Research Article

Dimension Result for the Polynomial Algebra $\mathbb{F}_2[x_1, \dots, x_n]$ as a Module over the Steenrod Algebra

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For $n \geq 1$, let $\mathbf{P}(n) = \mathbb{F}_2[x_1, \dots, x_n]$ be the polynomial algebra in n variables x_i , of degree one, over the field $\mathbb{F}_2$ of two elements. The mod-2 Steenrod algebra $\mathcal{A}$ acts on $\mathbf{P}(n)$ according to well-known rules. Let $\mathcal{A}^+\mathbf{P}(n)$ denote the image of the action of the positively graded part of $\mathcal{A}$. A major problem is that of determining a basis for the quotient vector space $\mathbf{Q}(n) = \mathbf{P}(n)/\mathcal{A}^+\mathbf{P}(n)$. Both $\mathbf{P}(n) = \bigoplus_{d \geq 0} \mathbf{P}^d(n)$ and $\mathbf{Q}(n)$ are graded where $\mathbf{P}^d(n)$ denotes the set of homogeneous polynomials of degree d . A spike of degree d is a monomial of the form $x_1^{2^{\lambda_1}-1} \dots x_n^{2^{\lambda_n}-1}$ where $\lambda_i \geq 0$ for each i . In this paper we show that if $n \geq 2$ and $d \geq 1$ can be expressed in the form $d = d(\lambda) = (n-1)(2^\lambda - 1)$ with $\lambda \geq 2$, then $\dim(\mathbf{Q}^{d(\lambda)}(n)) \geq B(n, d(\lambda)) + \{\sum_{q=2}^{\lambda} \binom{n}{q}\}$, if $\lambda < n$; $2^n - (n+1)$, if $\lambda \geq n$, where $B(n, d(\lambda))$ is the number of spikes of degree $d(\lambda)$.

1. Introduction

For $n > 1$ let $\mathbf{P}(n)$ be the mod-2 cohomology group of the n -fold product of $\mathbb{R}\mathbf{P}^\infty$ with itself. Then $\mathbf{P}(n)$ is the polynomial algebra

$$\mathbf{P}(n) = \mathbb{F}_2[x_1, \dots, x_n] \quad (1)$$

in n variables x_i , each of degree 1. $\mathbf{P}(n)$ is a module over the mod-2 Steenrod algebra $\mathcal{A}$ according to well-known rules. A polynomial u is said to be *hit* if it belongs to the set

$$\mathcal{A}^+\mathbf{P}(n) = \left\{ \sum_{i>0} Sq^i u_i \mid u_i \in \mathbf{P}(n), Sq^i \in \mathcal{A} \right\}. \quad (2)$$

The problem of determining $\mathcal{A}^+\mathbf{P}(n)$ is called the hit problem and has been studied by several authors [1–3]. A closely related problem is that of determining a basis for the quotient vector space

$$\mathbf{Q}(n) = \frac{\mathbf{P}(n)}{\mathcal{A}^+\mathbf{P}(n)} \quad (3)$$

which has also been studied by several authors [4–9]. Some of the motivation for studying these problems is mentioned in

[6]. It stems from the Peterson conjecture proved in [3] and various other sources [10, 11].

The following result is useful for determining $\mathcal{A}$ -generators for $\mathbf{P}(n)$. Let $\alpha(m)$ denote the number of digits 1 in the binary expansion of m .

Theorem 1 (Wood [3]). *Let $u \in \mathbf{P}(n)$ be a monomial of degree d . If $\alpha(n+d) > n$ then u is hit.*

Thus $\mathbf{Q}^d(n)$ is zero unless $\alpha(n+d) \leq n$ or, equivalently, unless d can be written in the form $d = \sum_{i=1}^n (2^{\lambda_i} - 1)$, where $\lambda_i \geq 0$. Thus $\mathbf{Q}^d(n) \neq 0$ only if $\mathbf{P}^d(n)$ contains monomials

$$v = x_1^{2^{\lambda_1}-1} \dots x_n^{2^{\lambda_n}-1} \quad (4)$$

called spikes.

We note that a spike can never appear as a term in a hit polynomial.

$\mathbf{Q}(n)$ has been explicitly calculated by Peterson [7] for $n = 1, 2$, by Kameko [12] for $n = 3$, and independently by Kameko [5] and Sum [8] for $n = 4$.

