Research Article

Multiresolution Expansion and Approximation Order of Generalized Tempered Distributions

Byung Keun Sohn

Department of Mathematics, Inje University, Kimhae 621-749, Republic of Korea

Correspondence should be addressed to Byung Keun Sohn; mathsohn@inje.ac.kr

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Let \( \mathcal{K}_M' \) be the generalized tempered distributions of \( e^{M(x)} \)-growth with restricted order \( r \in \mathbb{N}_0 \), where the function \( M(x) \) grows faster than any linear functions as \( |x| \to \infty \). We show the convergence of multiresolution expansions of \( \mathcal{K}_M' \) in the test function space \( \mathcal{K}_M' \). In addition, we show that the kernel of an integral operator \( K : \mathcal{K}_M' \to \mathcal{K}_M' \) provides approximation order in \( \mathcal{K}_M' \) in the context of shift-invariant spaces [4]. In this paper, we will show the uniform convergence on compact sets of the derivatives of multiresolution expansions of \( \mathcal{K}_M' \) and convergence of multiresolution expansions of \( \mathcal{K}_M' \) in the test function space \( \mathcal{K}_M' \). This is an extension of the works of Pilipović and Teofanov [4] in the context of generalized tempered distributions, \( \mathcal{K}_M \).

1. Introduction

Multiresolution analysis was shown to be very useful in extending the expansions in orthogonal wavelets from \( L^2(\mathbb{R}) \) to a certain class of tempered distributions. Some interactions between wavelets and tempered distributions have been presented by Walter’s work in [1–3]. Walter has found the analytic representation of tempered distributions of polynomial growth with restricted order, \( \mathcal{S}_r(\mathbb{R}) \), \( r \in \mathbb{N}_0 \), by wavelets [1] and the multiresolution expansions’ pointwise convergence of \( \mathcal{S}_r(\mathbb{R}) \) [3]. Pilipović and Teofanov have showed the uniform convergence on compact sets of the derivatives of multiresolution expansions of \( \mathcal{S}_r(\mathbb{R}) \) and the convergence of multiresolution expansions of \( \mathcal{S}_r(\mathbb{R}) \) in the test function space \( \mathcal{S}_r(\mathbb{R}) \) of \( \mathcal{S}_r(\mathbb{R}) \). As an application, Pilipović and Teofanov have shown that the kernel of an integral operator \( K : \mathcal{S}_r(\mathbb{R}) \to \mathcal{S}_r(\mathbb{R}) \) provides approximation order in \( \mathcal{S}_r(\mathbb{R}) \) in the context of shift-invariant spaces [4].

In the meantime, the tempered distributions of polynomial growth were extended to tempered distributions of \( e^{M(x)} \)-growth, \( \mathcal{K}_M(\mathbb{R}) \), in [5, 6] and \( e^{pM(x)} \)-growth, \( \mathcal{K}_p(\mathbb{R}) \), in [7, 8] or \( e^{M(x)} \)-growth, \( \mathcal{K}_M(\mathbb{R}) \), in [9, 10], where the function \( M(x) \) grows faster than any linear functions as \( |x| \to \infty \). We have considered the analytic representation of tempered distributions of \( e^{M(x)} \)-growth with restricted order, \( \mathcal{K}_M(\mathbb{R}) \), by wavelets [11]. Also, we have shown that the multiresolution expansions of \( \mathcal{K}_M(\mathbb{R}) \) converges pointwise to the value of the distribution where it exists [12].

In this paper, we will show the uniform convergence on compact sets of the derivatives of multiresolution expansions of \( \mathcal{K}_M(\mathbb{R}) \) and convergence of multiresolution expansions of \( \mathcal{K}_M(\mathbb{R}) \) in the test function space \( \mathcal{K}_M(\mathbb{R}) \). In addition, we will show that the kernel of an integral operator \( K : \mathcal{K}_M(\mathbb{R}) \to \mathcal{K}_M(\mathbb{R}) \) provides approximation order in \( \mathcal{K}_M(\mathbb{R}) \). This is an extension of the works of Pilipović and Teofanov [4] in the context of generalized tempered distributions, \( \mathcal{K}_M(\mathbb{R}) \).

2. The Generalized Tempered Distribution Spaces \( \mathcal{K}_M(\mathbb{R}) \)

Throughout this paper, we will use \( C \) or \( C_i \) to denote the positive constants, which are independent parameters and may be different at each occurrence.

Let \( \mu(\xi) \) (\( 0 \leq \xi \leq \infty \)) denote a continuous increasing function such that \( \mu(0) = 0 \) and \( \mu(\infty) = \infty \). For \( x \geq 0 \), we define

\[
M(x) = \int_0^x \mu(\xi) \, d\xi. \tag{1}
\]

The function \( M(x) \) is an increasing, convex, and continuous function with \( M(0) = 0, M(\infty) = \infty \) and satisfies the...
fundamental convexity inequality \(M(x_1) + M(x_2) \leq M(x_1 + x_2)\). Further, we define \(M(x)\) for negative \(x\) by means of the equality \(M(x) = M(-x)\). Note that since the derivative \(\mu(x)\) of \(M(x)\) is unbounded in \(\mathbb{R}\), the function \(M(x)\) will grow faster than any linear function as \(|x| \to \infty\). Now we list some properties of \(M(x)\) which will be frequently used later. Consider the following:

\[
\begin{align*}
M(x) + M(y) & \leq M(x + y) \quad \forall x, y \geq 0, \\
M(x + y) & \leq M(2x) + M(2y) \quad \forall x, y \geq 0.
\end{align*}
\]

