Research Article

QR-Submanifolds of $(p - 1)$ QR-Dimension in a Quaternionic Projective Space $QP^{(n+p)/4}$ under Some Curvature Conditions

Hyang Sook Kim¹ and Jin Suk Pak²

¹ Department of Applied Mathematics, Institute of Basic Science, Inje University, Gimhae 621-749, Republic of Korea

² Kyungpook National University, Daegu 702-701, Republic of Korea

Correspondence should be addressed to Hyang Sook Kim; mathkim@inje.ac.kr

Received 4 March 2013; Accepted 9 May 2013

Academic Editor: Luc Vrancken

Copyright © 2013 H. S. Kim and J. S. Pak. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The purpose of this paper is to study n -dimensional QR-submanifolds of $(p - 1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$ and especially to determine such submanifolds under some curvature conditions.

1. Introduction

Let M be a connected real n -dimensional submanifold of real codimension p of a quaternionic Kähler manifold $\bar{M}$ with quaternionic Kähler structure $\{F, G, H\}$. If there exists an r -dimensional normal distribution ν of the normal bundle $TM^\perp$ such that

$$\begin{aligned} F\nu_x &\subset \nu_x, & G\nu_x &\subset \nu_x, & H\nu_x &\subset \nu_x, \\ F\nu_x^\perp &\subset T_x M, & G\nu_x^\perp &\subset T_x M, & H\nu_x^\perp &\subset T_x M \end{aligned} \quad (1)$$

at each point x in M , then M is called a QR-submanifold of r QR-dimension, where $\nu^\perp$ denotes the complementary orthogonal distribution to ν in $TM^\perp$ (cf. [1–3]). Real hypersurfaces, which are typical examples of QR-submanifold with $r = 0$, have been investigated by many authors (cf. [2–9]) in connection with the shape operator and the induced almost contact 3-structure (for definition, see [10–13]). In their paper [2, 3], Kwon and Pak had studied QR-submanifolds of $(p - 1)$ QR-dimension isometrically immersed in a quaternionic projective space $QP^{(n+p)/4}$ and proved the following theorem as a quaternionic analogy to theorems given in [14, 15], which are natural extensions of theorems proved in [6] to the case

of QR-submanifolds with $(p - 1)$ QR-dimension and also extensions of theorems in [16].

Theorem K-P. *Let M be an n -dimensional QR-submanifold of $(p - 1)$ QR-dimension isometrically immersed in a quaternionic projective space $QP^{(n+p)/4}$, and let the normal vector field N_1 be parallel with respect to the normal connection. If the shape operator A_1 corresponding to N_1 satisfies*

$$A_1\phi = \phi A_1, \quad A_1\psi = \psi A_1, \quad A_1\theta = \theta A_1, \quad (2)$$

then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some $(4n_1 + 3)$ - and $(4n_2 + 3)$ -dimensional spheres (π is the Hopf fibration $S^{n+p+3}(1) \rightarrow QP^{(n+p)/4}$).

On the other hand, when M is a real hypersurface of $QP^{(n+p)/4}$, if $\pi^{-1}(M)$ is (1) an Einstein space or (2) a locally symmetric space, then $\pi^{-1}(M)$ has a parallel second fundamental form (cf. [4, 6, 7, 9]). Projecting the quantities on $\pi^{-1}(M)$ onto M in $QP^{(n+p)/4}$, we can consider QR-submanifolds of $(p - 1)$ QR-dimension with the conditions corresponding to (1) or (2). In this paper, we will study such QR-submanifolds isometrically immersed in $QP^{(n+p)/4}$ and obtain Theorem 3 and other results stated in the last Section 5 as quaternionic analogies to theorems given in [16, 17] and as the extensions of theorems given in [18] by using Theorem K-P.

2. Preliminaries

Let $\overline{M}$ be a real $(n + p)$ -dimensional quaternionic Kähler manifold. Then, by definition, there is a 3-dimensional vector bundle V consisting of tensor fields of type $(1, 1)$ over $\overline{M}$ satisfying the following conditions (a), (b), and (c).

- (a) In any coordinate neighborhood $\overline{\mathcal{U}}$, there is a local basis $\{F, G, H\}$ of V such that

$$\begin{aligned} F^2 &= -I, & G^2 &= -I, & H^2 &= -I, \\ FG &= -GF = H, & GH &= -HG = F, \\ HF &= -FH = G. \end{aligned} \quad (3)$$

- (b) There is a Riemannian metric g which is Hermite with respect to all of F, G , and H .

- (c) For the Riemannian connection $\overline{\nabla}$ with respect to g ,

$$\begin{pmatrix} \overline{\nabla} F \\ \overline{\nabla} G \\ \overline{\nabla} H \end{pmatrix} = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix} \begin{pmatrix} F \\ G \\ H \end{pmatrix}, \quad (4)$$

where p, q , and r are local 1-forms defined in $\overline{\mathcal{U}}$. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in $\overline{\mathcal{U}}$ (cf. [10, 19, 20]).

For canonical local bases $\{F, G, H\}$ and $\{F', G', H'\}$ of V in coordinate neighborhoods $\overline{\mathcal{U}}$ and $\overline{\mathcal{U}}'$, it follows that in $\overline{\mathcal{U}} \cap \overline{\mathcal{U}}'$

$$\begin{pmatrix} F' \\ G' \\ H' \end{pmatrix} = (s_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix} \quad (x, y = 1, 2, 3), \quad (5)$$

where s_{xy} are local differentiable functions with $(s_{xy}) \in \text{SO}(3)$ as a consequence of (3). As is well known (cf. [19]), every quaternionic Kähler manifold is orientable.

Now let M be an n -dimensional QR-submanifold of $(p - 1)$ QR-dimension isometrically immersed in $\overline{M}$. Then by definition, there is a unit normal vector field N such that $\nu_x^\perp = \text{Span}\{N\}$ at each point x in M . We set

$$U = -FN, \quad V = -GN, \quad W = -HN. \quad (6)$$

Denoting by $\mathcal{D}_x$ the maximal quaternionic invariant subspace $T_x M \cap FT_x M \cap GT_x M \cap HT_x M$ of $T_x M$, we have $\mathcal{D}_x^\perp = \text{Span}\{U, V, W\}$, where $\mathcal{D}_x^\perp$ means the complementary orthogonal subspace to $\mathcal{D}_x$ in $T_x M$ (cf. [1–3]). Thus, we have

$$T_x M = \mathcal{D}_x \oplus \text{Span}\{U, V, W\}, \quad x \in M, \quad (7)$$

which together with (3) and (6) implies

$$FT_x M, GT_x M, HT_x M \subset T_x M \oplus \text{Span}\{N\}. \quad (8)$$

Therefore, for any tangent vector field X and for a local orthonormal basis $\{N_\alpha\}_{\alpha=1,\dots,p}$ ($N_1 := N$) of normal vectors to M , we have

$$\begin{aligned} FX &= \phi X + u(X)N, \\ GX &= \psi X + v(X)N, \end{aligned} \quad (9)$$

$$\begin{aligned} HX &= \theta X + w(X)N, \\ FN_\alpha &= -U_\alpha + P_1 N_\alpha, \\ GN_\alpha &= -V_\alpha + P_2 N_\alpha, \\ HN_\alpha &= -W_\alpha + P_3 N_\alpha, \end{aligned} \quad (10)$$

