)v(t_{n+1}) + |\alpha_n - \alpha_{n+1}|\|GT(t_{n+1})x_n\| \\
&\quad + (1 - \alpha_n\eta) \sup_{z \in \{x_n\}, s \in \mathbb{R}^+} \|T(s + t_n)z - T(t_n)z\| \\
&\quad + \alpha_{n+1}\gamma\|f(x_{n+1}) - f(x_n)\| + |\alpha_{n+1} - \alpha_n|\gamma\|f(x_n)\|,
\end{aligned} \tag{3.12}$$

which implies that

$$\limsup_{n \rightarrow \infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0, \tag{3.13}$$

and by Lemma 2.3,

$$\lim_{n \rightarrow \infty} \|y_n - x_n\| = 0. \tag{3.14}$$

Thus,

$$\|x_{n+1} - x_n\| = (1 - \beta_n)\|y_n - x_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.15}$$

Next, we show that $\lim_{n \rightarrow \infty} \|y_n - T(t)y_n\| = 0$, for all $t \geq 0$.

Since

$$\begin{aligned}
\|x_n - T(t_n)x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - T(t_n)x_n\| \\
&\leq \|x_n - x_{n+1}\| + \beta_n\|x_n - T(t_n)x_n\| + (1 - \beta_n)\|y_n - T(t_n)x_n\|,
\end{aligned} \tag{3.16}$$

we have

$$\begin{aligned} (1 - \beta_n) \|x_n - T(t_n)x_n\| &\leq \|x_n - x_{n+1}\| + (1 - \beta_n) \|y_n - T(t_n)x_n\| \\ &= \|x_n - x_{n+1}\| + \alpha_n (1 - \beta_n) \|\gamma f(x_n) - GT(t_n)x_n\|. \end{aligned} \quad (3.17)$$

From $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$ and (3.15), we obtain

$$\lim_{n \rightarrow \infty} \|x_n - T(t_n)x_n\| = 0. \quad (3.18)$$

Also,

$$\begin{aligned} \|y_n - T(t_n)y_n\| &\leq \|y_n - x_n\| + \|x_n - T(t_n)x_n\| + \|T(t_n)x_n - T(t_n)y_n\| \\ &\leq (2 + u(t_n)) \|y_n - x_n\| + v(t_n) + \|x_n - T(t_n)x_n\| \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (3.19)$$

Since $\lim_{n \rightarrow \infty} t_n = \infty$ and $\{T(t) : t \geq 0\}$ is uniformly asymptotically regular,

$$\begin{aligned} \lim_{n \rightarrow \infty} \|T(t)T(t_n)x_n - T(t_n)x_n\| &\leq \lim_{n \rightarrow \infty} \sup_{x \in C} \|T(t)T(t_n)x - T(t_n)x\| = 0, \\ \lim_{n \rightarrow \infty} \|T(t)T(t_n)y_n - T(t_n)y_n\| &\leq \lim_{n \rightarrow \infty} \sup_{y \in C} \|T(t)T(t_n)y - T(t_n)y\| = 0, \end{aligned} \quad (3.20)$$

where C is any bounded subset of E containing $\{x_n\}$. Since $\{T(t)\}$ is continuous, we get that

$$\begin{aligned} \|y_n - T(t)y_n\| &\leq \|y_n - T(t_n)y_n\| + \|T(t_n)y_n - T(t)(T(t_n)y_n)\| \\ &\quad + \|T(t)(T(t_n)y_n) - T(t)y_n\|. \end{aligned} \quad (3.21)$$

This implies that

$$\lim_{n \rightarrow \infty} \|y_n - T(t)y_n\| = 0, \quad \forall t \geq 0. \quad (3.22)$$

Next, we show that

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - G)p, j(y_n - p) \rangle \leq 0. \quad (3.23)$$

Define a map $\phi : E \rightarrow \mathbb{R}$ by

$$\phi(y) := \mu_n \|y_n - y\|^2, \quad \forall y \in E. \quad (3.24)$$

Then, $\phi(y) \rightarrow \infty$ as $\|y\| \rightarrow \infty$, ϕ is continuous and convex, so as E is reflexive, there exists $q \in E$ such that $\phi(q) = \min_{u \in E} \phi(u)$. Hence, the set

$$K^* := \left\{ y \in E : \phi(y) = \min_{u \in E} \phi(u) \right\} \neq \emptyset. \quad (3.25)$$

