Research Article
Generalized Derivations on Prime Near Rings

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1. Introduction

Throughout the paper, N denotes a zero-symmetric left near ring with multiplicative center Z, and for any pair of elements x, y ∈ N, [x, y] denotes the commutator xy − yx, while the symbol (x, y) denotes the additive commutator x + y − x − y. A near ring N is called zero-symmetric if 0x = 0, for all x ∈ N (recall that left distributivity yields that x0 = 0). The near ring N is said to be 3-prime if xNx = 0 for x, y ∈ N implies that x = 0 or y = 0. A near ring N is called 2-torsion-free if (N, +) has no element of order 2. A nonempty subset A of N is called a semigroup right (resp., semigroup left) ideal if AN ⊆ A (resp., NA ⊆ A), and if A is both a semigroup right ideal and a semigroup left ideal, it is called a semigroup ideal. An additive mapping f : N → N is said to be a right (resp., left) generalized derivation with associated derivation d on N if f(xy) = f(x)y + xd(y) (resp., f(xy) = d(x)y + xf(y)), for all x, y ∈ N, and f is said to be a generalized derivation with associated derivation d on N if it is both a right generalized derivation and a left generalized derivation on N with associated derivation d. (Note that this definition differs from the one given by Hvala in [1]; his generalized derivations are our right generalized derivations.)

Every derivation on N is a generalized derivation.

In the case of rings, generalized derivations have received significant attention in recent years. We prove some theorems in the setting of a semigroup ideal of a 3-prime near ring admitting a generalized derivation, thereby extending some known results on derivations.

2. Preliminary Results

We begin with several lemmas, most of which have been proved elsewhere.

Lemma 1 (see [3, Lemma 1.3]). Let N be a 3-prime near ring and d be a nonzero derivation on N. Then

(i) If U is a nonzero semigroup right ideal or a nonzero semigroup left ideal of N, then d(U) ≠ {0}.

(ii) If U is a nonzero semigroup right ideal of N and x is an element of N which centralizes U, then x ∈ Z.

Lemma 2 (see [3, Lemma 1.2]). Let N be a 3-prime near ring.

(i) If z ∈ Z \ {0}, then z is not a zero divisor.

(ii) If Z \ {0} contains an element z for which z + z ∈ Z, then (N, +) is abelian.

(iii) If z ∈ Z \ {0} and x is an element of N such that xz ∈ Z, then x ∈ Z.
Lemma 3 (see [3, Lemmas 1.3 and 1.4]). Let $N$ be a 3-prime near ring and $U$ be a nonzero semigroup ideal of $N$. Let $d$ be a nonzero derivation on $N$.

(i) If $x, y \in N$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.

(ii) If $x \in N$ and $xU = \{0\}$ or $Ux = \{0\}$, then $x = 0$.

(iii) If $x \in N$ and $d(U)x = \{0\}$ or $xd(U) = \{0\}$, then $x = 0$.

Lemma 4 (see [3, Lemma 1.5]). If $N$ is a 3-prime near ring and $Z$ contains a nonzero semigroup left ideal or a semigroup right ideal, then $N$ is a commutative ring.

Lemma 5. If $f$ is a generalized derivation on $N$ with associated derivation $d$, then $(d(x)y + xf(y))z = d(x)yz + xf(y)z$, for all $x, y, z \in N$.

Proof. We prove only (ii), since (i) is proved in [2]. For all $x, y, z \in N$ we have $f(xy)z = f(xy)z = (d(x)y + xf(y))z = (d(x)yz + xf(y)z)$, for all $x, y, z \in N$.

(ii) If $x \in N$ and $f(U)x = \{0\}$, then $x = 0$.

Lemma 6. Let $N$ be a 3-prime near ring and $f$ a generalized derivation with associated derivation $d$.

(i) $f(x)y + xd(x) = yd(x) + f(x)y$ for all $x, y \in N$.

(ii) $d(x)y + xf(y) = f(x)y + d(x)y$ for all $x, y \in N$.

Proof. (i) $f(x)(y + y)) = f(x)(y + y) + xd(x) = (f(x)y + f(x)y + xd(x) + yd(x))$, for all $x, y \in N$.

(ii) Again, calculate $f(x(y + y))$ and compare.

Lemma 7. Let $N$ be a 3-prime near ring and $f$ a generalized derivation with associated derivation $d$. Then $f(Z) \subseteq Z$.