(2)

Using the function \(M(x)\), we define the space \(\mathbb{H}_M(\mathbb{R})\) as the space of all functions \(\varphi \in C^\infty(\mathbb{R})\) such that

\[
\|\varphi\|_{\mathbb{H}_M} = \sup_{x \in \mathbb{R}, \alpha \leq k} e^{M(k\alpha)} \left| \frac{d^\alpha}{dx^\alpha} \varphi(x) \right| < \infty, \quad k = 1, 2, \ldots.
\]

(3)

The topology in \(\mathbb{H}_M(\mathbb{R})\) is defined by the family of the seminorms \(\| \cdot \|_{\mathbb{H}_M}\). Then \(\mathbb{H}_M(\mathbb{R})\) become a Fréchet space and \(\mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{H}_M(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{H}_M(\mathbb{R})\) are continuous and dense inclusions; here \(\mathbb{S}(\mathbb{R})\) denotes the spaces of all \(C^\infty(\mathbb{R})\) functions with compact supports, \(\mathbb{S}(\mathbb{R})\) the spaces of polynomially decreasing functions (Schwartz functions), and \(\mathbb{H}_M(\mathbb{R})\) the space of all \(C^\infty(\mathbb{R})\) functions. By \(\mathbb{H}_M'(\mathbb{R})\), we mean the space of continuous linear functionals on \(\mathbb{H}_M(\mathbb{R})\).

**Definition 1.** We say that the elements of \(\mathbb{H}_M'(\mathbb{R})\) are generalized tempered distributions.

Clearly, when \(M(x) = \log(1 + |x|)\), \(\mathbb{H}_M'(\mathbb{R})\) are tempered distributions (Schwarz distributions), \(\mathbb{S}(\mathbb{R})\). When \(M(x) = |x|\), \(\mathbb{H}_M'(\mathbb{R})\) are tempered distributions, \(\mathbb{S}(\mathbb{R})\), which are introduced and characterized by Yoshinaga [6] and Hasumi [5], independently. When \(M(x) = |x|^p\), \(p > 1\), \(\mathbb{H}_M'(\mathbb{R})\) are tempered distributions, \(\mathbb{S}(\mathbb{R})\), which are introduced and characterized by Sznajder and Zielenszy [7, 8]. For details about \(\mathbb{H}_M'(\mathbb{R})\), we refer to [9, 10].

For a natural number \(r\), we define \(\mathbb{H}_M^r(\mathbb{R})\) the space of all \(\varphi \in C^r(\mathbb{R})\) such that

\[
\|\varphi\|_{\mathbb{H}_M^r} = \sup_{x \in \mathbb{R}, \alpha \leq kr} e^{M(kr)} \left| \frac{d^\alpha}{dx^\alpha} \varphi(x) \right| < \infty,
\]

\[
\lim_{x \to \infty} \sup_{\alpha \leq kr} e^{M(kr)} \left| \frac{d^\alpha}{dx^\alpha} \varphi(x) \right| = 0.
\]

(4)

The topology of \(\mathbb{H}_M^r(\mathbb{R})\) is defined by the family of \(\| \cdot \|_{\mathbb{H}_M^r}\) and the dual of \(\mathbb{H}_M^r(\mathbb{R})\) is denoted by \(\mathbb{H}_M^{r'}(\mathbb{R})\). Clearly, \(\mathbb{H}_M(\mathbb{R})\) is the projective limit of \(\mathbb{H}_M^r(\mathbb{R})\) when \(r \to \infty\) and \(\mathbb{H}_M'(\mathbb{R}) = \bigcup_{r \in \mathbb{N}} \mathbb{H}_M^r(\mathbb{R})\). Also, we have continuous and dense inclusion mapping as following:

\[
\mathbb{H}_M(\mathbb{R}) \hookrightarrow \cdots \hookrightarrow \mathbb{H}_M^{r+1}(\mathbb{R}) \hookrightarrow \mathbb{H}_M^r(\mathbb{R}) \hookrightarrow \cdots
\]

\[
\hookrightarrow \mathbb{H}_M'(\mathbb{R}) \hookrightarrow \mathbb{H}_M^{r+1'}(\mathbb{R}) \hookrightarrow \cdots \hookrightarrow \mathbb{H}_M^r(\mathbb{R}).
\]

(5)

**Definition 2.** We say that the elements of \(\mathbb{H}_M^{r'}(\mathbb{R})\) are generalized tempered distributions of order \(r\).

We define by \(\mathbb{H}_M^r(\mathbb{R})\) the space of all \(\psi \in C^r(\mathbb{R})\) such that

\[
\|\psi\|_{\mathbb{H}_M^r} = \sup_{x \in \mathbb{R}, 0 \leq \alpha \leq r} e^{M(k\alpha)} \left| \frac{d^\alpha}{dx^\alpha} \psi(x) \right| < \infty, \quad l = 1, 2, \ldots.
\]

(6)

The topology of \(\mathbb{H}_M^r(\mathbb{R})\) is defined by the family of \(\| \cdot \|_{\mathbb{H}_M^r}\) and the dual of \(\mathbb{H}_M^r(\mathbb{R})\) is denoted by \(\mathbb{H}_M^{r'}(\mathbb{R})\). Obviously, \(\mathbb{H}_M^r(\mathbb{R}) \subset \mathbb{H}_M^r(\mathbb{R})\).

