Research Article

Optimality and Duality for Multiobjective Fractional Programming Involving Nonsmooth Generalized $(\mathcal{F}, b, \phi, \rho, \theta)$ -Univex Functions

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We establish properly efficient necessary and sufficient optimality conditions for multiobjective fractional programming involving nonsmooth generalized $(\mathcal{F}, b, \phi, \rho, \theta)$ -univex functions. Utilizing the necessary optimality conditions, we formulate the parametric dual model and establish some duality results in the framework of generalized $(\mathcal{F}, b, \phi, \rho, \theta)$ -univex functions.

1. Introduction

In this paper, we consider the following nondifferentiable nonconvex multiobjective fractional programming problem:

$$\begin{aligned} \text{(MFP)} \quad & \text{Minimize} \quad \left(\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_m(x)}{g_m(x)} \right), \\ & \text{subject to} \quad x \in S = \{x \in \mathbb{R}^n \mid h_j(x) \leq 0, \\ & \quad \quad \quad \text{for } 1 \leq j \leq p\}, \end{aligned} \quad (1)$$

where

- (a1) $f_i : X_0 \mapsto \mathbb{R}$, $g_i : X_0 \mapsto \mathbb{R}$, $i = 1, 2, \dots, m$ and $h_j : X_0 \mapsto \mathbb{R}$, $j = 1, 2, \dots, p$ are Lipschitz on X_0 , and X_0 is an open subset of $\mathbb{R}^n$;
- (a2) $f_i(x) \geq 0$, $g_i(x) > 0$, $i = 1, 2, \dots, m$.

Minimize means obtaining *efficient* solution in the following sense. A point $\bar{x} \in S$ is said to be an *efficient* solution for (MFP) if there is no $x \in S$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(\bar{x})}{g_i(\bar{x})}, \quad 1 \leq i \leq m, \quad (2)$$

with at least one strict inequality.

A point $\bar{x} \in S$ is said to be a *properly efficient* solution for (MFP) which was introduced by Geoffrion [1] if and only if

- (a) $\bar{x}$ is an efficient solution for (MFP),
- (b) there exists a scalar $M > 0$ such that for each i , we have

$$\frac{f_i(\bar{x})/g_i(\bar{x}) - f_i(x)/g_i(x)}{f_j(x)/g_j(x) - f_j(\bar{x})/g_j(\bar{x})} \leq M, \quad (3)$$

for some j such that $f_j(\bar{x})/g_j(\bar{x}) < f_j(x)/g_j(x)$, whenever $x \in S$ and $f_i(\bar{x})/g_i(\bar{x}) > f_i(x)/g_i(x)$.

An efficient point for (MFP) that is not properly efficient is said to be *improperly efficient*. Thus, for $\bar{x}$ to be improperly efficient for (MFP) means that to every scalar $M > 0$, there is a point $x \in S$ and an i such that $f_i(\bar{x})/g_i(\bar{x}) > f_i(x)/g_i(x)$ and

$$\frac{f_i(\bar{x})/g_i(\bar{x}) - f_i(x)/g_i(x)}{f_j(x)/g_j(x) - f_j(\bar{x})/g_j(\bar{x})} > M, \quad (4)$$

for all j such that $f_j(\bar{x})/g_j(\bar{x}) < f_j(x)/g_j(x)$.

Many papers have been devoted to the multiobjective fractional programming problem in recent decades; see for example [1–13].

In [8], Preda introduced generalized $(\mathcal{F}, \rho)$ -convexity, an extension of $\mathcal{F}$ -convexity and generalized ρ -convexity defined by Vial [14, 15]. Bhatia and Jain [2] defined generalized $(\mathcal{F}, \rho)$ -convexity for nonsmooth functions, an extension of generalized $(\mathcal{F}, \rho)$ -convexity defined by Preda [8], and they derived some duality theorems for nonsmooth multiobjective programs. In [5, 6], Liu also established the Kuhn-Tucker type necessary and sufficient optimality conditions for multiobjective fractional programming involving nonsmooth pseudoinvex functions in [5] or $(\mathcal{F}, \rho)$ -convex functions in [6] and considered the parameter dual problem in the framework of generalized convex functions.