Since $\lim_{n \rightarrow \infty} \|y_n - T(t)y_n\| = 0$, $\lim_{t \rightarrow \infty} u(t) = 0$, $\lim_{t \rightarrow \infty} v(t) = 0$, and ϕ is continuous for all $z \in K^*$, we have

$$\begin{aligned} \phi\left(\lim_{t \rightarrow \infty} T(t)z\right) &= \lim_{t \rightarrow \infty} \phi(T(t)z) = \lim_{t \rightarrow \infty} \mu_n \|y_n - T(t)z\|^2 \\ &\leq \lim_{t \rightarrow \infty} \mu_n ((1 + u(t)) \|y_n - z\| + v(t))^2 = \mu_n \|y_n - z\|^2 = \phi(z). \end{aligned} \quad (3.26)$$

Hence, $\lim_{t \rightarrow \infty} T(t)z \in K^*$.

Let $p \in F$. Since K^* is a closed-convex set, there exists a unique $q \in K^*$ such that

$$\|p - q\| = \min_{x \in K^*} \|p - x\|. \quad (3.27)$$

Since $p = \lim_{t \rightarrow \infty} T(t)p$ and $\lim_{t \rightarrow \infty} T(t)q \in K^*$,

$$\begin{aligned} \left\|p - \lim_{t \rightarrow \infty} T(t)q\right\| &= \left\|\lim_{t \rightarrow \infty} T(t)p - \lim_{t \rightarrow \infty} T(t)q\right\| \\ &= \lim_{t \rightarrow \infty} \|T(t)p - T(t)q\| \\ &\leq \lim_{t \rightarrow \infty} ((1 + u(t)) \|p - q\| + v(t)) \\ &\leq \|p - q\|. \end{aligned} \quad (3.28)$$

Therefore, $\lim_{t \rightarrow \infty} T(t)q = q$. Since $T(s + h)x = T(s)T(h)x$ for all $x \in E$ and $s \geq 0$, we have

$$\begin{aligned} q &= \lim_{t \rightarrow \infty} T(t)q = \lim_{t \rightarrow \infty} T(s + t)q = \lim_{t \rightarrow \infty} T(s)T(t)q \\ &= T(s) \lim_{t \rightarrow \infty} T(t)q = T(s)q. \end{aligned} \quad (3.29)$$

Therefore, $q \in F$ and so $K^* \cap F \neq \emptyset$.

Let $p \in K^* \cap F(T)$ and $\tau \in (0, 1)$. Then, it follows that $\phi(p) \leq \phi(p - \tau(G - \gamma f)p)$, and using Lemma 2.2, we obtain that

$$\|y_n - p + \tau(G - \gamma f)p\|^2 \leq \|y_n - p\|^2 + 2\tau \langle (G - \gamma f)p, j(y_n - p + \tau(G - \gamma f)p) \rangle, \quad (3.30)$$

which implies that

$$\mu_n \langle (\gamma f - G)p, j(y_n - p + \tau(G - \gamma f)p) \rangle \leq 0. \quad (3.31)$$

Moreover,

$$\begin{aligned}\mu_n \langle (\gamma f - G)p, j(y_n - p) \rangle &= \mu_n \langle (\gamma f - G)p, j(y_n - p) - j(y_n - p + \tau(G - \gamma f)p) \rangle \\ &\quad + \mu_n \langle (\gamma f - G)p, j(y_n - p + \tau(G - \gamma f)p) \rangle \\ &\leq \mu_n \langle (\gamma f - G)p, j(y_n - p) - j(y_n - p + \tau(G - \gamma f)p) \rangle.\end{aligned}\tag{3.32}$$

Since j is norm-to-weak* uniformly continuous on bounded subsets of E , we have that

$$\mu_n \langle (\gamma f - G)p, j(y_n - p) \rangle \leq 0.\tag{3.33}$$

Observe that from (3.14) and (3.15), we have

$$\lim_{n \rightarrow \infty} \|y_{n+1} - y_n\| = 0.\tag{3.34}$$

This implies that

$$\limsup_{n \rightarrow \infty} [\langle (\gamma f - G)p, j(y_n - p) \rangle - \langle (\gamma f - G)p, j(y_{n+1} - p) \rangle] \leq 0,\tag{3.35}$$

and so we obtain by Lemma 2.4 that

$$\limsup_{n \rightarrow \infty} \langle (\gamma f - G)p, j(y_n - p) \rangle \leq 0.\tag{3.36}$$