Now, we give a theorem that will be used later.

**Theorem 3.** Let \(\phi\) and sequence \(\{\phi_n\}_{n \in \mathbb{N}}\) be given in \(\mathbb{H}_M^{r+1}(\mathbb{R})\) such that \(\{(d^n/dx^n)\phi_n\}_{n \in \mathbb{N}}\) converges uniformly to \((d^n/dx^n)\phi\) on every compact set \(K \subset \mathbb{R}\) and for \(\alpha = 0, 1, \ldots, r\). If \(\{\phi_n\}_{n \in \mathbb{N}}\) is bounded in \(\mathbb{H}_M^{r+1}(\mathbb{R})\), then the sequence \(\{\phi_n\}_{n \in \mathbb{N}}\) converges to \(\phi \in \mathbb{H}_M^{r+1}(\mathbb{R})\) in \(\mathbb{H}_M^{r'}(\mathbb{R})\).

**Proof.** Let \(\varepsilon > 0\) be given and let \(\alpha \in \{0, 1, \ldots, r\}\). Then there exist \(N\) such that

\[
sup_{x \in K} e^{M(r\alpha)} \left| \frac{d^n}{dx^n} (\phi_n - \phi) (x) \right| < \varepsilon, \quad n \geq N,
\]

(7)

for arbitrary \(K \subset \mathbb{R}\). Also, since the sequence \(\{\phi_n\}_{n \in \mathbb{N}}\) is bounded in \(\mathbb{H}_M^{r+1}(\mathbb{R})\), we can take a positive number \(A > 0\) and a compact set \(K\) such that \(|x| > A\) when \(x \notin K\) and

\[
sup_{x \in K} e^{M(r\alpha)} \left| \frac{d^n}{dx^n} (\phi_n - \phi) (x) \right| < \varepsilon.
\]

From (7) and (8), we have

\[
\lim_{n \to \infty} sup_{x \in \mathbb{R}} e^{M(r\alpha)} \left| \frac{d^n}{dx^n} (\phi_n - \phi) (x) \right| = 0, \quad 0 \leq \alpha \leq r.
\]

(9)

\[\square\]

3. Multiresolution Expansion of \(\mathbb{H}_M^r(\mathbb{R})\)

**Definition 4.** A multiresolution analysis (shortly MRA) consists of a sequence of closed subspaces \(V_n, n \in \mathbb{Z}\), of \(L^2(\mathbb{R})\) satisfying the following:

(i) \(\{\psi(t-n)\}_{n \in \mathbb{Z}}\) is an orthonormal basis of \(V_0\),

(ii) \(\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \subset L^2(\mathbb{R})\),

(iii) \(f \in V_n \iff f(2n) \in V_{n+1}\),

(iv) \(\bigcap_n V_n = \{0\}, \bigcup_n V_n = L^2(\mathbb{R})\).

The function \(\psi\) whose existence is asserted in (i) is called a scaling function of the given MRA.
Definition 5. We say that a multiresolution analysis \( V_n, n \in \mathbb{Z} \), is \((M, r-)\) regular MRA of \( L^2(\mathbb{R}) \) if the scaling function \( \psi \) is in \( \mathcal{F}_M(\mathbb{R}) \).

Example 6. It is impossible that the scaling function \( \psi \) has exponential decay and \( \psi \in C^\infty(\mathbb{R}) \), with all derivatives bounded, unless \( \psi = 0 \). Refer to [13, Corollary 5.5.3]. So we will restrict our attention to \( \mathcal{F}_M^r(\mathbb{R}) \) or \( \mathcal{F}_M(\mathbb{R}) \). From the remark in [13] or, page 152 [2, Example 4, page 48], Battle-Lemarié’s wavelets are in \( \mathcal{F}_M^r(\mathbb{R}) \) for some \( r \in \mathbb{N} \) when \( M(x) = |x| \), but not in \( \mathcal{F}_M(\mathbb{R}) \) even if they have exponential decay and smoothness. In [13], Daubechies shown that for an arbitrary nonnegative integer \( r \), there exists an \((M, r-)\) regular MRA of \( L^2(\mathbb{R}) \) such that the scaling function \( \psi \) has compact supports.

Let \( V_j \) be an \((M, r-)\) regular MRA of \( L^2(\mathbb{R}) \) and let \( \psi \) be a scaling function. The reproducing kernel of \( V_0 \) is given by

\[
q_0(x, y) = \sum_{n \in \mathbb{Z}} \psi(x - n) \overline{\psi(y - n)}. \tag{10}
\]

The series and its derivatives with respect to \( x \) or \( y \) of order \( \leq r \) converge uniformly on \( \mathbb{R} \) because of the regularity of \( \psi \in \mathcal{F}_M^r(\mathbb{R}) \). The reproducing kernel of the projection operator onto \( V_j \) is

\[
q_j(x, y) = 2^j q_0\left(2^j x, 2^j y\right), \quad x, y \in \mathbb{R}, \tag{11}
\]

and the projection of \( f \in L^2(\mathbb{R}) \) onto \( V_j \) is given by

\[
q_j f(x) = \langle f(y), q_j(x, y) \rangle = \int f(y) q_j(x, y) dy, \quad x \in \mathbb{R}. \tag{12}
\]

The sequence \( \{q_j\}_{j \in \mathbb{Z}} \), given in (12), is called the multiresolution expansion of \( f \in L^2(\mathbb{R}) \).