($\alpha = 1, \dots, p$). Then it is easily seen that $\{\phi, \psi, \theta\}$ and $\{P_1, P_2, P_3\}$ are skew-symmetric endomorphisms acting on $T_x M$ and $T_x M^\perp$, respectively. Moreover, the Hermitian property of $\{F, G, H\}$ implies

$$\begin{aligned} g(X, \phi U_\alpha) &= -u(X)g(N_1, P_1 N_\alpha), \\ g(X, \psi V_\alpha) &= -v(X)g(N_1, P_2 N_\alpha), \quad \alpha = 1, \dots, p, \end{aligned} \quad (11)$$

$$g(X, \theta W_\alpha) = -w(X)g(N_1, P_3 N_\alpha),$$

$$g(U_\alpha, U_\beta) = \delta_{\alpha\beta} - g(P_1 N_\alpha, P_1 N_\beta),$$

$$g(V_\alpha, V_\beta) = \delta_{\alpha\beta} - g(P_2 N_\alpha, P_2 N_\beta), \quad \alpha, \beta = 1, \dots, p, \quad (12)$$

$$g(W_\alpha, W_\beta) = \delta_{\alpha\beta} - g(P_3 N_\alpha, P_3 N_\beta).$$

Also, from the hermitian properties $g(FX, N_\alpha) = -g(X, FN_\alpha)$, $g(GX, N_\alpha) = -g(X, GN_\alpha)$, and $g(HX, N_\alpha) = -g(X, HN_\alpha)$, it follows that

$$\begin{aligned} g(X, U_\alpha) &= u(X)\delta_{1\alpha}, & g(X, V_\alpha) &= v(X)\delta_{1\alpha}, \\ g(X, W_\alpha) &= w(X)\delta_{1\alpha}, \end{aligned} \quad (13)$$

and hence,

$$\begin{aligned} g(U_1, X) &= u(X), & g(V_1, X) &= v(X), \\ g(W_1, X) &= w(X), & U_\alpha &= 0, \end{aligned} \quad (14)$$

$$V_\alpha = 0, \quad W_\alpha = 0, \quad \alpha = 2, \dots, p.$$

On the other hand, comparing (6) and (10) with $\alpha = 1$, we have $U_1 = U$, $V_1 = V$, and $W_1 = W$, which together with (6) and (14) implies

$$\begin{aligned} g(U, X) &= u(X), & g(V, X) &= v(X) \\ g(W, X) &= w(X), & u(U) &= 1, \quad v(V) = 1, \quad w(W) = 1. \end{aligned} \quad (15)$$

In the sequel, we will use the notations U, V , and W instead of U_1, V_1 , and W_1 .

Next, applying F to the first equation of (9) and using (10), (14), and (15), we have

$$\phi^2 X = -X + u(X)U, \quad u(X)P_1 N = -u(\phi X)N. \quad (16)$$

Similarly, we have

$$\phi^2 X = -X + u(X)U, \quad \psi^2 X = -X + v(X)V, \quad (17)$$

$$\theta^2 X = -X + w(X)W,$$

$$\begin{aligned} u(X)P_1N &= -u(\phi X)N, & v(X)P_2N &= -v(\psi X)N, \\ w(X)P_3N &= -w(\theta X)N, \end{aligned} \quad (18)$$

from which, taking account of the skew symmetry of P_1 , P_2 , and P_3 and using (11), we also have

$$\begin{aligned} u(\phi X) &= 0, & v(\psi X) &= 0, & w(\theta X) &= 0, \\ \phi U &= 0, & \psi V &= 0, & \theta W &= 0, & P_1N &= 0, \\ P_2N &= 0, & P_3N &= 0. \end{aligned} \quad (19)$$

So (10) can be rewritten in the form

$$\begin{aligned} FN &= -U, & GN &= -V, & HN &= -W, \\ FN_\alpha &= P_1N_\alpha, & GN_\alpha &= P_2N_\alpha, & HN_\alpha &= P_3N_\alpha \end{aligned} \quad (20)$$

($\alpha = 2, \dots, p$). Applying G and H to the first equation of (9) and using (3), (9), and (20), we have

$$\begin{aligned} \theta X + w(X)N &= -\psi(\phi X) - v(\phi X)N + u(X)V, \\ \psi X + v(X)N &= \theta(\phi X) + w(\phi X)N - u(X)W, \end{aligned} \quad (21)$$

and consequently,

$$\begin{aligned} \psi(\phi X) &= -\theta X + u(X)V, & v(\phi X) &= -w(X), \\ \theta(\phi X) &= \psi X + u(X)W, & w(\phi X) &= v(X). \end{aligned} \quad (22)$$

Similarly, the other equations of (9) yield

$$\begin{aligned} \phi(\psi X) &= \theta X + v(X)U, & u(\psi X) &= w(X), \\ \theta(\psi X) &= -\phi X + v(X)W, & w(\psi X) &= -u(X), \\ \phi(\theta X) &= -\psi X + w(X)U, & u(\theta X) &= -v(X), \\ \psi(\theta X) &= \phi X + w(X)V, & v(\theta X) &= u(X). \end{aligned} \quad (23)$$

From the first three equations of (20), we also have

$$\begin{aligned} \psi U &= -W, & v(U) &= 0, & \theta U &= V, \\ w(U) &= 0, & \phi V &= W, & u(V) &= 0, \\ \theta V &= -U, & w(V) &= 0, & \phi W &= -V, \\ u(W) &= 0, & \psi W &= U, & v(W) &= 0. \end{aligned} \quad (24)$$

Equations (14)–(17), (19), and (22)–(24) tell us that M admits the so-called almost contact 3-structure and consequently $n = 4m + 3$ for some integer m (cf. [12]).

Now let ∇ be the Levi-Civita connection on M , and let $\nabla^\perp$ be the normal connection induced from $\bar{\nabla}$ in the normal

bundle of M . Then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (25)$$

$$\bar{\nabla}_X N_\alpha = -A_\alpha X + \nabla_X^\perp N_\alpha, \quad \alpha = 1, \dots, p, \quad (26)$$

for X, Y tangent to M . Here h denotes the second fundamental form and A_α the shape operator corresponding to N_α . They are related by $h(X, Y) = \sum_{\alpha=1}^p g(A_\alpha X, Y)N_\alpha$. Furthermore, put

$$\nabla_X^\perp N_\alpha = \sum_{\beta=1}^p s_{\alpha\beta}(X)N_\beta, \quad (27)$$

where $(s_{\alpha\beta})$ is the skew-symmetric matrix of connection forms of $\nabla^\perp$.

Differentiating the first equation of (9) covariantly and using (4), (9), (10), (14) (25), and (26), we have

$$\begin{aligned} (\nabla_Y \phi)X &= r(Y)\psi X - q(Y)\theta X + u(X)A_1Y \\ &\quad - g(A_1Y, X)U, \end{aligned} \quad (28)$$

$$(\nabla_Y u)X = r(Y)v(X) - q(Y)w(X) + g(\phi A_1Y, X).$$

From the other equations of (9), we also have

$$\begin{aligned} (\nabla_Y \psi)X &= -r(Y)\phi X + p(Y)\theta X + v(X)A_1Y \\ &\quad - g(A_1Y, X)V, \end{aligned}$$

$$(\nabla_Y v)X = -r(Y)u(X) + p(Y)w(X) + g(\psi A_1Y, X), \quad (29)$$

$$\begin{aligned} (\nabla_Y \theta)X &= q(Y)\phi X - p(Y)\psi X + w(X)A_1Y \\ &\quad - g(A_1Y, X)W, \end{aligned}$$

$$(\nabla_Y w)X = q(Y)u(X) - p(Y)v(X) + g(\theta A_1Y, X).$$

Next, differentiating the first equation of (20) covariantly and comparing the tangential and normal parts, we have

$$\begin{aligned} \nabla_Y U &= r(Y)V - q(Y)W + \phi A_1Y, \\ g(A_\alpha U, Y) &= -\sum_{\beta=2}^p s_{1\beta}(Y)P_{1\beta\alpha}, \quad \alpha = 2, \dots, p. \end{aligned} \quad (30)$$