Recently, Zalmai [13, 16, 17] introduced generalized $(\mathcal{F}, b, \phi, \rho, \theta)$ -univex n -set functions and he also established sufficient efficiency conditions in multiobjective fractional subset programming [13] and sufficient optimality conditions in minimax fractional subset programming [16, 17] under various generalized $(\mathcal{F}, b, \phi, \rho, \theta)$ -univexity assumptions. In [10], Preda et al. obtained duality results for a dual model of Zalmai [13] replacing the assumptions of sublinearity or convexity by that of quasiconvexity in the third argument.

In this paper, we are inspired to consider the optimality and duality of properly efficient for (MFP) containing nonsmooth generalized $(\mathcal{F}, b, \phi, \rho, \theta)$ -univex functions. We organize this paper as follows. Some definitions and notations are given in Section 2. In Section 3, we establish some properly efficient sufficient optimality conditions for (MFP) involving nonsmooth generalized $(\mathcal{F}, b, \phi, \rho, \theta)$ -univex functions. We also derive some duality results in Section 4.

2. Notations and Preliminary

Throughout the paper, let $\mathbb{R}^n$ be the n -dimensional Euclidean space and let $\mathbb{R}_+^n$ be its nonnegative orthant.

Definition 1. The function $f : X_0 \mapsto \mathbb{R}$ is said to be Lipschitz on X_0 if there exists $c > 0$ such that for all $y, x \in X_0$,

$$|f(y) - f(x)| \leq c \|y - x\|, \quad (5)$$

where $\|\cdot\|$ denotes any norm in $\mathbb{R}^n$.

For each d in $\mathbb{R}^n$, $f^\circ(x; d)$ is the generalized directional derivative of Clarke [18] defined by

$$f^\circ(x; d) = \limsup_{\substack{y \rightarrow x \\ t \downarrow 0}} \frac{f(y + td) - f(y)}{t}. \quad (6)$$

It then follows that

$$f^\circ(x; d) = \max \{\xi^\top d \mid \xi \in \partial f(x)\} \text{ for any } x \text{ and } d. \quad (7)$$

where $\partial f(\cdot)$ denotes Clarke's generalized gradient [18].

Definition 2. A functional $\mathcal{F} : \mathbb{R}^n \mapsto \mathbb{R}$ is said to be sublinear (superlinear) if $\mathcal{F}(a_1 + a_2) \leq (\geq) \mathcal{F}(a_1) + \mathcal{F}(a_2)$ for all $a_1, a_2 \in \mathbb{R}^n$, and $\mathcal{F}(\alpha a) = \alpha \mathcal{F}(a)$ for all $\alpha \in \mathbb{R}_+$ and $a \in \mathbb{R}^n$.

Let X be an open subset of $\mathbb{R}^n$. We suppose that the function $f : X \mapsto \mathbb{R}$ is Lipschitz on X and $d(\cdot, \cdot) : X \times X \mapsto \mathbb{R}$ such that $d(x, x) = 0$ for any x .

Definition 3 (cf. [13, 16, 17, 19, 20]). (1) Let $\mathcal{F}(x, x^*; \cdot) : \mathbb{R}^n \mapsto \mathbb{R}$ be a sublinear function, $b : X \times X \mapsto \mathbb{R}$ a function with positive values, $\phi : \mathbb{R} \mapsto \mathbb{R}$ a function, ρ a real number, and $\theta : X \times X \mapsto X \times X$ a function such that $x \neq x^* \Rightarrow \theta(x, x^*) \neq (0, 0)$. The function f is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$ -univex at x^* if

$$\begin{aligned} \phi(f(x) - f(x^*)) &\geq \mathcal{F}(x, x^*; b(x, x^*)\xi) \\ &\quad + \rho d^2(\theta(x, x^*)) \end{aligned} \quad (8)$$

for all $x \in X$ and $\xi \in \partial f(x^*)$.

(2) The function f is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$ -pseudounivex at x^* if

$$\begin{aligned} \mathcal{F}(x, x^*; b(x, x^*)\xi) &\geq -\rho d^2(\theta(x, x^*)) \\ \Rightarrow \phi(f(x) - f(x^*)) &\geq 0 \end{aligned} \quad (9)$$

for all $x \in X$ and $\xi \in \partial f(x^*)$.

(3) The function f is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$ -strictly-pseudounivex at x^* if

$$\begin{aligned} \mathcal{F}(x, x^*; b(x, x^*)\xi) &\geq -\rho d^2(\theta(x, x^*)) \\ \Rightarrow \phi(f(x) - f(x^*)) &> 0 \end{aligned} \quad (10)$$

for all $x \in X$ and $\xi \in \partial f(x^*)$.