Finally, we show that $x_n \rightarrow p$ as $n \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} (u(t_n)/\alpha_n) = 0$, if we denote by $\sigma(t_n)$ the value $2u(t_n) + u(t_n)^2$, then we clearly have $\lim_{n \rightarrow \infty} (\sigma(t_n)/\alpha_n) = 0$. Let $N_0 \in \mathbb{N}$ be large enough such that $(1 - \alpha_n\eta)(\sigma(t_n)/\alpha_n) < (\eta - 2\gamma)/2$, for all $n \geq N_0$, and let M be

a positive real number such that $\|x_n - p\| \leq M$ for all $n \geq 0$. Then, using the recursion formula (3.2) and for $n \geq N_0$, we have

$$\begin{aligned}
 \|y_n - p\|^2 &= \|\alpha_n(\gamma f(x_n) - G(p)) + (I - \alpha_n G)(T(t_n)x_n - p)\|^2 \\
 &\leq (1 - \alpha_n \eta) \|T(t_n)x_n - p\|^2 + 2\alpha_n \langle \gamma f(x_n) - G(p), j(y_n - p) \rangle \\
 &\leq (1 - \alpha_n \eta) [(1 + u(t_n)) \|x_n - p\| + v(t_n)]^2 \\
 &\quad + 2\alpha_n \langle \gamma f(x_n) - \gamma f(p) + \gamma f(p) - G(p), j(y_n - p) \rangle \\
 &\leq (1 - \alpha_n \eta) \left[(1 + u(t_n))^2 \|x_n - p\|^2 + 2(1 + u(t_n))v(t_n) \|x_n - p\|^2 + v(t_n)^2 \right] \\
 &\quad + 2\alpha_n \langle \gamma f(p) - G(p), j(y_n - p) \rangle - 2\alpha_n \gamma \|y_n - p\| \psi(\|x_n - p\|) \\
 &\quad + 2\alpha_n \gamma \|(y_n - x_n) + (x_n - p)\| \|x_n - p\| \\
 &\leq [(1 - \alpha_n \eta)(1 + \sigma(t_n)) + 2\alpha_n \gamma] \|x_n - p\|^2 \\
 &\quad + \alpha_n \left[2 \langle \gamma f(p) - G(p), j(y_n - p) \rangle + 2(1 - \alpha_n \eta)(1 + u(t_n)) \frac{v(t_n)}{\alpha_n} \|x_n - p\|^2 \right. \\
 &\quad \left. + (1 - \alpha_n \eta) \frac{v(t_n)^2}{\alpha_n} + 2\gamma \|y_n - x_n\| \|x_n - p\| \right] \\
 &= \left[1 - \alpha_n \left((\eta - 2\gamma) - (1 - \alpha_n \eta) \frac{\sigma_n}{\alpha_n} \right) \right] \|x_n - p\|^2 \\
 &\quad + \alpha_n \left[2 \langle \gamma f(p) - G(p), j(y_n - p) \rangle + 2(1 - \alpha_n \eta)(1 + u(t_n)) \frac{v(t_n)}{\alpha_n} \|x_n - p\|^2 \right. \\
 &\quad \left. + (1 - \alpha_n \eta) \frac{v(t_n)^2}{\alpha_n} + 2\gamma \|y_n - x_n\| \|x_n - p\| \right],
 \end{aligned} \tag{3.37}$$

so that

$$\begin{aligned}
 \|x_{n+1} - p\|^2 &\leq \beta_n \|x_n - p\|^2 + (1 - \beta_n) \|y_n - p\|^2 \\
 &\leq \left(\beta_n + (1 - \beta_n) \left[1 - \alpha_n \left((\eta - 2\gamma) - (1 - \alpha_n \eta) \frac{\sigma_n}{\alpha_n} \right) \right] \right) \|x_n - p\|^2 \\
 &\quad + \alpha_n (1 - \beta_n) \left[2 \langle \gamma f(p) - G(p), j(y_n - p) \rangle + 2(1 - \alpha_n \eta)(1 + u(t_n)) \right. \\
 &\quad \left. \times \frac{v(t_n)}{\alpha_n} \|x_n - p\|^2 + (1 - \alpha_n \eta) \frac{v(t_n)^2}{\alpha_n} + 2\gamma \|y_n - x_n\| \|x_n - p\| \right]
 \end{aligned}$$