Proof. Let $z \in Z$ and $x \in N$. Then $f(zx) = f(zx)$; that is, $f(z)x + zd(x) = d(z)x + xf(z)$. Applying Lemma 6(i), we get $zd(x) + f(z)x = d(z)x + xf(z)$. It follows that $f(z)x = xf(z)$ for all $x \in N$, so $f(z) \in Z$.

Lemma 8. Let $N$ be a 3-prime near ring and $U$ a nonzero semigroup ideal of $N$. If $f$ is a nonzero right generalized derivation of $N$ with associated derivation $d$, then $f(U) \neq \{0\}$.

Proof. Suppose $f(U) = \{0\}$. Then $f(ux) = f(u)x + ud(x) = 0 = ud(x)$ for all $u \in U$ and $x \in N$, and it follows by Lemma 3(ii) that $d = 0$. Therefore $f(ux) = f(u)x = 0$ for all $u \in U$ and $x \in N$, and another appeal to Lemma 3(ii) gives $f = 0$, which is a contradiction.

Lemma 9 (see [2, Theorem 2.1]). Let $N$ be a 3-prime near ring with a nonzero right generalized derivation $f$ with associated derivation $d$. If $f(N) \subseteq Z$ then $(N, +)$ is abelian. Moreover, if $N$ is 2-torsion-free, then $N$ is a commutative ring.

Lemma 10 (see [2, Theorem 4.1]). Let $N$ be a 2-torsion-free 3-prime near ring and $f$ a nonzero generalized derivation on $N$ with associated derivation $d$. If $f(N)$, $(N) = \{0\}$, then $N$ is a commutative ring.

3. Main Results

The theorems that we prove in this section extend the results proved in [2, Theorems 2.1 and 3.1], [3, Theorems 2.1, 3.1, and 3.3], and [5, Theorem 3.3].

Theorem 11. Let $N$ be a 3-prime near ring and $U$ be a nonzero semigroup ideal of $N$. Let $f$ be a nonzero right generalized derivation with associated derivation $d$. If $f(U) \subseteq Z$, then $(N, +)$ is abelian. Moreover, if $N$ is 2-torsion-free, then $N$ is a commutative ring.

Proof. We begin by showing that $(N, +)$ is abelian, which by Lemma 2(ii) is accomplished by producing $z \in Z \setminus \{0\}$ such that $z + z \in Z$. Let $a$ be an element of $U$ such that $f(a) \neq 0$. Then for all $x \in N$, $ax \in U$ and $ax + ax = a(x + x) \in U$, so that $f(ax) \in Z$ and $f(ax) + f(ax) \in U$; hence we need only to show that there exists $x \in N$ such that $f(ax) \neq 0$. Suppose that this is not the case, so that $f(ax) = 0 = f(ax) + axd(a) = axd(a)$, for all $x \in N$. By Lemma 3(i) either $a = 0$ or $d(a) = 0$.

If $d(a) = 0$, then $f(ax) = f(ax) + axd(a)$; that is, $f(ax) = f(ax) + f(a)x$, for all $x \in N$. Thus $f(ax) = f(ax) + f(ax) = f(ax) + f(ax) = f(ax)$, for all $x \in N$, and $u \in U$. This implies that $f(u)a, y \neq 0$, for all $u \in U$ and $y \in N$ and Lemma 2(i) gives $a \in Z$. Thus $f(ax) = f(ax) + f(ax)$, for all $ax \in N$. Replacing $x$ by $u \in U$, we have $f(U) = \{0\}$, and by Lemma 2(i) and 8, we get that $a = 0$. Thus, we have a contradiction.

To complete the proof, we show that if $N$ is 2-torsion-free, then $N$ is commutative.

Consider first the case $d = 0$. This implies that $f(u)x = f(u)x \in Z$ for all $u \in U$ and $x \in N$. By Lemma 8, we have $u \in U$ such that $f(u) \in Z \setminus \{0\}$, so $N$ is commutative by Lemma 2(ii).

Now consider the case $d \neq 0$. Let $c \in Z \setminus \{0\}$. This implies that if $x \in U$, $f(x) = f(x)c + xd(c)$, for all $x \in U$ and $c \in Z$. Therefore by Lemma 5(i), $f(x)c + xd(c) = yf(x)c + yd(c)$, for all $x \in U$ and $c \in Z$. Since $d(c) \in Z$ and $f(x) \in Z$, we obtain $d(c) = 0$, for all $x \in U$ and $c \in Z$. Let $d(Z) \neq 0$. Choosing $c$ such that $d(c) \neq 0$ and noting that $d(c)$ is not a zero divisor, we have $[x, y] = 0$, for all $x, y \in U$. By Lemma 1(ii), $U \subseteq Z$; hence $N$ is commutative by Lemma 4.