From the other equations of (20), we have similarly

$$\begin{aligned} \nabla_Y V &= -r(Y)U + p(Y)W + \psi A_1Y, \\ g(A_\alpha V, Y) &= -\sum_{\beta=2}^p s_{1\beta}(Y)P_{2\beta\alpha}, \quad \alpha = 2, \dots, p, \\ \nabla_Y W &= q(Y)U - p(Y)V + \theta A_1Y, \end{aligned} \quad (31)$$

$$g(A_\alpha W, Y) = -\sum_{\beta=2}^p s_{1\beta}(Y)P_{3\beta\alpha}, \quad \alpha = 2, \dots, p.$$

Finally the equation of Gauss is given as follows (cf. [21]):

$$\begin{aligned} g(\bar{R}(X, Y)Z, W) \\ = g(R(X, Y)Z, W) \\ + \sum_{\alpha} \{g(A_{\alpha}X, Z)g(A_{\alpha}Y, W) \\ - g(A_{\alpha}Y, Z)g(A_{\alpha}X, W)\}, \end{aligned} \quad (32)$$

for X, Y , and Z tangent to M , where $\bar{R}$ and R denote the Riemannian curvature tensor of $\bar{M}$ and M , respectively.

In the rest of this paper we assume that the distinguished normal vector field $N_1 := N$ is parallel with respect to the normal connection $\nabla^{\perp}$. Then it follows from (27) that $s_{1\beta} = 0$, and consequently, (30)-(31) imply

$$A_{\alpha}U = 0, \quad A_{\alpha}V = 0, \quad A_{\alpha}W = 0, \quad \alpha = 2, \dots, p. \quad (33)$$

On the other hand, since the curvature tensor $\bar{R}$ of $Q^{(n+p)/4}$ is of the form

$$\begin{aligned} \bar{R}(\bar{X}, \bar{Y})\bar{Z} &= g(\bar{Y}, \bar{Z})\bar{X} - g(\bar{X}, \bar{Z})\bar{Y} \\ &+ g(F\bar{Y}, \bar{Z})F\bar{X} - g(F\bar{X}, \bar{Z})F\bar{Y} \\ &- 2g(F\bar{X}, \bar{Y})F\bar{Z} + g(G\bar{Y}, \bar{Z})G\bar{X} \\ &- g(G\bar{X}, \bar{Z})G\bar{Y} - 2g(G\bar{X}, \bar{Y})G\bar{Z} \\ &+ g(H\bar{Y}, \bar{Z})H\bar{X} - g(H\bar{X}, \bar{Z})H\bar{Y} \\ &- 2g(H\bar{X}, \bar{Y})H\bar{Z} \end{aligned} \quad (34)$$

for $\bar{X}, \bar{Y}$, and $\bar{Z}$ tangent to $Q^{(n+p)/4}$, (32) reduces to

$$\begin{aligned} R(X, Y)Z &= g(Y, Z)X - g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z + g(\psi Y, Z)\psi X \\ &- g(\psi X, Z)\psi Y - 2g(\psi X, Y)\psi Z \\ &+ g(\theta Y, Z)\theta X - g(\theta X, Z)\theta Y \\ &- 2g(\theta X, Y)\theta Z \\ &+ \sum_{\alpha} \{g(A_{\alpha}Y, Z)A_{\alpha}X - g(A_{\alpha}X, Z)A_{\alpha}Y\}. \end{aligned} \quad (35)$$

3. Fibrations and Immersions

From now on n -dimensional QR-submanifolds of $(p-1)$ QR-dimension isometrically immersed in $Q^{(n+p)/4}$ only will be considered. Moreover, we will use the assumption and the notations as in Section 2.

Let $S^{n+p+3}(a)$ be the hypersphere of radius a (>0) in $Q^{(n+p+4)/4}$ the quaternionic space of quaternionic dimension

$(n+p+4)/4$, which is identified with the Euclidean $(n+p+4)$ -space $\mathbb{R}^{n+p+4}$. The unit sphere $S^{n+p+3}(1)$ will be briefly denoted by S^{n+p+3} . Let $\tilde{\pi} : S^{n+p+3} \rightarrow Q^{(n+p)/4}$ be the natural projection of S^{n+p+3} onto $Q^{(n+p)/4}$ defined by the Hopf fibration $S^3 \rightarrow S^{n+p+3} \rightarrow Q^{(n+p)/4}$. As is well known (cf. [10, 11, 20]), S^{n+p+3} admits a Sasakian 3-structure whereby $\tilde{\xi}, \tilde{\eta}$, and $\tilde{\zeta}$ are mutually orthogonal unit Killing vector fields. Thus it follows that

$$\begin{aligned} \tilde{\nabla}_{\tilde{\xi}}\tilde{\xi} &= 0, \quad \tilde{\nabla}_{\tilde{\eta}}\tilde{\eta} = 0, \quad \tilde{\nabla}_{\tilde{\zeta}}\tilde{\zeta} = 0, \\ \tilde{\nabla}_{\tilde{\xi}}\tilde{\eta} &= -\tilde{\nabla}_{\tilde{\eta}}\tilde{\xi} = \tilde{\zeta}, \quad \tilde{\nabla}_{\tilde{\xi}}\tilde{\zeta} = -\tilde{\nabla}_{\tilde{\zeta}}\tilde{\xi} = \tilde{\eta}, \\ \tilde{\nabla}_{\tilde{\eta}}\tilde{\zeta} &= -\tilde{\nabla}_{\tilde{\zeta}}\tilde{\eta} = \tilde{\xi}, \end{aligned} \quad (36)$$

where $\tilde{\nabla}$ denotes the Riemannian connection with respect to the canonical metric $\tilde{g}$ on S^{n+p+3} (cf. [6, 9-13]). Moreover, each fibre $\tilde{\pi}^{-1}(x)$ of x in $Q^{(n+p)/4}$ is a maximal integral submanifold of the distribution spanned by $\tilde{\xi}, \tilde{\eta}$, and $\tilde{\zeta}$. Thus the base space $Q^{(n+p)/4}$ admits the induced quaternionic Kähler structure of constant Q -sectional curvature 4 (cf. [10, 11]). We have especially a fibration $\pi : \pi^{-1}(M) \rightarrow M$ which is compatible with the Hopf fibration $\tilde{\pi}$. More precisely speaking, $\pi : \pi^{-1}(M) \rightarrow M$ is a fibration with totally geodesic fibers such that the following diagram is commutative:

$$\begin{array}{ccc} \pi^{-1}(M) & \xrightarrow{i'} & S^{n+p+3} \\ \pi \downarrow & & \downarrow \tilde{\pi} \\ M & \xrightarrow{i} & Q^{(n+p)/4} \end{array} \quad (37)$$

where $i' : \pi^{-1}(M) \rightarrow S^{n+p+3}$ and $i : M \rightarrow Q^{(n+p)/4}$ are isometric immersions.