Definition 7. For a given \( f \in \mathcal{F}_M^r(\mathbb{R}) \), the sequence \( \{q_j\}_{j \in \mathbb{Z}} \) defined by

\[
\langle q_j f, \phi \rangle = \langle f, q_j \phi \rangle, \quad \phi \in \mathcal{F}_M(\mathbb{R}) \tag{13}
\]

is called the multiresolution expansion of \( f \in \mathcal{F}_M^r(\mathbb{R}) \).

We deduce the following properties of the reproducing kernel \( q_0 \) with scaling function \( \psi \in \mathcal{F}_M^r(\mathbb{R}) \):

(a) \( q_0(x, y) = q_0(y, x) \) and \( q_0(x + k, y + k) = q_0(x, y) \) for all \( k \in \mathbb{Z} \).

(b) For every \( l \in \mathbb{N} \) and \( 0 \leq \alpha, \beta \leq r \), there exist \( C_l^\alpha \) such that

\[
\left| \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} q_0(x, y) \right| \leq \sum_j \left| \frac{\partial^\alpha}{\partial x^\alpha} \psi(x - j) \right| \left| \frac{\partial^\beta}{\partial y^\beta} \psi(y - j) \right|
\leq \sum_j C_l^\alpha e^{-M((l+1)(x-y))} e^{-M((l+1)(y-j))}
\leq \sum_j C_l^\alpha e^{-M(l(y-j))} e^{-M(j-y)}
\leq C_l^\alpha e^{-M((x-y))} \sum_j C_l^\alpha e^{-M(x-j)} e^{-M(j-y)}
\leq C_l^\alpha e^{-M((y-x))}, \tag{14}
\]

where we used the properties (2).

(c) \( \int_{-\infty}^{\infty} q_0(x, y) y^\alpha dy = x^\alpha, \quad y \in \mathbb{R}, \quad 0 \leq \alpha \leq r \).

Let \( V_j \) be an \((M, r-)\) regular MRA of \( L^2(\mathbb{R}) \). We fix a function \( g \in \mathcal{D}(\mathbb{R}) \) with \( \int g(x) dx = 1 \). We let \( g_j \) denote the function \( 2^j g(2^j x) \) and let \( G_j \) denote the operation of convolution by \( g_j \). For each fixed \( x \), we consider the function \( \partial_x^\alpha q_0(x, y) \) of the variable \( y \). From (c), we have

\[
\int \partial_x^\alpha q_0(x, y) y^\beta dy = 0, \tag{15}
\]

for \( 0 \leq \beta < \alpha \), whereas

\[
\int \partial_x^\alpha q_0(x, y) y^\alpha dy = \alpha!. \tag{16}
\]

Now, it follows from integration by parts that the kernel \( g(x-y) \) of the operator \( G \) shares these properties (15) and (16) with \( q_0(x, y) \).

Let

\[
R^\alpha(x, y) = \partial_x^\alpha q_0(x, y) - \partial^\alpha x g(x - y). \tag{17}
\]

From (b) and the fact that \( g \in \mathcal{D}(\mathbb{R}) \subseteq \mathcal{F}_M(\mathbb{R}) \), we have

\[
|R^\alpha(x, y)| \leq c_l e^{-M((x-y))}, \quad x, y \in \mathbb{R}, \quad k \in \mathbb{N}, \tag{18}
\]

and these functions also satisfy

\[
\int R^\alpha(x, y) dy = 0 \tag{19}
\]

identically in \( x \) for every \( \alpha = 1, 2, \ldots, r \). They, for every \( j \in \mathbb{Z} \) and \( f \in C(\mathbb{R}) \) with at most \( e^{M(x)} \)-growth, define operator \( R^\alpha_j \) by

\[
R^\alpha_j f(x) = 2^j \int R^\alpha(2^j x, 2^j y) f(y) dy \tag{20}
\]
which are such that
\[
q_j \frac{d^\alpha}{dx^\alpha} f(x) = G_j \frac{d^\alpha}{dy^\alpha} f(y) + R_j \frac{d^\alpha}{dy^\alpha} f(y),
\]
that is,
\[
\left( q_j (x, y) \frac{d^\alpha}{dy^\alpha} f(y) \right) dy = 2^j \left( g(2^j (x - y)) \frac{d^\alpha}{dy^\alpha} f(y) \right) dy + 2^j \left( R^\alpha (2^j x, 2^j y) \frac{d^\alpha}{dy^\alpha} f(y) \right) dy.
\]
From Theorem 1.1 in [14], we have
\[
\lim_{j \to \infty} G_j \frac{d^\alpha}{dy^\alpha} f(y) dy = \frac{d^\alpha}{dx^\alpha} f(x), \quad x \in \mathbb{R}, \quad \alpha \geq 0,
\]
uniformly on compact sets. Now we will show the uniform convergence on compact sets of the derivatives of multiresolution expansions of \( \mathcal{M}^\alpha \).