$$\begin{aligned}
&\leq \left[1 - \alpha_n(1 - \beta_n) \left((\eta - 2\gamma) - (1 - \alpha_n\eta) \frac{\sigma_n}{\alpha_n} \right) \right] \|x_n - p\|^2 \\
&\quad + \alpha_n(1 - \beta_n) \left[2\langle \gamma f(p) - G(p), j(y_n - p) \rangle + 2(1 - \alpha_n\eta)(1 + u(t_n)) \frac{v(t_n)}{\alpha_n} M^2 \right. \\
&\quad \quad \left. + (1 - \alpha_n\eta) \frac{v(t_n)^2}{\alpha_n} + 2\gamma \|y_n - x_n\| M \right] \\
&= \left[1 - \alpha_n(1 - \beta_n) \left((\eta - 2\gamma) - (1 - \alpha_n\eta) \frac{\sigma_n}{\alpha_n} \right) \right] \|x_n - p\|^2 \\
&\quad + \alpha_n(1 - \beta_n) \left((\eta - 2\gamma) - (1 - \alpha_n\eta) \frac{\sigma_n}{\alpha_n} \right) \\
&\quad \times \frac{\left[2\langle \gamma f(p) - G(p), j(y_n - p) \rangle + 2(1 - \alpha_n\eta)(1 + u(t_n)) \left(\frac{v(t_n)}{\alpha_n} \right) M^2 + \mathcal{A}_n \right]}{\left((\eta - 2\gamma) - (1 - \alpha_n\eta) \left(\frac{\sigma_n}{\alpha_n} \right) \right)},
\end{aligned} \tag{3.38}$$

where $\mathcal{A}_n$ denotes $(1 - \alpha_n\eta)(v(t_n)^2/\alpha_n) + 2\gamma\|y_n - x_n\|M$.

Observe that $\sum_{n=1}^{\infty} \alpha_n(1 - \beta_n)((\eta - 2\gamma) - (1 - \alpha_n\eta)(\sigma_n/\alpha_n)) = \infty$ and

$$\limsup_{n \rightarrow \infty} \left(\frac{2\langle \gamma f(p) - G(p), j(y_n - p) \rangle + 2(1 - \alpha_n\eta)(1 + u(t_n))(v(t_n)/\alpha_n) M^2 + \mathcal{A}_n}{((\eta - 2\gamma) - (1 - \alpha_n\eta)(\sigma_n/\alpha_n))} \right) \leq 0. \tag{3.39}$$

Applying Lemma 2.5, we obtain $\|x_n - p\| \rightarrow 0$ as $n \rightarrow \infty$. This completes the proof. $\square$

The following corollaries follow from Theorem 3.1.

Corollary 3.2. *Let E be a real uniformly convex and uniformly smooth Banach space, $\mathfrak{J} = \{T(t) : t \geq 0\}$, and let $F, f, G, \delta, \lambda, \eta, \gamma, \{\beta_n\}, \{\alpha_n\}, \{t_n\}$ and $\{x_n\}$ be as in Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which solves the variational inequality (3.3).*

Corollary 3.3. *Let $E = H$ be a real Hilbert space, and let $\mathfrak{J} = \{T(t) : t \geq 0\}$, $F, f, G, \delta, \lambda, \eta, \gamma, \{\beta_n\}, \{\alpha_n\}, \{t_n\}$ and $\{x_n\}$ be as in Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which solves the variational inequality*

$$\langle (G - \gamma f)q, x - q \rangle \geq 0, \quad \forall x \in F. \tag{3.40}$$

Corollary 3.4. *Let $\mathfrak{J} = \{T(t) : t \geq 0\}$ be a family of nonexpansive semigroup of a real reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm E , and let $F, f, G, \delta, \lambda, \eta, \gamma, \{\beta_n\}, \{\alpha_n\}, \{t_n\}$, and $\{x_n\}$ be as in Theorem 3.1. Then, the sequence $\{x_n\}$ converges strongly to a common fixed point of the family $\mathfrak{J}$ which solves the variational inequality (3.3).*

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