(4) The function f is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$ -prestrictly-pseudounivex at x^* if

$$\begin{aligned} \mathcal{F}(x, x^*; b(x, x^*)\xi) &> -\rho d^2(\theta(x, x^*)) \\ \Rightarrow \phi(f(x) - f(x^*)) &\geq 0 \end{aligned} \quad (11)$$

for all $x \in X$ and $\xi \in \partial f(x^*)$.

(5) The function f is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$ -quasiunivex at x^* if

$$\begin{aligned} \phi(f(x) - f(x^*)) \leq 0 &\Rightarrow \mathcal{F}(x, x^*; b(x, x^*)\xi) \\ &\leq -\rho d^2(\theta(x, x^*)) \end{aligned} \quad (12)$$

for all $x \in X$ and $\xi \in \partial f(x^*)$.

(6) The function f is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$ -strictly-quasiunivex at x^* if

$$\begin{aligned} \phi(f(x) - f(x^*)) \leq 0 &\Rightarrow \mathcal{F}(x, x^*; b(x, x^*)\xi) \\ &< -\rho d^2(\theta(x, x^*)) \end{aligned} \quad (13)$$

for all $x \in X$ and $\xi \in \partial f(x^*)$.

(7) The function f is said to be $(\mathcal{F}, b, \phi, \rho, \theta)$ -prestrictly-quasiunivex at x^* if

$$\begin{aligned} \phi(f(x) - f(x^*)) < 0 &\Rightarrow \mathcal{F}(x, x^*; b(x, x^*)\xi) \\ &\leq -\rho d^2(\theta(x, x^*)) \end{aligned} \quad (14)$$

for all $x \in X$ and $\xi \in \partial f(x^*)$.

In order to establish some Kuhn-Tucker conditions for efficient optimality of (MFP), we give the scalar minimization problem as follows:

$$\begin{aligned}
 (\text{SFP})_i \quad & \text{Minimize} \quad \frac{f_i(x)}{g_i(x)}, \\
 & \text{subject to} \quad \frac{f_k(x)}{g_k(x)} \leq \frac{f_k(\bar{x})}{g_k(\bar{x})}, \quad k \neq i, \\
 & \quad h_j(x) \leq 0, \quad j = 1, 2, \dots, p.
 \end{aligned} \quad (15)$$

In [6], we established the following necessary efficient optimality conditions for (MFP).

Lemma 4. Let $\bar{x} \in S$. If $\bar{x}$ is an efficient solution for (MFP) and satisfies an appropriate constraint qualification [21] for (SFP)_i, $i = 1, 2, \dots, m$. Then there exist $\bar{v}_i = f_i(\bar{x})/g_i(\bar{x}) \geq 0$, $\bar{y}_i > 0$, $1 \leq i \leq m$, and $\bar{\beta} \in \mathbb{R}_+^p$ such that

$$\begin{aligned}
 0 & \in \sum_{i=1}^m \bar{y}_i T_i(\bar{x}) + \sum_{j=1}^p \bar{\beta}_j \partial h_j(\bar{x}), \\
 \bar{\beta}_j h_j(\bar{x}) &= 0, \quad 1 \leq j \leq p,
 \end{aligned} \quad (16)$$

where

$$T_i(\bar{x}) = \frac{\partial f_i(\bar{x}) + \bar{v}_i \partial(-g_i)(\bar{x})}{g_i(\bar{x})}, \quad 1 \leq i \leq m. \quad (17)$$

If we choose $\bar{\lambda}_i = \bar{y}_i/g_i(\bar{x}) > 0$, the result of Lemma 4 can be restated as follows.