**Theorem 8.** Let \( f \in C^\alpha(\mathbb{R}) \) such that the corresponding derivatives \( (d^\alpha / dx^\alpha)f \) are bounded by a \( e^{M(k_0 x)} \) when \( |x| \to \infty \), for every \( \alpha = 0, 1, \ldots, r \) and some \( k_0 \in \mathbb{N} \). If \( q_j f \), given by (12), be the projection of \( f \) onto an \( (M, r) \)-regular MRA of \( L^2(\mathbb{R}) \), then the sequence \( \{(d^\alpha / dx^\alpha)q_j f\} \) converges uniformly on compact sets to \( (d^\alpha / dx^\alpha)f \) as \( j \to \infty \), for every \( \alpha = 0, 1, \ldots, r \).

**Proof.** If \( |y - x| \leq c \), we have
\[
\left| \frac{d^\alpha}{dy^\alpha} f(y) - \frac{d^\alpha}{dx^\alpha} f(x) \right|_{y=x} \leq e^{M(\alpha)} (y - x),
\]
where \( e^{M(\alpha)}(x) \) is a continuous function with \( e^{M(k_0 x)} \) growth and \( e^{M(0)}(0) = 0 \). From (18), given a compact set \( K \), we have
\[
2^j \left| \int R^\alpha (2^j x, 2^j y) \frac{d^\alpha}{dy^\alpha} f(y) dy \right|
\leq 2^j \left| \int R^\alpha (2^j x, 2^j y) \left( \frac{d^\alpha}{dy^\alpha} f(y) - \frac{d^\alpha}{dx^\alpha} f(y) \right) dy \right|
\leq 2^j \left| c_k e^{-M(2^j k |x-y|)} \right| e^{M(\alpha)} (y - x) dy,
\]
for large enough \( j \) and \( x \in K \). Since \( k \) can be chosen arbitrary, we obtain by dominated convergence theorem,
\[
\lim_{j \to \infty} 2^j \left| \int R^\alpha (2^j x, 2^j y) \frac{d^\alpha}{dy^\alpha} f(y) dy \right|
\leq 2 \left| c_k e^{-M(2^j k |x-y|)} \right| e^{M(\alpha)} (y - x) dy = 0
\]
uniformly for \( x \in K \). From (21) and (23), we have the conclusion.

We now ready to show the main theorem.

**Theorem 9.** Let \( \phi \in \mathcal{K}(\mathbb{R}) \) and let \( q \phi(x) \), given by (7), be a projection of \( \phi \) onto an \( (M, r) \)-regular MRA of \( L^2(\mathbb{R}) \). If \( \phi \in \mathcal{K}^{2(r+1)}(\mathbb{R}) \), then the sequence \( \{q \phi(x)\} \) converges to \( \phi(x) \) in \( \mathcal{K}^{2(r+1)}(\mathbb{R}) \) as \( j \to \infty \).

**Proof.** Let \( g \) and \( R^\alpha \) be given in (21) such that \( g \in \mathcal{K}(\mathbb{R}) \) and \( \int g(x) dx = 1 \). From Theorems 3 and 8 and (21), it suffices to show that
\[
\sup_{x \in \mathbb{R}} e^{M((r+1)x)} \left| \frac{1}{h} \int q_j \left( \frac{x-y}{h} \right) \phi(y) dy \right|
= \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \left| \frac{1}{h} \int g \left( \frac{x-y}{h} \right) \frac{d^\alpha}{dy^\alpha} \phi(y) dy \right|
+ \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \left| \frac{1}{h} \int R^\alpha \left( \frac{x}{h}, \frac{y}{h} \right) \frac{d^\alpha}{dy^\alpha} \phi(y) dy \right|
\]
is bounded for every \( \alpha \in \{0, 1, \ldots, r\} \) and \( h > 0 \). Since \( g \) has a compact support, then
\[
\sup_{x \in \mathbb{R}} e^{M((r+1)x)} \left| \frac{1}{h} \int g \left( \frac{x-y}{h} \right) e^{M((r+1)x)} e^{-M((2r+1)y)} dy \right|
\leq \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \left| \frac{1}{h} \int g \left( \frac{x-y}{h} \right) e^{M((2r+1)(x-y))} dy \right| \leq C.
\]
Hence we have only to show that
\[
K = \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \left| \frac{1}{h} \int R^\alpha \left( \frac{x}{h}, \frac{y}{h} \right) \frac{d^\alpha}{dy^\alpha} \phi(y) dy \right| \leq C,
\]
for every \( \alpha \in \{0, 1, \ldots, r\} \) and \( h > 0 \). Let \( S_1 = \{ |y| : |x-y| \leq 1 \} \), \( S_2 = \{ |y| : |x-y| > 1 \} \), \( S_3 = \{ |y| : |x-y| > 1 \} \) and \( S_4 = \{ |y| : |x-y| > 1 \} \). Then, by (18), we have
\[
I = \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \left| \frac{1}{h} \int R^\alpha \left( \frac{x}{h}, \frac{y}{h} \right) \frac{d^\alpha}{dy^\alpha} \phi(y) dy \right|
\leq c_1 \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \left| \frac{1}{h} \int e^{-M(k |x-y|)} e^{-M((2r+1)y)} dy \right|
= c_1 \sup_{x \in \mathbb{R}} e^{M((r+1)x)} \left( \frac{1}{h} \int S_1 + \frac{1}{h} \int S_2 + \frac{1}{h} \int S_3 \right)
\times e^{-M(k |x-y|)} e^{-M((2r+1)y)} dy
= c_1 (I_1 + I_2 + I_3).\]
By a simple change of variable, we have