Now, let ξ, η , and ζ be the unit vector fields tangent to the fibers of $\pi^{-1}(M)$ such that $i'_*\xi = \tilde{\xi}$, $i'_*\eta = \tilde{\eta}$, and $i'_*\zeta = \tilde{\zeta}$. (In what follows, we will again delete the i' and i'_* in our notation.) Furthermore, we denote by X^* the horizontal lift of a vector field X tangent to M . Then, the horizontal lifts N_{α}^* ($\alpha = 1, \dots, p$) of the normal vectors N_{α} to M form an orthonormal basis of normal vectors to $\pi^{-1}(M)$ in S^{n+p+3} . Let A'_{α} and $s'_{\alpha\beta}$ be the corresponding shape operators and normal connection forms, respectively. Then, as shown in [3, 9, 22], the fundamental equations for the submersion π are given by

$$\begin{aligned} \nabla_{X^*}Y^* &= (\nabla_X Y)^* + g'((\phi X)^*, Y^*)\xi + g'((\psi X)^*, Y^*)\eta \\ &+ g'((\theta X)^*, Y^*)\zeta, \end{aligned} \quad (38)$$

$$[X^*, Y^*] = [X, Y]^* + 2g'((\phi X)^*, Y^*)\xi + 2g'((\psi X)^*, Y^*)\eta + 2g'((\theta X)^*, Y^*)\zeta, \quad (39)$$

$$\begin{aligned} {}'\nabla_{X^*}\xi &= {}'\nabla_{\xi}X^* = -(\phi X)^*, \\ {}'\nabla_{X^*}\eta &= {}'\nabla_{\eta}X^* = -(\psi X)^*, \end{aligned} \quad (40)$$

$${}'\nabla_{X^*}\zeta = {}'\nabla_{\zeta}X^* = -(\theta X)^*,$$

$$[X^*, \xi] = 0, \quad [X^*, \eta] = 0, \quad [X^*, \zeta] = 0, \quad (41)$$

where g' denotes the Riemannian metric of $\pi^{-1}(M)$ induced from $\tilde{g}$ in S^{n+p+3} and ${}'\nabla$ the Levi-Civita connection with respect to g' . The same equations are valid for the submersion $\tilde{\pi}$ by replacing ϕ, ψ , and θ (resp., ξ, η , and ζ) with F, G , and H (resp., $\tilde{\xi}, \tilde{\eta}$, and $\tilde{\zeta}$), respectively. We denote by ${}'\nabla^\perp$ the normal connection of $\pi^{-1}(M)$ induced from $\tilde{\nabla}$. Since the diagram is commutative, $\tilde{\nabla}_{X^*}N_\alpha^*$ implies

$$\begin{aligned} {}'\nabla_{X^*}^\perp N_\alpha^* - A'_\alpha X^* &= (\tilde{\nabla}_X N_\alpha)^* + \tilde{g}((FX)^*, N_\alpha^*)\tilde{\xi} \\ &\quad + \tilde{g}((GX)^*, N_\alpha^*)\tilde{\eta} + \tilde{g}((HX)^*, N_\alpha^*)\tilde{\zeta} \\ &= -(A_\alpha X)^* + g(U_\alpha, X)^*\xi + g(V_\alpha, X)^*\eta \\ &\quad + g(W_\alpha, X)^*\zeta + (\nabla_X^\perp N_\alpha)^* \end{aligned} \quad (42)$$

because of (10), (26), and (38), from which, comparing the tangential part, we have

$$\begin{aligned} A'_\alpha X^* &= (A_\alpha X)^* - g(U_\alpha, X)^*\xi \\ &\quad - g(V_\alpha, X)^*\eta - g(W_\alpha, X)^*\zeta. \end{aligned} \quad (43)$$

Next, calculating $\tilde{\nabla}_\xi N_\alpha^*$ and using (10), (26), and (40), we have

$${}'\nabla_\xi^\perp N_\alpha^* - A'_\alpha \xi = -(FN_\alpha)^* = U_\alpha^* - (P_1 N_\alpha)^*, \quad (44)$$

which yields

$$A'_\alpha \xi = -U_\alpha^* \quad (45)$$

and similarly

$$A'_\alpha \eta = -U_\alpha^*, \quad A'_\alpha \eta = -V_\alpha^*, \quad A'_\alpha \zeta = -W_\alpha^*. \quad (46)$$

Hence, (43) and (46) with $\alpha = 1$ imply

$$\begin{aligned} A'_1 X^* &= (A_1 X)^* - g(U, X)^*\xi - g(V, X)^*\eta - g(W, X)^*\zeta, \\ A'_1 \xi &= -U^*, \quad A'_1 \eta = -V^*, \quad A'_1 \zeta = -W^*. \end{aligned} \quad (47)$$

4. Co-Gauss Equations for the Submersion

$$\pi: \pi^{-1}(M) \rightarrow M$$

In this section, we derive the co-Gauss and co-Codazzi equations of the submersion $\pi: \pi^{-1}(M) \rightarrow M$ for later use.

Differentiating (38) with $Y = U$ covariantly along $\pi^{-1}(M)$ and using (24), (38), and (39), we have

$$\begin{aligned} {}'\nabla_{Y^*} {}'\nabla_{X^*} U^* &= (\nabla_Y \nabla_X U)^* + \{v(X)\theta Y - w(X)\psi Y\}^* \\ &\quad + g(\phi Y, \nabla_X U)^*\xi \\ &\quad + \{g(\psi Y, \nabla_X U) + g(\nabla_Y X, W) + g(X, \nabla_Y W)\}^*\eta \\ &\quad + \{g(\theta Y, \nabla_X U) - g(\nabla_Y X, V) - g(X, \nabla_Y V)\}^*\zeta. \end{aligned} \quad (48)$$

Similarly (38) with $Y = V$ and (38) with $Y = W$ give

$$\begin{aligned} {}'\nabla_{Y^*} {}'\nabla_{X^*} V^* &= (\nabla_Y \nabla_X V)^* + \{w(X)\phi Y - u(X)\theta Y\}^* \\ &\quad + \{g(\phi Y, \nabla_X V) - g(\nabla_Y X, W) - g(X, \nabla_Y W)\}^*\xi \\ &\quad + g(\psi Y, \nabla_X V)^*\eta \\ &\quad + \{g(\theta Y, \nabla_X V) + g(\nabla_Y X, U) + g(X, \nabla_Y U)\}^*\zeta, \end{aligned} \quad (49)$$

$$\begin{aligned} {}'\nabla_{Y^*} {}'\nabla_{X^*} W^* &= (\nabla_Y \nabla_X W)^* \\ &\quad - \{v(X)\phi Y - u(X)\psi Y\}^* \\ &\quad + \{g(\phi Y, \nabla_X W) + g(\nabla_Y X, V) + g(X, \nabla_Y V)\}^*\xi \\ &\quad + \{g(\psi Y, \nabla_X W) - g(\nabla_Y X, U) - g(X, \nabla_Y U)\}^*\eta \\ &\quad + g(\theta Y, \nabla_X W)^*\zeta, \end{aligned} \quad (50)$$

respectively. On the other hand, it follows from (19), (24), (38), and (39) that

$$\begin{aligned} {}'\nabla_{[Y^*, X^*]} U^* &= (\nabla_{[Y, X]} U)^* + 2g(\psi Y, X)^* W^* \\ &\quad - 2g(\theta Y, X)^* V^* + g([Y, X], W)^*\eta \\ &\quad - g([Y, X], V)^*\zeta, \end{aligned} \quad (51)$$

$$\begin{aligned} {}^{\prime}\nabla_{[Y^*, X^*]} V^* &= (\nabla_{[Y, X]} V)^* - 2g(\phi Y, X)^* W^* \\ &\quad + 2g(\theta Y, X)^* U^* - g([Y, X], W)^* \xi \\ &\quad + g([Y, X], U)^* \zeta, \end{aligned} \quad (52)$$

$$\begin{aligned} {}^{\prime}\nabla_{[Y^*, X^*]} W^* &= (\nabla_{[Y, X]} W)^* + 2g(\phi Y, X)^* V^* \\ &\quad - 2g(\psi Y, X)^* U^* + g([Y, X], V)^* \xi \\ &\quad - g([Y, X], U)^* \eta. \end{aligned} \quad (53)$$