2. Preliminaries

In this section we recall some results in Kameko [12] and Singer [2] on admissible monomials and hit monomials in $\mathbf{P}(n)$.

If $b = x_1^{e_1} \cdots x_n^{e_n}$ is a monomial in $\mathbf{P}(n)$, write $e_i = \sum_{j \geq 0} \alpha_j(e_i) 2^j$ for the binary expansion of each exponent e_i . The expansions are then assembled into a matrix

$$\beta(b) = (\alpha_j(e_i)) \quad (5)$$

of digits 0 or 1 with $\alpha_j(e_i)$ in the (i, j) th position of the matrix.

We will associate with a monomial b two sequences

$$w(b) = (w_0(b), w_1(b), \dots, w_j(b), \dots), \quad (6)$$

$$e(b) = (e_1, e_2, \dots, e_n),$$

where $w_j(b) = \sum_{i=1}^n \alpha_j(e_i)$ for each $j \geq 0$. $w(b)$ is called the *weight vector* of the monomial b and $e(b)$ is called the *exponent vector* of the monomial b . Note that $w_j(b) \leq n$ for all j . The monomial b is said to have length l if $w_l(b) \neq 0$ and $w_j(b) = 0$ for all $j > l$.

Given two sequences

$$p = (u_0, u_1, \dots, u_l, 0, 0, \dots), \quad q = (v_0, v_1, \dots, v_l, 0, 0, \dots), \quad (7)$$

we say that $p < q$ if there is a positive integer k such that $u_i = v_i$ for all $i < k$ and $u_k < v_k$. We are now in a position to define an order relation on monomials.

Definition 2. Let a, b be monomials in $\mathbf{P}(n)$. We say that $a < b$ if one of the following holds:

- (i) $w(a) < w(b)$,
- (ii) $w(a) = w(b)$ and $e(a) < e(b)$.

Note that the order relation on the set of sequences is the lexicographical one.

Following Kameko [4] we define the following.

Definition 3. A monomial $b \in \mathbf{P}(n)$ is said to be *inadmissible* if there exist monomials $b_1, b_2, \dots, b_r \in \mathbf{P}(n)$ with $b_j < b$ for each j , $1 \leq j \leq r$, such that

$$b \equiv \left(\sum_{j=1}^r b_j \right) \bmod \mathcal{A}^+ \mathbf{P}(n). \quad (8)$$

b is said to be *admissible* if it is not inadmissible.

Clearly the set of all admissible monomials in $\mathbf{P}(n)$ is a minimal set of $\mathcal{A}$ -generators of $\mathbf{P}(n)$. Clearly a spike is admissible.

We shall require the following result due to Kameko.

Definition 4. Let d be a positive integer. Define a linear mapping $h : \mathbf{P}^{2d+n}(n) \rightarrow \mathbf{P}^d(n)$ by

$$h(b) = \begin{cases} c & \text{if } b = x_1 x_2 \cdots x_n c^2 \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

for any monomial $b \in \mathbf{P}^{2d+n}(n)$.

Then h induces a homomorphism $h_* : \mathbf{Q}^{2d+n}(n) \rightarrow \mathbf{Q}^d(n)$.

Let $\beta(d) = \min\{m \in \mathbf{Z} \mid \alpha(m+d) \leq m\}$. In [5, Theorem 4.2] Kameko proved the following.

Theorem 5 (Kameko). Let d be a positive integer. If $\beta(2d+n) = n$, then $h_* : \mathbf{Q}^{2d+n}(n) \rightarrow \mathbf{Q}^d(n)$ is an isomorphism.

From Wood's theorem and the above result of Kameko the problem of determining $\mathcal{A}$ -generators for $\mathbf{P}(n)$ is reduced to the cases $\beta(d) < n$.

We recall the following result of Singer on hit polynomials in $\mathbf{P}(n)$.

Definition 6. A spike $v = x_1^{2^{\lambda_1-1}} \cdots x_n^{2^{\lambda_n-1}}$ is called a *minimal spike* if its weight order is minimal with respect to other spikes of degree d or, equivalently, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s \geq 0$ and $\lambda_{j-1} = \lambda_j$ only if $j = s$ or $\lambda_{j+1} = 0$.

In [11, Theorem 1.2] Singer proved the following.