The remaining case is $d \neq 0$ and $d(Z) = \{0\}$. Suppose we can show that $U \subseteq Z$. Taking $z \in U \setminus Z \setminus \{0\}$ and $x \in N$, we have $f(xz) = f(zx) \in Z$; therefore $f(N) \subseteq Z$ by Lemma 2(iii) and $N$ is commutative by Lemma 9.

Assume, then, that $U \subseteq Z$ and $f(u) = f(u)u + ud(u) = uf(u) + d(u)$, for all $u \in U$, so $f(u) = 0$. Since $f(x) = f(x) + xd(u) \in Z$ for all $u \in U$ and $x \in N$, we have $f(x) + xd(u) = uf(x) + xd(u)$, and right multiplying by $u$ gives $uxd(u) = 0$. Consequently $d(u)ud(u) = 0$, so that $d(u) = 0$ for all $u \in U$. Since $u^2 = 0$, $d(u^2) = d(u)u + ud(u) = 0$ for all $u \in U$, so
Let \( N \) be a 3-prime near ring with a nonzero generalized derivation \( f \) with associated nonzero derivation \( d \). Let \( U \) be a nonzero semigroup ideal of \( N \). If \( [f(U), f(U)] = 0 \), then \( (N, +) \) is abelian.

**Proof.** Assume that \( x \in N \) is such that \([x, f(U)] = 0\) and \( U \subseteq Z \). Let \( d(x) = d(0) = 0 \) for all \( x \in U \). Then \( [f(U), f(U)] = 0 \) and by Lemma 1(ii), \( f(U) \subseteq Z \). Our result now follows by Theorem 11.

We have already observed that if \( f \) is a generalized derivation with \( d = 0 \), then \( f(x) = xf(y) \) for all \( x, y \in N \). For 3-prime near rings, we have the following converse.

**Theorem 13.** Let \( N \) be a 2-torsion-free 3-prime near ring and \( U \) be a nonzero semigroup ideal of \( N \). If \( f \) is a nonzero generalized derivation with associated derivation \( d \) such that \([f(U), f(U)] = 0\), then \( N \) is a commutative ring if it satisfies one of the following: (i) \( d(Z) \neq \{0\} \); (ii) \( U \cap Z \neq \{0\} \); (iii) \( d = 0 \) and \( f(Z) \neq \{0\} \).

**Proof.** (i) Let \( a \in N \) centralizes \( f(U) \), and let \( z \in Z \) such that \( d(z) \neq 0 \). Then a centralizes \( f(uz) \) for all \( u \in U \), so that \( d(f(uz) + ud(z)a) = d(f(uz) + ud(z)a) \). Since \( d(z) \neq 0 \) and \( [u, a] = 0 \) for all \( u \in U \). Therefore a centralizes \( U \), and by Lemma 1(ii), \( a \in Z \). Since \( f(U) \) centralizes \( f(U) \), \( f(U) \subseteq Z \) and our result follows by Theorem 11.

(ii) We may assume \( d(Z) = \{0\} \). Let \( z \in (U \cap Z) \). For all \( x, y \in N \), \( f(xy) = f(x)z \) and \( f(yz) = f(y) \) commute; hence \( z^2 [f(x), f(y)] = 0 = [f(x), f(y)] \). Our result now follows from Lemma 10.

(iii) Let \( u, v \in U \) and \( z \in Z \) such that \( f(z) \neq 0 \). Then \( f(xz), f(u) = 0 = [f(x)u, f(v)] \), and since \( f(z) \in Z \), \( f(u) = 0 \). Thus \( f(U) \) centralizes \( U \), and by Lemma 1(ii), \( f(U) \subseteq Z \). Our result now follows by Theorem 11.