\[
I_1 = \sup_{x \in \mathbb{R}} e^{M[(r+1)x]} \frac{1}{h} \int_{S_1} e^{-M((x-y)/h)} e^{-M(2(r+1)y)} \, dy \\
\leq \sup_{x \in \mathbb{R}} \frac{1}{h} \int_{S_1} e^{-M((x-y)/h)} e^{-M(2(r+1)(x-y))} \, dy \\
\leq 2e^{M(2(r+1))} \int_0^{1/h} e^{-M(t/h)} \, dt = 2e^{M(2(r+1))} \\
\times \int_0^{1/h} e^{-M(t/h)} \, dt \leq C_1.
\]

Since \((1/2)|x| < |y|\) and \((1/2)|y| \leq |x - y|\) on \(S_2\), then

\[
I_2 = \sup_{x \in \mathbb{R}} e^{M[(r+1)x]} \frac{1}{h} \int_{S_1} e^{-M((x-y)/h)} e^{-M(2(r+1)y)} \, dy \\
\leq \sup_{x \in \mathbb{R}} \frac{1}{h} \int_{S_2} e^{-M((x-y)/h)} e^{-M((x+1)y)} \, dy \\
\leq \frac{1}{h} \int_{S_2} e^{-M((2y)/h)} \, dy \leq C_2,
\]

for sufficiently large \(I\). Since \((1/2)|x| > |x - y|\) on \(S_3\), then

\[
I_3 = \sup_{x \in \mathbb{R}} e^{M[(r+1)x]} \frac{1}{h} \int_{S_1} e^{-M((x-y)/h)} e^{-M(2(r+1)y)} \, dy \\
\leq \sup_{x \in \mathbb{R}} \frac{1}{h} e^{-M(2(x/h))} \int_{S_1} e^{-M(2(r+1)y)} \, dy \\
\leq C_3 \frac{1}{h} \int_{S_1} e^{-M(2(r+1)y)} \, dy \leq C_3,
\]

for sufficiently large \(I\).}

4. Approximation Order of \(\mathcal{H}^r_M(\mathbb{R})\)

A space of functions \(S\) is called shift invariant if it is invariant under all integer translate, that is,

\[ f \in S \iff f(\cdot + k) \in S \quad \forall k \in \mathbb{Z}. \quad (34) \]

The principal shift-invariant subspaces \(S = S(\phi)\) are generated by the closure of the linear span of the shifts of \(\phi\). The stationary ladder of spaces \(S^h(\phi) : h > 0\) is given by

\[ S^h(\phi) = \left\{ f \left( \frac{x}{h} \right) : f \in S \right\}. \quad (35) \]

To rate the efficiency for approximation of such spaces, the concept of approximation order is widely used. We say that the scale of the space \(S^h(\phi)\) provides approximation order \(k\) in \(F\) if for every sufficiently smooth \(f\),

\[
\inf_{g \in S^h(\phi)} \|f - g\|_F \leq C h^k, \quad h > 0, \quad (36)
\]

where \(C = C(f) > 0\). For further details about the theory on the approximation order provided by shift-invariant spaces, we refer to [15, 16]. We will focus our attention to the so-called approximation order of an integral operator.

Let \(K\) be an integral operator of the following form

\[
(Kf)(x) = \int K(x, y) f(y) \, dy, \quad x \in \mathbb{R}. \quad (37)
\]

We assume that \(K(x - k, y) = K(x, y + k), \ h \in \mathbb{Z}, \ x, y \in \mathbb{R}\). For \(h > 0\), we define

\[
K_h = \varphi_h K \varphi_{1/h}, \quad (38)
\]

where \(\varphi\) is the scaling operator \(\varphi_h f = f(\cdot/h)\). We say that the integral operator \(K\) defined by (37) provides approximation order \(k\) in \(F\) if for every sufficiently smooth \(f\),

\[
\|K_h f - f\|_F \leq C h^k, \quad h > 0, \quad (39)
\]

where \(C = C(f) > 0\). For further details about the theory on the approximation order provided by integral or kernel operator, we refer to [17, 18].

**Definition 10** (see [4]). Let \(f \in \mathcal{H}^r_M(\mathbb{R})\). Let \(K(x, y), x, y, \in \mathbb{R}\), be the kernel of an integral operator \(K : \mathcal{H}^r_M(\mathbb{R}) \rightarrow \mathcal{H}^r_M(\mathbb{R})\). \(Kf\) is given by \(\langle Kf, \phi \rangle = \langle f, K\phi \rangle\). We say that the operator \(K\) provides approximation order \(k\) in \(\mathcal{H}^r_M(\mathbb{R})\) if

\[
\|K_h f - f\|_{\mathcal{H}^r_M} \leq \sup_{\|\phi\|_{\mathcal{H}^{r+1}_M} \leq 1} \| \langle K_h f, \phi \rangle - \langle f, \phi \rangle \| \leq C h^k, \quad (40)
\]

where the constant \(C = C(f) > 0\). We will now show that the kernel of an integral operator \(K : \mathcal{H}^{r+k}_M(\mathbb{R}) \rightarrow \mathcal{H}^r_M(\mathbb{R})\) provides approximation order in \(\mathcal{H}^{r+k}_M(\mathbb{R})\).