Theorem 7 (Singer). Let $b \in \mathbf{P}(n)$ be a monomial of degree d , where $\alpha(n+d) \leq n$. Let v be the minimal spike of degree d . If $w(b) < w(v)$, then b is hit.

3. Main Result

In this section we state our main result, Theorem 8, which was obtained in [13]. The proof is deferred until Section 5. Theorem 8 is the basis of a more general result, obtained in [14], which we state at the conclusion of this paper.

Let $\lambda \geq 2$ be an integer and let $d(\lambda) = (n-1)(2^\lambda - 1)$. Let $B(n, d(\lambda))$ denote the number of spikes of degree $d(\lambda)$. In [13] it is shown that one has the following.

Theorem 8. If $n \geq 2$, then

$$\dim(\mathbf{Q}^{d(\lambda)}(n)) \geq B(n, d(\lambda)) + \begin{cases} \sum_{q=2}^{\lambda} \binom{n}{q} & \text{if } \lambda < n \\ 2^n - (n+1) & \text{if } \lambda \geq n. \end{cases} \quad (10)$$

A general formula for computing $B(n, d)$ for an arbitrary value of d can be found in [15].

The result is a consequence of the following lemma.

For each q , $2 \leq q \leq n$, and any given $\lambda \geq q$, let c_q be the monomial

$$c_q = x_1^{2^{q-1}-1} \cdots x_i^{(2^\lambda-2^{q-i})-1} \cdots x_q^{2^\lambda-2} x_{q+1}^{2^\lambda-1} \cdots x_n^{2^\lambda-1} \quad (11)$$

in $\mathbf{P}^{d(\lambda)}(n)$ and let

$$W(n, q) = \{\sigma \in S_n \mid \sigma(j) < \sigma(k) \text{ if } j < k \leq q \text{ and } \sigma(s) < \sigma(r) \text{ if } q < s < r\}. \quad (12)$$

Lemma 9. Let q, λ be integers such that $\lambda \geq q$ and let $c_q \in \mathbf{P}^{d(\lambda)}(n)$. Then for any given $\sigma \in W(n, q)$ the monomial

$$c_q(\sigma) = x_{\sigma(1)}^{2^{q-1}-1} \cdots x_{\sigma(i)}^{(2^\lambda-2^{q-i})-1} \cdots x_{\sigma(q)}^{2^\lambda-2} x_{\sigma(q+1)}^{2^\lambda-1} \cdots x_{\sigma(n)}^{2^\lambda-1} \quad (13)$$

is admissible.

The following example illustrates the definitions previously mentioned and typical monomials defined in the lemma.

Example 10. If $n = 3$, $\lambda = 3$, and $q = 2$, then $W(3, 2) = \{(1), (2\ 3), (1\ 2\ 3)\}$ and in matrix notation

$$c_2(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad c_2(2\ 3) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (14)$$

$$c_2(1\ 2\ 3) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

Note that

$$|W(n, q)| = \binom{n}{q} \quad (15)$$

and that if $n = q$ then the only element σ in $W(n, q)$ is the identity (13).

4. Proof of Lemma 9

We first show that if $q = n$, then $c_n = c_n(1)$ is admissible, that is, that

$$b = b(n) = c_n(1) = x_1^{2^{n-1}-1} x_2^{2^\lambda-2^{n-2}-1} \cdots x_i^{2^\lambda-2^{n-i}-1} \cdots x_n^{2^\lambda-2} \quad (16)$$

is admissible. We will suppress mentioning the fact that $b(n)$ is also a function of λ .

Suppose that $\lambda \geq n$ is given and let $m \in \mathbf{P}^{d(\lambda)}(n)$ be the spike

$$m = m(n) = x_1^0 x_2^{2^\lambda-1} \cdots x_j^{2^\lambda-1} \cdots x_n^{2^\lambda-1} \quad (17)$$

so that

$$w(m) = \left(\frac{\lambda}{n-1, n-1, \dots, n-1} \right). \quad (18)$$

Let $\mathbf{D}^{d(\lambda)}(n)$ denote the subspace of $\mathbf{P}^{d(\lambda)}(n)$ spanned by monomials $a = x_1^{t_1} \cdots x_n^{t_n}$ such that $w(a) = w(m)$. Further let $\mathbf{D}^{d(\lambda)}(n, b)$ denote the subspace of $\mathbf{D}^{d(\lambda)}(n)$ for which $t_i \neq 2^\lambda - 1$ for any i . Let