4. **Generalized Derivations: Acting as a Homomorphism or an Antihomomorphism**

In [4], Bell and Kappe proved that if \( R \) is a semiprime ring and \( d \) is a derivation on \( R \) which is either an endomorphism or an antiderivation on \( R \), then \( d = 0 \). Of course, derivations which are not endomorphisms or antiderivations on \( R \) may behave as such on certain subsets of \( R \); for example, any derivation \( d \) behaves as the zero endomorphism on the subring \( C \) consisting of all constants (i.e., the elements \( x \) for which \( d(x) = 0 \)). In fact in a semiprime ring \( R \), \( d \) may behave as an endomorphism on a proper ideal of \( R \). However as noted in [4], the behaviour of \( d \) is somewhat restricted in the case of a prime ring. Recently the authors in [6] considered \( (\theta, \phi) \)-derivation \( d \) acting as a homomorphism or an antiderivation on a nonzero Lie ideal of a prime ring and concluded that \( d = 0 \). In this section we establish similar results in the setting of a semigroup ideal of a 3-prime near ring admitting a generalized derivation.
homomorphism on $U$, then $f$ is the identity map on $N$ and $d = 0$.

Proof. By the hypothesis
\[
 f(xy) = d(x)y + xf(y) = f(x)f(y) \quad \forall x, y \in U. \tag{4}
\]
Replacing $y$ by $yz$ in the above relation, we get
\[
 f(xyz) = d(x)yz + x(d(y)z + yf(z)) \quad \forall x, y, z \in U, \tag{5}
\]
or
\[
 f(xy)f(z) = d(x)yz + x(d(y)z + yf(z)) \quad \forall x, y, z \in U. \tag{6}
\]
This implies that
\[
 (d(x)y + xf(y))f(z) = d(x)yz + x(d(y)z + yf(z)) \quad \forall x, y, z \in U. \tag{7}
\]
Using Lemma 5(ii), we get
\[
 d(x)yz + xd(y)z + xyf(z) \quad \forall x, y, z \in U, \tag{8}
\]
or
\[
 d(x)yz + xd(y)z + xyf(z) \quad \forall x, y, z \in U. \tag{9}
\]
This implies that
\[
 d(x)yz + xd(y)z + xyf(z) = d(x)yz \quad \forall x, y, z \in U. \tag{10}
\]
That is,
\[
 d(x)yz = d(x)yz \quad \forall x, y, z \in U. \tag{11}
\]
Therefore
\[
 d(x)y(f(z) - z) = 0 \quad \forall x, y, z \in U, \tag{12}
\]
which implies that
\[
 d(x)U(f(z) - z) = \{0\} \quad \forall x, z \in U. \tag{13}
\]
It follows by Lemma 3(i) that either $d(U) = 0$ or $f(z) = z$ for all $z \in U$.

In fact, as we now show, both of these conditions hold.

Suppose that $f(u) = u$ for all $u \in U$. Then for all $u \in U$ and $x \in N$, $f(xu) = xu = d(x)u + xf(u) = d(x)u + xu$; hence $d(x)U = \{0\}$ for all $x \in N$, and thus $d = 0$.

On the other hand, suppose that $d(U) = \{0\}$, so that $d = 0$.

Then for all $x, y \in U$, $f(xy) = f(x)y = f(x)f(y)$, so that $f(x)(y - f(y)) = 0$. Replacing $y$ by $z$, $z \in N$, and noting that $f(zy) = zf(y)$, we see that $f(x)N(y - f(y)) = \{0\}$ for all $x, y \in U$. Therefore, $f(U) = \{0\}$ or $f$ is the identity on $U$. But $f(U) = \{0\}$ contradicts Lemma 8, so $f$ is the identity on $U$.

We now know that $f$ is the identity on $U$ and $f(xy) = xf(y)$ for all $x, y \in N$. Consequently, $f(uu) = uu = uf(x)$ for all $u \in U$ and $x \in N$, so that $U(x - f(x)) = \{0\}$ for all $x \in N$. It follows that $f$ is the identity on $N$.

\[\Box\]

**Theorem 16.** Let $N$ be a 3-prime near ring and $U$ be a nonzero semigroup ideal of $N$. If $f$ is a nonzero generalized derivation on $N$ with associated derivation $d$. If $f$ acts as an antihomomorphism on $U$, then $d = 0$, $f$ is the identity map on $N$, and $N$ is a commutative ring.

Proof. We begin by showing that $d = 0$ if and only if $f$ is the identity map on $N$.

Clearly if $f$ is the identity map on $N$, $xd(y) = 0$ for all $x, y \in N$, and hence $d = 0$.