**Theorem 11.** Let \(\phi \in \overline{\mathcal{H}^{r+k}_M(\mathbb{R})}\) with compact support such that the integral shifts of \(\phi\) form an orthogonal basis of \(S(\phi)\) with respect to the inner product in \(L^2(\mathbb{R})\). Assume that \(\phi(x) = \sum_{n \in \mathbb{N}} c_n \phi(2x - k)\) for some sequence \(\{c_n\}_{n \in \mathbb{N}}\). Let

\[
K(x, y) = \sum_{l \in \mathbb{Z}} \phi(x - l) \bar{\phi}(y - l), \quad x, y \in \mathbb{R} \quad (41)
\]

be the kernel of the integral operator given by (37). Then \(K\) provides approximation order \(k\) in \(\mathcal{H}^r_M(\mathbb{R})\).

**Proof.** Firstly, we will show that

\[
J = \|K_h \phi - \phi\|_{\mathcal{H}^r_M} = \sup_{x \in \mathbb{R}} e^{M(|x|)} \left| \frac{d^n}{dx^n} \left( \langle K_h (x, y), \phi (y) \rangle - \langle f, \phi \rangle \right) \right| \leq C \|\phi\|_{\mathcal{H}^{r+1}_M} h^k, \quad (42)
\]
where $\phi \in \mathcal{H}^{r,k}_M$, $k \in \mathbb{N}$. If we accept the result (42) for a moment, it follows that for $f \in \mathcal{H}^{r,k}_M(\mathbb{R}) \subset \mathcal{H}^{r,k}_M(\mathbb{R})$, we have

$$
|\langle K_h f - f, \phi \rangle| = |\langle f, K_h \phi - \phi \rangle| 
\leq \|f\|_{\mathcal{H}^{r,k}_M} \|K_h \phi - \phi\|_{\mathcal{H}^{r,k}_M} \leq Ch^k \|\phi\|_{\mathcal{H}^{r,k}_M}, 
$$

hence

$$
\|K_h f - f\|_{\mathcal{H}^{r,k}_M} = \sup_{\|\phi\|_{\mathcal{H}^{r,k}_M}} |\langle K_h f, \phi \rangle - \langle f, \phi \rangle| \leq Ch^k, 
$$

(44)

$h > 0$,

which implies the conclusion.

Since $\{S^j(\phi) : j \in \mathbb{Z}\}$ satisfy the conditions of $(M, r)$-regular MRA of $L^2(\mathbb{R})$ with $S(\phi) = V_0$, we can apply (21) to the operator $K$, that is,

$$
\frac{d^\alpha}{dx^\alpha} K_h f(x) = \frac{1}{h} \int g\left(\frac{x-y}{h}\right) \frac{d^\alpha}{dy^\alpha} f(y) dy + \frac{1}{h} \int R^\alpha\left(\frac{x-y}{h}\right) \frac{d^\alpha}{dy^\alpha} f(y) dy,
$$

(45)

where $g$ and $R^\alpha$ are given in (21). For $0 \leq \alpha \leq r$,

$$
\begin{align*}
J &= \sup_{x \in \mathbb{R}} e^{M(\alpha)} \left| \frac{d^\alpha}{dx^\alpha} \left( \langle K_h(x, y), \phi(y) \rangle - \phi(x) \right) \right|
\leq \sup_{x \in \mathbb{R}} e^{M(\alpha)} \left| \frac{1}{h} \int g\left(\frac{x-y}{h}\right) \left( \frac{d^\alpha}{dy^\alpha} \phi(y) - \frac{d^\alpha}{dy^\alpha} \phi(y) \right) \bigg|_{y=x} \right| dy
+ \sup_{x \in \mathbb{R}} e^{M(\alpha)} \left| \frac{1}{h} \int R^\alpha\left(\frac{x-y}{h}\right) \left( \frac{d^\alpha}{dy^\alpha} \phi(y) - \frac{d^\alpha}{dy^\alpha} \phi(y) \right) \bigg|_{y=x} \right| dy
= J_1 + J_2.
\end{align*}
$$

(46)

In order to estimate $J_1$, we consider $g \in \mathcal{D}(\mathbb{R})$ with $\int g(x) dx = 1$ and $\int g(x)x^\alpha dx = 0, 0 < |\alpha| < \max\{r, k - 1\}$.

Let $c$ be a constant such that $\sup g \subset [-c, c]$. If we assume $h \in (0, 1)$, the smoothness of $\phi \in \mathcal{H}^{r,k}_M(\mathbb{R}) \subset C^{r,k}(\mathbb{R})$ implies