$$\pi : \mathbf{P}^{d(\lambda)}(n) \longrightarrow \mathbf{D}^{d(\lambda)}(n) \quad (19)$$

denote the projection of $\mathbf{P}^{d(\lambda)}(n)$ onto its summand $\mathbf{D}^{d(\lambda)}(n)$ and let

$$\pi_b : \mathbf{D}^{d(\lambda)}(n) \longrightarrow \mathbf{D}^{d(\lambda)}(n, b) \quad (20)$$

denote the projection of $\mathbf{D}^{d(\lambda)}(n)$ onto $\mathbf{D}^{d(\lambda)}(n, b)$. Let $\mathbf{H}^{d(\lambda)}(n, b)$ be the subspace of $\mathbf{D}^{d(\lambda)}(n, b)$ spanned by

$$\{b_1 + b_2 \mid b_1, b_2 \text{ monomials in } \mathbf{D}^{d(\lambda)}(n, b)\}. \quad (21)$$

We claim that

$$\pi_b \pi (\mathcal{A}^+ \mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n)) = \mathbf{H}^{d(\lambda)}(n, b). \quad (22)$$

Clearly this shall suffice to show that b is admissible (it being the monomial of least order in $\mathbf{D}^{d(\lambda)}(n, b)$).

Our proof of (22) is by induction on n and λ and splits into two cases depending on whether $\lambda = n$ or $\lambda > n$.

Case 1 ($\lambda = n$). In this case our argument is by induction on n , starting with the case $n = 2$. If $n = 2$ then (22) is easily seen to be true since

$$\pi_{b(2)} \pi (\mathcal{A}^+ \mathbf{P}(2) \cap \mathbf{P}^3(2)) \quad (23)$$

is spanned by $\{x_1^2 x_2 + x_1 x_2^2\}$.

Assume that (22) is true when the number of variables is $n - 1$, where n is fixed and $n - 1 \geq 2$.

Let $\tau = (1\ 2 \cdots n) \in S_n$ and for each j , $1 \leq j \leq n$, let $\mathbf{D}_j^{d(\lambda)}(n, b)$ be the vector subspace of $\mathbf{D}^{d(\lambda)}(n, b)$ generated by monomials

$$(x_1 \cdots \hat{x}_j \cdots x_n) (x_{\tau^j(1)}^{e_1} \cdots x_{\tau^j(n-1)}^{e_{n-1}} x_{\tau^j(n)}^{2^{\lambda-1}-1})^2, \quad (24)$$

where $x_1^{e_1} \cdots x_{n-1}^{e_{n-1}} \in \mathbf{D}^{d(\lambda-1)}(n-1, b(n-1))$ and $x_1 \cdots \hat{x}_j \cdots x_n$ is obtained from the monomial $x_1 \cdots x_j \cdots x_n$ by deleting the factor x_j . It is easy to see that $\mathbf{D}^{d(\lambda-1)}(n-1, b(n-1))$ is isomorphic to $\mathbf{D}_j^{d(\lambda)}(n, b)$ for each j . In fact the linear mapping

$$h_j : \mathbf{D}^{d(\lambda-1)}(n-1, b(n-1)) \longrightarrow \mathbf{D}_j^{d(\lambda)}(n, b) \quad (25)$$

given on monomials by

$$h_j(x_1^{e_1} \cdots x_{n-1}^{e_{n-1}}) = (x_1 \cdots \hat{x}_j \cdots x_n) (x_{\tau^j(1)}^{e_1} \cdots x_{\tau^j(n-1)}^{e_{n-1}} x_{\tau^j(n)}^{2^{\lambda-1}-1})^2 \quad (26)$$

is an isomorphism for each j . It is easy to check that

$$\mathbf{D}^{d(\lambda)}(n, b) = \bigoplus_{j=1}^n h_j(\mathbf{D}^{d(\lambda-1)}(n-1, b(n-1))) \quad (27)$$

and further that $\pi_{b(n)} \pi (\mathcal{A}^+ \mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n))$ is equal to

$$\pi_{b(n)} \pi (Sq^1(\mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n))) = \bigoplus_{j=1}^n h_j(\mathbf{H}^{d(\lambda-1)}(n-1, b(n-1))). \quad (28)$$

It is therefore sufficient to show that we can find monomials $u_j \in D_j^{d(\lambda)}(n)$ such that

$$u_1 + u_2, \dots, u_j + u_{j+1}, \dots, u_{n-1} + u_n \quad (29)$$

all belong to $\pi_{b(n)}\pi(\mathcal{A}^+\mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n))$.