Conversely, assume that $d = 0$, in which case $f(xy) = f(x)y = xf(y)$ for all $x, y \in N$. It follows that for any $x, y, z \in U$,
\[
 f(yxz) = f(z)f(yx) = f(z)yf(x) = zf(y)f(x) = zf(xy). \tag{14}
\]
On the other hand,
\[
 f(yxz) = f(xz)f(y) = f(x)zf(y) = f(x)f(zy) = f(y)f(xz) = f(y)f(x)z = f(zy)f(x)z = f(zy)f(x)z = zf(xy). \tag{15}
\]
Comparing (14) and (15) shows that $f(U^2)$ centralizes $U$, so that $f(U^2) \subseteq Z$ by Lemma 1(ii).

Now $U^2$ is a nonzero semigroup ideal by Lemma 3(iii); hence $f(U^2) \neq 0$ by Lemma 8. Choosing $x, y \in U$ such that $f(xy) \neq 0$, we see that for any $z \in U$, $f(xy)z = f(xy)z = f(zy)f(xz) = f(y)f(xz) = f(y)f(x)f(z) = f(xy)f(z)$, and hence $f(xy)(z - f(z)) = 0$. Since $f(xy) \in Z \setminus \{0\}$, we conclude that $f(z) = z$ for all $z \in U$, and it follows easily that $f$ is the identity map on $N$.

We note now that if the identity map on $N$ acts as an antihomomorphism on $U$, then $U$ is commutative, so that by Lemmas 1(ii) and 4 $N$ is a commutative ring.

To complete the proof of our theorem, we need only to argue that $d = 0$. By our antihomomorphism hypothesis
\[
 f(xy) = d(x)y + xf(y) = f(y)f(x) \quad \forall x, y \in U. \tag{16}
\]
Replacing $y$ by $xy$ in the above relation, we get
\[
 f(xy)f(x) = f(xxy) = d(x)xy + xf(xy) \quad \forall x, y \in U. \tag{17}
\]
This implies that
\[
 (d(x)y + xf(y))f(x) = d(x)xy + xf(xy) \quad \forall x, y \in U. \tag{18}
\]
Using Lemma 5(ii), we get
\[ d(x) y f(x) + x f(y) f(x) = d(x) x y + x f(y) f(x) \quad \forall x, y \in U. \] (19)

Thus
\[ d(x) y f(x) = d(x) x y \quad \forall x, y \in U. \] (20)

Replacing \( y \) by \( yr \) in (20) and using (20), we get
\[ d(x) yr f(x) = d(x) x yr, \] and so
\[ d(x) [r, f(x)] = 0 \quad \forall x, y \in U, r \in N. \] (21)

Application of Lemma 3(i) yields that for each \( x \in U \) either \( d(x) = 0 \) or \([r, f(x)] = 0\); that is \( d(x) = 0 \) or \( f(x) \in Z \).

Suppose that there exists \( w \in U \) such that \( f(w) \in Z \setminus \{0\} \).

Then for all \( v \in U \) such that \( d(v) = 0 \), \( f(wv) = f(w)v = f(v) f(w) = f(w) f(v) \), and hence \( f(w)(v - f(v)) = 0 \) = \( v - f(v) \). Now consider arbitrary \( x, y \in U \). If one of \( f(x), f(y) \) is in \( Z \), then \( f(xy) = f(x)f(y) \). If \( d(x) = 0 = d(y) \), then
\[ d(xy) = d(x)y + xd(y) = 0, \] so \( f(xy) = xy = f(x)f(y) \). Therefore \( f(xy) = f(x)f(y) \) for all \( x, y \in U \), and by Theorem 15, \( f \) is the identity map on \( N \), and therefore \( d = 0 \).

The remaining possibility is that for each \( x \in U \), either \( d(x) = 0 \) or \( f(x) = 0 \). Let \( u \in U \setminus \{0\} \), and let \( U_1 = uN \). Then \( U_1 \) is a nonzero semigroup right ideal contained in \( U \) and \( U_1 \) is an additive subgroup of \( N \). The sets \( \{x \in U_1 | d(x) = 0\} \) and \( \{x \in U_1 | f(x) = 0\} \) are additive subgroups of \( U_1 \) with union equal to \( U_1 \), so \( d(U_1) = \{0\} \) or \( f(U_1) = \{0\} \).\( d(U_1) = \{0\} \), then \( d = 0 \) by Lemma 1(i). Suppose, then, that \( f(U_1) = \{0\} \).

Then for arbitrary \( x, y \in N \) \( f(uxy) = f(ux)y + uxd(y) = 0 = uxd(y) \), so \( uNf(y) = \{0\} \), and again \( d = 0 \). This completes the proof. \( \square \)

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