By means of (48) and (51), we have

$$\begin{aligned} {}^{\prime}R(Y^*, X^*) U^* &= \{R(Y, X) U\}^* \\ &\quad + \{w(Y) \psi X - w(X) \psi Y - v(Y) \theta X \\ &\quad + v(X) \theta Y + 2g(\theta Y, X) V \\ &\quad - 2g(\psi Y, X) W\}^* \\ &\quad + \{g(\phi Y, \nabla_X U) - g(\phi X, \nabla_Y U)\}^* \xi \\ &\quad + \{g(\psi Y, \nabla_X U) - g(\psi X, \nabla_Y U) \\ &\quad + g(X, \nabla_Y W) - g(Y, \nabla_X W)\}^* \eta \\ &\quad + \{g(\theta Y, \nabla_X U) - g(\theta X, \nabla_Y U) \\ &\quad - g(X, \nabla_Y V) + g(Y, \nabla_X V)\}^* \zeta, \end{aligned} \quad (54)$$

where ${}^{\prime}R$ denotes the curvature tensor of $\pi^{-1}(M)$ with respect to the connection ${}^{\prime}\nabla$. Using (30), (31), (33), and (35), we can easily see that

$$\begin{aligned} {}^{\prime}R(Y^*, X^*) U^* &= \{u(X) Y - u(Y) X \\ &\quad + u(A_1 X) A_1 Y - u(A_1 Y) A_1 X\}^* \\ &\quad + \{r(Y) w(X) - r(X) w(Y) \\ &\quad + q(Y) v(X) - q(X) v(Y) \\ &\quad + u(X) u(A_1 Y) - u(Y) u(A_1 X)\}^* \xi \\ &\quad + \{p(X) v(Y) - p(Y) v(X) \\ &\quad + v(X) u(A_1 Y) - v(Y) u(A_1 X)\}^* \eta \\ &\quad + \{p(X) w(Y) - p(Y) w(X) \\ &\quad + w(X) u(A_1 Y) - w(Y) u(A_1 X)\}^* \zeta. \end{aligned} \quad (55)$$

By the same method, we can easily verify that (49), (50), (52), and (53) yield

$$\begin{aligned} {}^{\prime}R(Y^*, X^*) V^* &= \{v(X) Y - v(Y) X + v(A_1 X) A_1 Y \\ &\quad - v(A_1 Y) A_1 X\}^* \\ &\quad + \{q(X) u(Y) - q(Y) u(X) \\ &\quad - u(Y) v(A_1 X) + u(X) v(A_1 Y)\}^* \xi \\ &\quad + \{r(Y) w(X) - r(X) w(Y) \\ &\quad + p(Y) u(X) - p(X) u(Y) \\ &\quad + v(X) v(A_1 Y) - v(Y) v(A_1 X)\}^* \eta \\ &\quad + \{q(X) w(Y) - q(Y) w(X) \\ &\quad - w(Y) v(A_1 X) + w(X) v(A_1 Y)\}^* \zeta, \\ {}^{\prime}R(Y^*, X^*) W^* &= \{w(X) Y - w(Y) X + w(A_1 X) A_1 Y \\ &\quad - w(A_1 Y) A_1 X\}^* \\ &\quad + \{r(X) u(Y) - r(Y) u(X) \\ &\quad - u(Y) w(A_1 X) + u(X) w(A_1 Y)\}^* \xi \\ &\quad + \{r(X) v(Y) - r(Y) v(X) \\ &\quad - v(Y) w(A_1 X) + v(X) w(A_1 Y)\}^* \eta \\ &\quad + \{q(Y) v(X) - q(X) v(Y) \\ &\quad + p(Y) u(X) - p(X) u(Y) \\ &\quad + w(X) w(A_1 Y) - w(Y) w(A_1 X)\}^* \zeta. \end{aligned} \quad (56)$$

Differentiating (38) with $X = U$ covariantly along $\pi^{-1}(M)$ and using (24), we have

$$\begin{aligned} {}^{\prime}\nabla_{Y^*} {}^{\prime}\nabla_{U^*} X^* &= (\nabla_Y \nabla_U X)^* + \{w(X) \psi Y - v(X) \theta Y\}^* \\ &\quad + g(\phi Y, \nabla_U X)^* \xi \\ &\quad + \{g(\psi Y, \nabla_U X) - g(\nabla_Y W, X) - g(W, \nabla_Y X)\}^* \eta \\ &\quad + \{g(\theta Y, \nabla_U X) + g(\nabla_Y V, X) + g(V, \nabla_Y X)\}^* \zeta. \end{aligned} \quad (57)$$

Similarly, (38) with $X = V$ and (38) with $X = W$, respectively, give

$$\begin{aligned} {}'\nabla_{Y^*} {}'\nabla_{V^*} X^* &= (\nabla_Y \nabla_V X)^* - \{w(X) \phi Y - u(X) \theta Y\}^* \\ &\quad + g(\psi Y, \nabla_V X)^* \eta \\ &\quad + \{g(\phi Y, \nabla_V X) + g(\nabla_Y W, X) + g(W, \nabla_Y X)\}^* \xi \\ &\quad + \{g(\theta Y, \nabla_V X) - g(\nabla_Y U, X) - g(U, \nabla_Y X)\}^* \zeta, \end{aligned} \quad (58)$$

$$\begin{aligned} {}'\nabla_{Y^*} {}'\nabla_{W^*} X^* &= (\nabla_Y \nabla_W X)^* + \{v(X) \phi Y - u(X) \psi Y\}^* \\ &\quad + g(\theta Y, \nabla_W X)^* \zeta \\ &\quad + \{g(\phi Y, \nabla_W X) - g(\nabla_Y V, X) - g(V, \nabla_Y X)\}^* \xi \\ &\quad + \{g(\psi Y, \nabla_W X) + g(\nabla_Y U, X) + g(U, \nabla_Y X)\}^* \eta. \end{aligned} \quad (59)$$

Differentiating (38) also covariantly in the direction of U^* and using (24), we have

$$\begin{aligned} {}'\nabla_{U^*} {}'\nabla_{Y^*} X^* &= (\nabla_U \nabla_Y X)^* \\ &\quad + g(\psi Y, X)^* W^* - g(\theta Y, X)^* V^* \\ &\quad + \{g(\nabla_U (\phi Y), X) + g(\phi Y, \nabla_U X)\}^* \xi \\ &\quad + \{g(\nabla_U (\psi Y), X) + g(\psi Y, \nabla_U X) - g(W, \nabla_Y X)\}^* \eta \\ &\quad + \{g(\nabla_U (\theta Y), X) + g(\theta Y, \nabla_U X) + g(V, \nabla_Y X)\}^* \zeta. \end{aligned} \quad (60)$$

Similarly, differentiating (38) covariantly in the direction of V^* and W^* , respectively, we have

$$\begin{aligned} {}'\nabla_{V^*} {}'\nabla_{Y^*} X^* &= (\nabla_V \nabla_Y X)^* \\ &\quad - g(\phi Y, X)^* W^* - g(\theta Y, X)^* U^* \\ &\quad + \{g(\nabla_V (\phi Y), X) + g(\phi Y, \nabla_V X) + g(W, \nabla_Y X)\}^* \xi \\ &\quad + \{g(\nabla_V (\psi Y), X) + g(\psi Y, \nabla_V X)\}^* \eta \\ &\quad + \{g(\nabla_V (\theta Y), X) + g(\theta Y, \nabla_V X) - g(U, \nabla_Y X)\}^* \zeta, \end{aligned} \quad (61)$$

$$\begin{aligned} {}'\nabla_{W^*} {}'\nabla_{Y^*} X^* &= (\nabla_W \nabla_Y X)^* \\ &\quad - g(\psi Y, X)^* U^* + g(\phi Y, X)^* V^* \\ &\quad + \{g(\nabla_W (\phi Y), X) + g(\phi Y, \nabla_W X) - g(V, \nabla_Y X)\}^* \xi \\ &\quad + \{g(\nabla_W (\psi Y), X) + g(\psi Y, \nabla_W X) + g(U, \nabla_Y X)\}^* \eta \\ &\quad + \{g(\nabla_W (\theta Y), X) + g(\theta Y, \nabla_W X)\}^* \zeta. \end{aligned} \quad (62)$$