We choose $u_j = h_j(b(n-1))$ for each j ; that is,

$$u_j = (x_1 \cdots \hat{x}_j \cdots x_n) \left(x_{\tau^j(1)}^{2^{n-2}-1} \cdots x_{\tau^j(n-1)}^{2^{\lambda-1}-2} x_{\tau^j(n)}^{2^{\lambda-1}-1} \right)^2. \quad (30)$$

We first illustrate the argument in the special case $n = \lambda = 4$. In this case $b(3) = x_1^3 x_2^5 x_3^6$, $u_1 = x_1^{14} x_2^7 x_3^{11} x_4^{13}$, $u_2 = x_1^{13} x_2^{14} x_3^7 x_4^{11}$, $u_3 = x_1^{11} x_2^{13} x_3^{14} x_4^7$, and $u_4 = x_1^7 x_2^{11} x_3^{13} x_4^{14} = b(4)$.

Consider $u_1 \in \mathbf{D}_1^{45}(4, b(4))$. Put

$$r_1 = x_1^{14} x_2^{13} x_3^7 x_4^{11} \in \mathbf{D}_1^{45}(4, b(4)). \quad (31)$$

By the induction hypothesis,

$$u_1 + r_1 \in \pi_{b(4)}\pi(\mathcal{A}^+\mathbf{P}(4) \cap \mathbf{P}^{45}(4)). \quad (32)$$

But $r_1 + u_2 \in \pi_{b(4)}\pi(\mathbf{Sq}^1(\mathbf{P}(4)) \cap \mathbf{P}^{45}(4))$. Put

$$r_2 = x_1^{11} x_2^{14} x_3^{13} x_4^7 \in \mathbf{D}_2^{45}(4, b(4)). \quad (33)$$

By the induction hypothesis,

$$u_2 + r_2 \in \pi_{b(4)}\pi(\mathcal{A}^+\mathbf{P}(4) \cap \mathbf{P}^{45}(4)). \quad (34)$$

But $r_2 + u_3 \in \pi_{b(4)}\pi(\mathbf{Sq}^1(\mathbf{P}(4)) \cap \mathbf{P}^{45}(4))$. Put

$$r_3 = x_1^7 x_2^{11} x_3^{14} x_4^{13} \in \mathbf{D}_3^{45}(4, b(4)). \quad (35)$$

By the induction hypothesis

$$u_3 + r_3 \in \pi_{b(4)}\pi(\mathcal{A}^+\mathbf{P}(4) \cap \mathbf{P}^{45}(4)). \quad (36)$$

But $r_3 + u_4 \in \pi_{b(4)}\pi(\mathbf{Sq}^1(\mathbf{P}(4)) \cap \mathbf{P}^{45}(4))$, which concludes the argument in this special case.

The argument to show that $u_j + u_{j+1} \in \pi_{b(n)}\pi(\mathcal{A}^+\mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n))$ in general proceeds as follows.

Consider the element

$$u_j = (x_1 \cdots \hat{x}_j \cdots x_n) \left(x_{\tau^j(1)}^{2^{n-2}-1} \cdots x_{\tau^j(n-1)}^{2^{\lambda-1}-2} x_{\tau^j(n)}^{2^{\lambda-1}-1} \right)^2 \quad (37)$$

in $\mathbf{D}_j^{d(\lambda)}(n, b)$. Let $r_j \in \mathbf{D}_j^{d(\lambda)}(n, b)$ be the monomial

$$r_j = (x_1 \cdots \hat{x}_j \cdots x_n) \times \left(x_{\tau^{j+1}(1)}^{2^{n-2}-1} \cdots x_{\tau^{j+1}(n-2)}^{2^{\lambda-1}-4} x_{\tau^{j+1}(n-1)}^{2^{\lambda-1}-2} x_{\tau^{j+1}(n)}^{2^{\lambda-1}-1} \right)^2. \quad (38)$$

By the induction hypothesis,

$$u_j + r_j \in \pi_{b(n)}\pi(\mathcal{A}^+\mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n)). \quad (39)$$

But $r_j + u_{j+1} \in \pi_{b(n)}\pi(\mathbf{Sq}^1(\mathbf{P}(n)) \cap \mathbf{P}^{d(\lambda)}(n))$ which shows that $u_j + u_{j+1} \in \pi_{b(n)}\pi(\mathcal{A}^+\mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n))$ as required.