$$
J_1 = \sup_{x \in \mathbb{R}} e^{M(\alpha)} \left| \frac{1}{h} \int_{|x-y| \leq c} g(x-y) \left( \frac{d^\alpha}{dy^\alpha} \phi(y) - \frac{d^\alpha}{dy^\alpha} \phi(y) \right) \bigg|_{y=x} dy \right|
\times \left( \frac{d^\alpha}{dy^\alpha} \phi(y) \bigg|_{y=hy} - \frac{d^\alpha}{dy^\alpha} \phi(y) \bigg|_{y=hx} \right) dy
= \sup_{x \in \mathbb{R}} e^{M(\alpha)} \left| \frac{1}{h} \int_{|x-y| \leq c} g(x-y) \left( \frac{d^\alpha}{dy^\alpha} \phi(y) \bigg|_{y=hy} - \frac{d^\alpha}{dy^\alpha} \phi(y) \bigg|_{y=hx} \right) dy \right|
\times \left( (y-x) \times h \frac{d^{\alpha+1}}{dy^{\alpha+1}} \phi(y) \bigg|_{y=hx} + \cdots + (y-x)^{k-1} \times h^{k-1} \frac{d^{\alpha+k-1}}{dy^{\alpha+k-1}} \phi(y) \bigg|_{y=hy} 
+ \frac{(y-x)^k}{k!} \times h^k \frac{d^{\alpha+k}}{dy^{\alpha+k}} \phi(y) \bigg|_{y=\xi(y)} \right) dy
\leq \sup_{x \in \mathbb{R}} e^{M(\alpha)} \frac{c^k h^k}{k!} \sup_{\xi(y) \in [hx-hx+h\xi+hc]} \left| \frac{d^{\alpha+k}}{dy^{\alpha+k}} \phi(y) \right|
= C \frac{c^k h^k}{k!} \sup_{x \in \mathbb{R}} \sup_{t \in [-c+c]} \left| \frac{d^{\alpha+k}}{dt^{\alpha+k}} \phi(t) \right|
= C \frac{c^k h^k}{k!} \sup_{x \in \mathbb{R}} \sup_{t \in [-c+c]} \left( e^{-M(\alpha)} e^{M(\alpha)} \frac{d^{\alpha+k}}{dt^{\alpha+k}} \phi(t) \right)
\leq C \frac{c^k h^k}{k!} \sup_{x \in \mathbb{R}} e^{M(\alpha)} e^{-M(\alpha)} \sup_{t \in [c-c+c]} \left( e^{M(\alpha)} \frac{d^{\alpha+k}}{dt^{\alpha+k}} \phi(t) \right)
= C_1 \|\phi\|_{\mathcal{H}^{r,k}_M},
$$

(47)

where $\xi(y) = hx + \theta h(y-x) \in [hx-hc, hx+hc]$ for some $\theta \in (0, 1)$ and $C_1 = C(c^k/k!)h^k \sup_{x \in \mathbb{R}} e^{M(\alpha)} e^{-M(\alpha)} < \infty$.

To show the finiteness of $C_1$ in the last statement, we use

$$
\sup_{x \in \mathbb{R}} e^{M(\alpha)} e^{-M(\alpha)} \leq \sup_{|\alpha|<c} e^{M(\alpha)} e^{-M(\alpha)} < \infty.
$$
\[
+ \sup_{|s|>c} e^{M(r)} e^{-M(|s| - c)}
\leq \sup_{|s|<c} e^{M(r)} e^{-M(|s| - c)}
+ \sup_{|s|>c} e^{M(r)} e^{-M(|s| - c)}
\leq e^{M(r_c)} + e^{-M(r_c)}.
\]

(48)

We will estimate \( J_2 \) by using the following facts. Since \( \phi \) has a compact support, there exists \( M > 0 \) such that \( K(x, y) = 0 \) for \( |x - y| > M \). Also, by the choice of \( g \) and property (c) of the reproducing kernel \( q \), we have

\[
\int R^a (x, y) y^s dy = \frac{d^a}{dx^a} \int K(x, y) y^s dy
- \frac{d^a}{dx^a} \int g(x - y) y^s dy
= \frac{d^a}{dx^a} x^s - \frac{d^a}{dx^a} x^s dt
= \frac{d^a}{dx^a} x^s - \frac{d^a}{dx^a} x^s = 0, \quad 0 \leq s \leq r + k - 1.
\]

Hence

\[
J_2 = \sup_{h \in \mathbb{R}} e^{M(rh_x)} \left| \int R^a (x, y) \right|
\times \left( \frac{d^{a+1}}{dx^{a+1}} \phi(y) \right)_{y=hy} \left| \int R^a (x, y) \right|
= \sup_{h \in \mathbb{R}} e^{M(rh_x)} \left| \int R^a (x, y) \right|
\times \left( (y - x) \times h \frac{d^{a+1}}{dx^{a+1}} \phi(y) \right)_{y=hy}
+ \cdots + \left( (y - x)^{k-1} \frac{k!}{(k-1)!} \times h^{k-1} \frac{d^{a+k-1}}{dx^{a+k-1}} \phi(y) \right)_{y=hy}
+ \left( (y - x)^k \frac{k!}{k!} \times h^k \frac{d^{a+k}}{dx^{a+k}} \phi(y) \right)_{y=hy}
= \frac{h^k}{k!} \sup_{h \in \mathbb{R}} e^{M(rh_x)} \left| \int_{|x-y|\leq M} R^a (x, y) (y - x)^k \right|
\times \left( \frac{d^{a+k}}{dx^{a+k}} \phi(y) \right)_{y=hy} \left| \int_{|x-y|\leq M} R^a (x, y) (y - x)^k \right|
\leq \frac{M^k}{k!} \sup_{h \in \mathbb{R}} \sup_{z \in \mathbb{R}} e^{M(rh_x)} \left| \frac{d^{a+k}}{dx^{a+k}} \phi(y) \right|_{y=z(y)}
\]

(50)

\[\square\]

References