On the other hand, (38) and (39) with $X = U$ imply

$$\begin{aligned} {}'\nabla_{[Y^*, U^*]} X^* &= (\nabla_{[Y, U]} X)^* - 2\{w(Y) \psi X - v(Y) \theta X\}^* \\ &\quad + g(\phi[Y, U], X)^* \xi + g(\psi[Y, U], X)^* \eta \\ &\quad + g(\theta[Y, U], X)^* \zeta. \end{aligned} \quad (63)$$

Similarly, from (39) with $X = V$ and (39) with $X = W$, respectively, we find that

$$\begin{aligned} {}'\nabla_{[Y^*, V^*]} X^* &= (\nabla_{[Y, V]} X)^* + 2\{w(Y) \phi X - u(Y) \theta X\}^* \\ &\quad + g(\phi[Y, V], X)^* \xi + g(\psi[Y, V], X)^* \eta \\ &\quad + g(\theta[Y, V], X)^* \zeta, \end{aligned} \quad (64)$$

$$\begin{aligned} {}'\nabla_{[Y^*, W^*]} X^* &= (\nabla_{[Y, W]} X)^* - 2\{v(Y) \phi X - u(Y) \psi X\}^* \\ &\quad + g(\phi[Y, W], X)^* \xi + g(\psi[Y, W], X)^* \eta \\ &\quad + g(\theta[Y, W], X)^* \zeta. \end{aligned} \quad (65)$$

Using (28)–(31), it follows from (57), (60), and (63) that

$$\begin{aligned} {}'R(Y^*, U^*) X^* &= \{R(Y, U) X\}^* \\ &\quad + \{w(X) \psi Y - v(X) \theta Y \\ &\quad - g(\psi Y, X) W + g(\theta Y, X) V \\ &\quad + 2w(Y) \psi X - 2v(Y) \theta X\}^* \\ &\quad - \{r(U) g(\psi Y, X) - q(U) g(\theta Y, X) \\ &\quad + u(Y) u(A_1 X) + r(Y) w(X) \\ &\quad + q(Y) v(X) - g(A_1 Y, X)\}^* \xi \\ &\quad - \{-p(Y) v(X) - r(U) g(\phi Y, X) \\ &\quad + p(U) g(\theta Y, X) + v(Y) u(A_1 X)\}^* \eta \\ &\quad - \{-p(Y) w(X) + q(U) g(\phi Y, X) \\ &\quad - p(U) g(\psi Y, X) + w(Y) u(A_1 X)\}^* \zeta, \end{aligned} \quad (66)$$

from which, taking account of (35) and using (24) and (33), we obtain

$$\begin{aligned}
 {}^1R(Y^*, U^*)X^* &= \{u(X)Y - g(Y, X)U \\
 &\quad + u(A_1X)A_1Y - g(A_1Y, X)A_1U\}^* \\
 &\quad - \{r(U)g(\psi Y, X) - q(U)g(\theta Y, X) \\
 &\quad + u(Y)u(A_1X) + r(Y)w(X) \\
 &\quad + q(Y)v(X) - g(A_1Y, X)\}^*\xi \\
 &\quad - \{-p(Y)v(X) - r(U)g(\phi Y, X) \\
 &\quad + p(U)g(\theta Y, X) + v(Y)u(A_1X)\}^*\eta \\
 &\quad - \{-p(Y)w(X) + q(U)g(\phi Y, X) \\
 &\quad - p(U)g(\psi Y, X) + w(Y)u(A_1X)\}^*\zeta.
 \end{aligned} \tag{67}$$

Similarly, by using (58), (59), (61), (62), (64), and (65), we can easily obtain

$$\begin{aligned}
 {}^1R(Y^*, V^*)X^* &= \{v(X)Y - g(Y, X)V \\
 &\quad + v(A_1X)A_1Y - g(A_1Y, X)A_1V\}^* \\
 &\quad - \{-r(V)g(\phi Y, X) + p(V)g(\theta Y, X) \\
 &\quad + v(Y)v(A_1X) + r(Y)w(X) \\
 &\quad + p(Y)u(X) - g(A_1Y, X)\}^*\eta \\
 &\quad + \{q(Y)u(X) - r(V)g(\psi Y, X) \\
 &\quad + q(V)g(\theta Y, X) - u(Y)v(A_1X)\}^*\xi \\
 &\quad - \{-q(Y)w(X) + q(V)g(\phi Y, X) \\
 &\quad - p(V)g(\psi Y, X) + w(Y)v(A_1X)\}^*\zeta,
 \end{aligned}$$

$$\begin{aligned}
 {}^1R(Y^*, W^*)X^* &= \{w(X)Y - g(Y, X)W \\
 &\quad + w(A_1X)A_1Y - g(A_1Y, X)A_1W\}^* \\
 &\quad - \{q(W)g(\phi Y, X) - p(W)g(\psi Y, X) \\
 &\quad + w(Y)w(A_1X) + q(Y)v(X) \\
 &\quad + p(Y)u(X) - g(A_1Y, X)\}^*\zeta \\
 &\quad - \{-r(Y)u(X) + r(W)g(\psi Y, X) \\
 &\quad - q(W)g(\theta Y, X) + u(Y)w(A_1X)\}^*\xi \\
 &\quad + \{r(Y)v(X) + r(W)g(\phi Y, X) \\
 &\quad - p(W)g(\theta Y, X) - v(Y)w(A_1X)\}^*\eta.
 \end{aligned} \tag{68}$$

5. Main Results

It is well known [3] that if $\pi^{-1}(M)$ is locally symmetric then ${}^1\nabla A_1 = 0$ which implies identities (2) in Theorem K-P. In this point of view, we consider the following assumptions in (69) which are weaker conditions than the locally symmetry of $\pi^{-1}(M)$.

In order to obtain our main results, let M be n -dimensional QR-submanifolds of $(p-1)$ QR-dimension in $Q\mathbb{P}^{(n+p)/4}$ with the assumptions

$$\begin{aligned}
 ({}^1\nabla_\xi {}^1R)(Y^*, X^*)U^* &= 0, & ({}^1\nabla_\eta {}^1R)(Y^*, X^*)V^* &= 0, \\
 ({}^1\nabla_\zeta {}^1R)(Y^*, X^*)W^* &= 0, \\
 ({}^1\nabla_\xi {}^1R)(Y^*, U^*)X^* &= 0, & ({}^1\nabla_\eta {}^1R)(Y^*, V^*)X^* &= 0, \\
 ({}^1\nabla_\zeta {}^1R)(Y^*, W^*)X^* &= 0.
 \end{aligned} \tag{69}$$

We notice here that the above curvature conditions in (69) are different from those in [18] due to Pak and Sohn.