Case 2 ($\lambda \geq n$). Our argument is by induction on both n and λ . In the case $n = 2 < \lambda$,

$$\pi_{b(2)}\pi(\mathcal{A}^+\mathbf{P}(2) \cap \mathbf{P}^{d(\lambda)}(2)) \quad (40)$$

is known to be spanned by $\mathbf{H}^{d(\lambda)}(2, b(2))$.

We have seen that (22) is true if $\lambda = n$. Assume therefore that (22) is true when the number of variables is $n-1$ where n is fixed and $n-1 \geq 2$ and for $\lambda-1$, where λ is fixed and $\lambda-1 \geq n$.

Let $\tau \in S_n$ be defined as above, let $\mathbf{D}_j^{d(\lambda)}(n, b)$ be the vector subspace of $\mathbf{D}^{d(\lambda)}(n, b)$ defined in the same manner as above, and let

$$h_j : \mathbf{D}^{d(\lambda-1)}(n-1, b(n-1)) \longrightarrow \mathbf{D}_j^{d(\lambda)}(n, b) \quad (41)$$

be the mapping given by (26).

For each i , $1 \leq i \leq n$, let

$$f_i : \mathbf{D}^{d(\lambda-1)}(n, b(n)) \longrightarrow \mathbf{D}^{d(\lambda)}(n, b(n)) \quad (42)$$

be the linear mapping given on monomials by

$$f_i(x_1^{e_1} \cdots x_n^{e_n}) = x_1^{2e_1+1} \cdots x_{i-1}^{2e_{i-1}+1} x_i^{2e_i} x_{i+1}^{2e_{i+1}+1} \cdots x_n^{2e_n+1}. \quad (43)$$

It is easy to check that

$$\mathbf{D}^{d(\lambda)}(n, b(n)) = \bigoplus_{j=1}^n h_j(\mathbf{D}^{d(\lambda-1)}(n-1, b(n-1))) \quad (44)$$

$$\bigoplus_{i=1}^n \bigoplus_{j=1}^n f_i(\mathbf{D}^{d(\lambda-1)}(n, b(n)))$$

and further that $\pi_{b(n)}\pi(\mathcal{A}^+\mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n))$ is equal to

$$\pi_{b(n)}\pi(\mathbf{Sq}^1(\mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n))) \bigoplus \left[\bigoplus_{j=1}^n h_j(\mathbf{H}^{d(\lambda-1)}(n-1, b(n-1))) \right] \bigoplus \left[\bigoplus_{i=1}^n f_i(\mathbf{H}^{d(\lambda-1)}(n, b(n))) \right]. \quad (45)$$

It is therefore sufficient to show that we can find monomials $u_j \in \mathbf{D}_j^{d(\lambda)}(n)$ and monomials $v_i \in f_i(\mathbf{D}^{d(\lambda-1)}(n, b(n)))$ such that

$$u_1 + u_2, \dots, u_j + u_{j+1}, \dots, u_{n-1} + u_n, \quad (46)$$

$$v_1 + v_2, \dots, v_i + v_{i+1}, \dots, v_{n-1} + v_n,$$

and $v_1 + u_1$ all belong to $\pi_{b(n)}\pi(\mathcal{A}^+\mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n))$. But this is a direct calculation, similar to the one for the case $\lambda = n$, which makes use of induction on n and λ and is omitted.

Finally we show that if $q < n$ and $\lambda \geq q$, then $c_q(\sigma)$ is admissible. Let $d(\lambda)$ denote $d(\lambda, q) = (q-1)(2^\lambda - 1)$. We first note that

$$u_q = c_q(1) = x_1^{2^{q-1}-1} x_2^{2^\lambda-2^{q-2}-1} \cdots x_i^{2^\lambda-2^{q-i}-1} \cdots x_q^{2^\lambda-2} \quad (47)$$

is an admissible monomial in $\mathbf{P}^{d(\lambda)}(q)$ (by Cases 1 and 2).