We first consider the assumption

$$({}^1\nabla_\xi {}^1R)(Y^*, X^*)U^* = 0. \tag{70}$$

Differentiating (55) covariantly in the direction of ξ and using (19), (36), and (40), and the assumption $({}^1\nabla_\xi {}^1R)(Y^*, X^*)U^* = 0$, we have

$$\begin{aligned}
 &-{}^1R((\phi Y)^*, X^*)U^* - {}^1R(Y^*, (\phi X)^*)U^* \\
 &= \{-u(X)\phi Y + u(Y)\phi X \\
 &\quad - u(A_1X)\phi A_1Y + u(A_1Y)\phi A_1X\}^* \\
 &\quad + \{p(X)w(Y) - p(Y)w(X) \\
 &\quad + w(X)u(A_1Y) - w(Y)u(A_1X)\}^*\eta \\
 &\quad - \{p(X)v(Y) - p(Y)v(X) \\
 &\quad + v(X)u(A_1Y) - v(Y)u(A_1X)\}^*\zeta,
 \end{aligned} \tag{71}$$

from which, taking the vertical component of ξ , η , and ζ , respectively, and using (22)–(24) and (55) itself, we can get

$$\begin{aligned}
 &r(X)v(Y) - r(Y)v(X) - q(X)w(Y) \\
 &\quad + q(Y)w(X) - r(\phi Y)w(X) + r(\phi X)w(Y) \\
 &\quad - q(\phi Y)v(X) + q(\phi X)v(Y)
 \end{aligned} \tag{72}$$

$$\begin{aligned}
 &-u(X)u(A_1\phi Y) + u(Y)u(A_1\phi X) = 0, \\
 &-u(A_1\phi Y)v(X) + u(A_1\phi X)v(Y) + p(\phi Y)v(X) \\
 &\quad - p(\phi X)v(Y) = 0,
 \end{aligned} \tag{73}$$

$$\begin{aligned}
 &-u(A_1\phi Y)w(X) + u(A_1\phi X)w(Y) + p(\phi Y)w(X) \\
 &\quad - p(\phi X)w(Y) = 0.
 \end{aligned} \tag{74}$$

Putting $Y = U$ in (72) and using (19) and (24), we have

$$\phi A_1 U + r(U) V - q(U) W = 0, \quad (75)$$

and consequently,

$$\begin{aligned} r(U) &= w(A_1 U) = u(A_1 W), \\ q(U) &= v(A_1 U) = u(A_1 V). \end{aligned} \quad (76)$$

Putting $Y = W$ and $X = V$ in (73) and using (15) and (24) yield

$$p(V) = v(A_1 U) = u(A_1 V). \quad (77)$$

Also, putting $Y = V$ and $X = W$ in (74) and using (15) and (24), we have

$$p(W) = w(A_1 U) = u(A_1 W). \quad (78)$$

Summing up, we have

$$\begin{aligned} A_1 U &= u(A_1 U) U + p(V) V + p(W) W, \\ p(V) &= v(A_1 U) = u(A_1 V) = q(U), \\ p(W) &= w(A_1 U) = u(A_1 W) = r(U). \end{aligned} \quad (79)$$

Thus we get the following lemma.

Lemma 1. *Let M be an n -dimensional QR-submanifold of $(p-1)$ QR-dimension in a quaternionic projective space $Q\mathbb{P}^{(n+p)/4}$, and let the normal vector field N_1 be parallel with respect to the normal connection. If the equalities in (69) are established, then*

$$\begin{aligned} A_1 U &= u(A_1 U) U + p(V) V + p(W) W, \\ A_1 V &= q(U) U + v(A_1 V) V + q(W) W, \\ A_1 W &= r(U) U + r(V) V + w(A_1 W) W, \\ p(V) &= v(A_1 U) = u(A_1 V) = q(U), \\ p(W) &= w(A_1 U) = u(A_1 W) = r(U), \\ q(W) &= w(A_1 V) = v(A_1 W) = r(V). \end{aligned} \quad (80)$$

Next, we assume the additional condition

$$(' \nabla_{\xi} ' R)(Y^*, U^*) X^* = 0. \quad (81)$$

Differentiating (67) covariantly in the direction of ξ and using (36), (40) and the assumption $(' \nabla_{\xi} ' R)(Y^*, U^*) X^* = 0$, we have

$$\begin{aligned} & - ' R((\phi Y)^*, U^*) X^* - ' R(Y^*, U^*)(\phi X)^* \\ &= \{-u(X) \phi Y - u(A_1 X) \phi A_1 Y \\ & \quad + g(A_1 Y, X) \phi A_1 U\}^* \\ & \quad - \{-p(Y) w(X) + q(U) g(\phi Y, X) \\ & \quad - p(U) g(\psi Y, X) + w(Y) u(A_1 X)\}^* \eta \\ & \quad + \{-p(Y) v(X) - r(U) g(\phi Y, X) \\ & \quad + p(U) g(\theta Y, X) + v(Y) u(A_1 X)\}^* \zeta, \end{aligned} \quad (82)$$

from which, taking the vertical component of ξ , η , and ζ , respectively, and using (22)–(24) and (67) itself, we can find

$$\begin{aligned} & r(U) \{-2g(\theta Y, X) + u(Y) v(X) - v(Y) u(X)\} \\ & \quad - q(U) \{2g(\psi Y, X) + u(Y) w(X) - w(Y) u(X)\} \\ & \quad + u(Y) u(A_1 \phi X) + r(\phi Y) w(X) + r(Y) v(X) \\ & \quad + q(\phi Y) v(X) - q(Y) w(X) \end{aligned} \quad (83)$$

$$\begin{aligned} & + g(\phi A_1 Y - A_1 \phi Y, X) = 0, \\ & - q(U) g(\phi Y, X) + p(\phi Y) v(X) - v(Y) u(A_1 \phi X) \\ & \quad - p(U) \{g(\psi Y, X) + u(Y) w(X) \\ & \quad - w(Y) u(X)\} = 0, \end{aligned} \quad (84)$$

$$\begin{aligned} & - r(U) g(\phi Y, X) + p(\phi Y) w(X) - w(Y) u(A_1 \phi X) \\ & \quad - p(U) \{g(\theta Y, X) + v(Y) u(X) - u(Y) v(X)\} = 0. \end{aligned} \quad (85)$$

Taking the skew-symmetric part of (83) with respect to X and Y , we have

$$\begin{aligned} & 2r(U) \{-2g(\theta Y, X) + u(Y) v(X) - v(Y) u(X)\} \\ & \quad - 2q(U) \{2g(\psi Y, X) + u(Y) w(X) - w(Y) u(X)\} \\ & \quad + u(Y) u(A_1 \phi X) - u(X) u(A_1 \phi Y) + r(\phi Y) w(X) \\ & \quad - r(\phi X) w(Y) + r(Y) v(X) - r(X) v(Y) \\ & \quad + q(\phi Y) v(X) - q(\phi X) v(Y) \\ & \quad - q(Y) w(X) + q(X) w(Y) = 0. \end{aligned} \quad (86)$$

Replacing Y with θY in (86) and using (19) and (22)–(24), we have

$$\begin{aligned} & r(U) \{4g(Y, X) - 3w(Y) w(X) \\ & \quad - 2u(Y) u(X) - 2v(Y) v(X)\} \\ & \quad - q(U) \{4g(\phi Y, X) + 3w(Y) v(X) - 2w(X) v(Y)\} \\ & \quad - q(\theta Y) w(X) - q(\psi Y) v(X) - q(\phi X) u(Y) \\ & \quad + r(\theta Y) v(X) - r(X) u(Y) - r(\psi Y) w(X) \\ & \quad - v(Y) u(A_1 \phi X) + u(X) u(A_1 \psi Y) \\ & \quad - u(A_1 U) u(X) w(Y) = 0. \end{aligned} \quad (87)$$

Now we consider the following orthonormal basis:

$$\begin{aligned} & \{U, V, W, e_1, \dots, e_m, \phi(e_1), \dots, \phi(e_m), \\ & \quad \psi(e_1), \dots, \psi(e_m), \theta(e_1), \dots, \theta(e_m)\}, \end{aligned} \quad (88)$$

which will be called Q -basis, where $4m + 3 = \dim M$. Taking the trace of the above equation with respect to the Q -basis and using (76), we can easily see $4mr(U) = 0$; that is,

$$r(U) = 0. \quad (89)$$

Replacing also Y with ψY in (86) and using (19) and (22)–(24), we have

$$q(U) = 0. \quad (90)$$

Substituting (89) and (90) into (75), we have $\phi A_1 U = 0$ and hence

$$A_1 U = u(A_1 U) U. \quad (91)$$

On the other hand, replacing Y with ψY in (84) and using (19), (22), (23), and (90), we obtain

$$\begin{aligned} p(U) \{g(Y, X) - u(Y)u(X) - w(Y)w(X)\} \\ + p(\theta Y)v(X) = 0, \end{aligned} \quad (92)$$

from which, taking the trace with respect to the Q -basis and using (15) and (24), we find $4mp(U) = 0$; that is,

$$p(U) = 0 \quad (93)$$

which together with (84) and (90) implies

$$p(\phi Y) = 0. \quad (94)$$

Replacing Y with ϕY in the above equation and using (17) and (93), we can easily see that

$$p(Y) = 0 \quad (95)$$

for any vector Y tangent to M .

Summing up, we have the following lemma.

Lemma 2. *Let M be as in Lemma 1, and let the normal vector field N_1 be parallel with respect to the normal connection. If the equalities in (69) and (5.2) are established, then*

$$\begin{aligned} A_1 U = u(A_1 U) U, \quad A_1 V = v(A_1 V) V, \\ A_1 W = w(A_1 W) W, \quad p = q = r = 0. \end{aligned} \quad (96)$$

Finally, we will prove our main theorem.

Theorem 3. *Let M be an n -dimensional QR-submanifold of $(p - 1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$, and let the normal vector field N_1 be parallel with respect to the normal connection. If the equalities in (69) and (5.2) are established, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some $(4n_1 + 3)$ - and $(4n_2 + 3)$ -dimensional spheres (π is the Hopf fibration $S^{n+p+3}(1) \rightarrow QP^{(n+p)/4}$).*

Proof. By means of (96), it follows easily from (83) that

$$\phi A_1 = A_1 \phi. \quad (97)$$

By the quite same method, we can obtain

$$\psi A_1 = A_1 \psi, \quad \theta A_1 = A_1 \theta. \quad (98)$$

Combining with those equalities and Theorem K-P, we complete the proof. $\square$

Corollary 4. *Let M be an n -dimensional QR-submanifold of $(p - 1)$ QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$, and let the normal vector field N_1 be parallel with respect to the normal connection. If the following equalities:*

$$' \nabla_{\xi} ' R = 0, \quad ' \nabla_{\eta} ' R = 0, \quad ' \nabla_{\zeta} ' R = 0 \quad (99)$$

are established, then $\pi^{-1}(M)$ is locally a product of $M_1 \times M_2$ where M_1 and M_2 belong to some $(4n_1 + 3)$ - and $(4n_2 + 3)$ -dimensional spheres.

Acknowledgment

This work was supported by the 2013 Inje University research grant.

References

- [1] A. Bejancu, *Geometry of CR-Submanifolds*, D. Reidel Publishing Company, Dordrecht, The Netherlands, 1986.
- [2] J.-H. Kwon and J. S. Pak, "Scalar curvature of QR-submanifolds immersed in a quaternionic projective space," *Saitama Mathematical Journal*, vol. 17, pp. 89–116, 1999.
- [3] J.-H. Kwon and J. S. Pak, "QR-submanifolds of Q QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$," *Acta Mathematica Hungarica*, vol. 86, no. 1-2, pp. 89–116, 2000.
- [4] H. B. Lawson, Jr., "Rigidity theorems in rank- $(n - 1)$ symmetric spaces," *Journal of Differential Geometry*, vol. 4, pp. 349–357, 1970.
- [5] A. Martínez and J. D. Pérez, "Real hypersurfaces in quaternionic projective space," *Annali di Matematica Pura ed Applicata*, vol. 145, pp. 355–384, 1986.
- [6] J. S. Pak, "Real hypersurfaces in quaternionic Kaehlerian manifolds with constant $QP^{(n+p)/4}$ -sectional curvature," *Kōdai Mathematical Seminar Reports*, vol. 29, no. 1-2, pp. 22–61, 1977.
- [7] J. D. Pérez and F. G. Santos, "On pseudo-Einstein real hypersurfaces of the quaternionic projective space," *Kyungpook Mathematical Journal*, vol. 25, no. 1, pp. 15–28, 1985.
- [8] J. D. Pérez and F. G. Santos, "On real hypersurfaces with harmonic curvature of a quaternionic projective space," *Journal of Geometry*, vol. 40, no. 1-2, pp. 165–169, 1991.
- [9] Y. Shibuya, "Real submanifolds in a quaternionic projective space," *Kodai Mathematical Journal*, vol. 1, no. 3, pp. 421–439, 1978.
- [10] S. Ishihara, "Quaternion Kählerian manifolds and fibred Riemannian spaces with Sasakian $(p - 1)$ -structure," *Kōdai Mathematical Seminar Reports*, vol. 25, pp. 321–329, 1973.
- [11] S. Ishihara and M. Konishi, "Fibred riemannian space with sasakian 3-structure," in *Differential Geometry in Honor of K. Yano*, pp. 179–194, Kinokuniya, Tokyo, Japan, 1972.
- [12] Y.-Y. Kuo, "On almost contact $(p - 1)$ -structure," *The Tohoku Mathematical Journal*, vol. 22, pp. 325–332, 1970.

- [13] Y. Y. Kuo and S.-I. Tachibana, "On the distribution appeared in contact $(p - 1)$ -structure," *Taiwanese Journal of Mathematics*, vol. 2, pp. 17–24, 1970.
- [14] Y.-W. Choe and M. Okumura, "Scalar curvature of a certain CR-submanifold of complex projective space," *Archiv der Mathematik*, vol. 68, no. 4, pp. 340–346, 1997.
- [15] M. Okumura and L. Vanhecke, " n -dimensional real submanifolds with $(p - 1)$ -dimensional maximal holomorphic tangent subspace in complex projective spaces," *Rendiconti del Circolo Matematico di Palermo*, vol. 43, no. 2, pp. 233–249, 1994.
- [16] J. S. Pak and W.-H. Sohn, "Some curvature conditions of n -dimensional CR-submanifolds of $(p - 1)$ CR-dimension in a complex projective space," *Bulletin of the Korean Mathematical Society*, vol. 38, no. 3, pp. 575–586, 2001.
- [17] W.-H. Sohn, "Some curvature conditions of n -dimensional CR-submanifolds of $(n-1)$ CR-dimension in a complex projective space," *Communications of the Korean Mathematical Society*, vol. 25, pp. 15–28, 2000.
- [18] J. S. Pak and W.-H. Sohn, "Some curvature conditions of n -dimensional QR-submanifolds of Q QR-dimension in a quaternionic projective space $QP^{(n+p)/4}$," *Bulletin of the Korean Mathematical Society*, vol. 40, no. 4, pp. 613–631, 2003.
- [19] S. Ishihara, "Quaternion Kählerian manifolds," *Journal of Differential Geometry*, vol. 9, pp. 483–500, 1974.
- [20] K. Yano and S. Ishihara, "Fibred spaces with invariant Riemannian metric," *Kōdai Mathematical Seminar Reports*, vol. 19, pp. 317–360, 1967.
- [21] B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, NY, USA, 1973.
- [22] B. O'Neill, "The fundamental equations of a submersion," *The Michigan Mathematical Journal*, vol. 13, pp. 459–469, 1966.