Now given that $\sigma \in W(n, q)$, let

$$g_q^\sigma : \mathbf{P}(q) \longrightarrow \mathbf{P}(n) \quad (48)$$

be the linear mapping given on monomials by

$$g_q^\sigma(x_1^{e_1} \cdots x_q^{e_q}) = x_{\sigma(1)}^{e_1} \cdots x_{\sigma(q)}^{e_q} x_{\sigma(q+1)}^{2^\lambda-1} \cdots x_{\sigma(j)}^{2^\lambda-1} \cdots x_{\sigma(n)}^{2^\lambda-1}. \quad (49)$$

Then for all $u \in \mathbf{P}(q)$ and $\theta \in \mathcal{A}^+$ for which $\theta(u) \in \mathbf{P}^{d(\lambda)}(q)$ we have

$$(g_q^\sigma(\pi_{u_q} \pi(\theta(u)))) = \pi_b \pi(\theta(g_q^\sigma(u))) \quad (50)$$

so that

$$\pi(\mathcal{A}^+ \mathbf{P}(n) \cap \mathbf{P}^{d(\lambda)}(n)) = \bigoplus_{\sigma, q} g_q^\sigma(\mathbf{H}^{d(\lambda)}(q, u_q)). \quad (51)$$

Since

$$\mathbf{D}^{d(\lambda)}(n) = \bigoplus_{\sigma, q} g_q^\sigma(\mathbf{D}^{d(\lambda)}(q, u_q)) \quad (52)$$

and each g_q^σ preserves the order of monomials, we see that $c_q(\sigma) = g_q^\sigma(u_q)$ is admissible for each pair σ, q . This completes the proof of the lemma.

5. Proof of Theorem 8

Let $\mathbf{C}^{d(\lambda)}(n)$ denote the subspace of $\mathbf{Q}^{d(\lambda)}(n)$ with basis the union of $B(n, d(\lambda))$ (the spikes of degree $d(\lambda)$) and the collection

$$\mathcal{B}_n(d(\lambda)) = \{c_q(\sigma) \mid 2 \leq q \leq n, \sigma \in W(n, q)\} \quad (53)$$

of monomials defined in Lemma 9. Then

$$\begin{aligned} \dim(\mathbf{Q}^{d(\lambda)}(n)) &\geq \dim(\mathbf{C}^{d(\lambda)}(n)) \\ &= B(n, d(\lambda)) + |\mathcal{B}_n(d(\lambda))| \\ &= B(n, d(\lambda)) + \sum_{q=2}^{\min\{\lambda, n\}} |W(n, q)| \\ &= B(n, d(\lambda)) + \begin{cases} \sum_{q=2}^{\lambda} \binom{n}{q} & \text{if } \lambda < n \\ 2^n - (n+1) & \text{if } \lambda \geq n. \end{cases} \end{aligned} \quad (54)$$

We note that if $\lambda \geq n$ and $\mathbf{C}_*(n)$ is the subspace of $\mathbf{C}(n)$ generated by $\mathcal{B}_n$, then the mapping

$$f : \mathbf{C}_*^{d(\lambda)}(n) \longrightarrow \mathbf{C}_*^{d(\lambda+1)}(n) \quad (55)$$

defined on monomials by $f(c_q(\sigma)) = c_q(\sigma)(x_{\sigma(2)} \cdots x_{\sigma(n)})^{2^\lambda}$ is an isomorphism between the vector subspaces $\mathbf{C}_*(n)$.

This completes the proof of Theorem 8.

Remark 11. The result of this paper in part fulfills the requirements of the following theorem, obtained in [14], whose proof needed a more detailed but separate treatment of Theorem 8.

Theorem 12. Suppose that $n \geq 2$ and $d > 0$ may be expressed in the form $d = \sum_{i=1}^{n-1} (2^{\lambda_i} - 1)$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_{n-2} \geq \lambda_{n-1} \geq 1$. Then

$$\dim(\mathbf{Q}^d(n)) \geq \left(\sum_{q=1}^{\min\{\lambda_{n-1}, n\}} \binom{n}{q} \right) (\dim(\mathbf{Q}^{d'}(n-1))), \quad (56)$$

where $d' = \sum_{i=1}^{n-1} (2^{\lambda_i - \lambda_{n-1}} - 1)$.

Disclosure

The result in Section 3 is based on work in the author's Ph.D. thesis [13] authored under the supervision of Dr. R. M. W. Wood.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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