Theorem 5 (necessary optimality conditions). Let $\bar{x} \in S$. If $\bar{x}$ is an (properly) efficient solution for (MFP) and satisfies an appropriate constraint qualification [21] for (SFP)_i, $i = 1, 2, \dots, m$. Then there exist $\bar{\lambda}_i > 0$, $1 \leq i \leq m$, $\bar{v} \in \mathbb{R}_+^m$, and $\bar{\beta} \in \mathbb{R}_+^p$ such that

$$0 \in \sum_{i=1}^m \bar{\lambda}_i [\partial f_i(\bar{x}) + \bar{v}_i \partial(-g_i)(\bar{x})] + \sum_{j=1}^p \bar{\beta}_j \partial h_j(\bar{x}), \quad (18)$$

$$f_i(\bar{x}) - \bar{v}_i g_i(\bar{x}) = 0, \quad 1 \leq i \leq m, \quad (19)$$

$$\bar{\beta}_j h_j(\bar{x}) = 0, \quad 1 \leq j \leq p. \quad (20)$$

In order to simplify the complication of (MFP), we consider the following nonfractional multiobjective programming problem with a parameter $v \in \mathbb{R}_+^m$:

$$\begin{aligned}
 (\text{MP})_v \quad & \text{minimize} \quad (f_1(x) - v_1 g_1(x), \\
 & \quad f_2(x) - v_2 g_2(x), \dots, \\
 & \quad f_m(x) - v_m g_m(x)) \\
 & \text{subject to} \quad x \in S.
 \end{aligned} \quad (21)$$

We obtained the equivalence between (MFP) and (MP)_v as follows.

Lemma 6. Let $\bar{x} \in S$ and $\bar{v}_i = f_i(\bar{x})/g_i(\bar{x})$, $1 \leq i \leq m$. Then $\bar{x}$ is a properly efficient solution for (MFP) if and only if $\bar{x}$ is a properly efficient solution for (MP) _{$\bar{v}$} .

Proof. (a) Let $\bar{x}$ be an efficient solution for (MP) _{$\bar{v}$} . Assume that $\bar{x}$ is not an efficient solution for (MFP). Then there exists $x \in S$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(\bar{x})}{g_i(\bar{x})}, \quad 1 \leq i \leq m, \quad (22)$$

with at least one strict inequality. Thus, we have

$$\begin{aligned}
 f_i(x) - \frac{f_i(\bar{x}) g_i(x)}{g_i(\bar{x})} \\
 \leq 0 = f_i(\bar{x}) - \frac{f_i(\bar{x}) g_i(\bar{x})}{g_i(\bar{x})}, \quad 1 \leq i \leq m,
 \end{aligned} \quad (23)$$

or equivalently,

$$f_i(x) - \bar{v}_i g_i(x) \leq f_i(\bar{x}) - \bar{v}_i g_i(\bar{x}), \quad 1 \leq i \leq m, \quad (24)$$

with at least one strict inequality which contradicts the efficient solution of $\bar{x}$ for (MP) _{$\bar{v}$} . Therefore, we have $\bar{x}$ as an efficient solution for (MFP). Conversely, if $\bar{x}$ is an efficient solution for (MFP), along with the same lines of above, we can obtain that $\bar{x}$ is an efficient solution for (MP) _{$\bar{v}$} .

(b) Let $\bar{x}$ be a properly efficient solution for (MFP). Assume that $\bar{x}$ is not a properly efficient solution for (MP) _{$\bar{v}$} . Then every scalar $M > 0$, there is a point $x \in S$ and an i such that $f_i(\bar{x}) - \bar{v}_i g_i(\bar{x}) > f_i(x) - \bar{v}_i g_i(x)$ and

$$\frac{(f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})) - (f_i(x) - \bar{v}_i g_i(x))}{(f_j(x) - \bar{v}_j g_j(x)) - (f_j(\bar{x}) - \bar{v}_j g_j(\bar{x}))} > M \frac{g_i(x)}{g_j(x)}, \quad (25)$$

for all j such that $f_j(\bar{x}) - \bar{v}_j g_j(\bar{x}) < f_j(x) - \bar{v}_j g_j(x)$.

It follows that for some i such that $0 > f_i(x) - f_i(\bar{x})/g_i(\bar{x})g_i(x)$ and

$$\frac{(f_i(\bar{x})/g_i(\bar{x}))g_i(x) - f_i(x)}{f_j(x) - (f_j(\bar{x})/g_j(\bar{x}))g_j(x)} > M \frac{g_i(x)}{g_j(x)}, \quad (26)$$

for all j such that $0 < f_j(x) - f_j(\bar{x})/g_j(\bar{x})g_j(x)$, or equivalently, for some i such that $f_i(\bar{x})/g_i(\bar{x}) > f_i(x)/g_i(x)$ and

$$\frac{f_i(\bar{x})/g_i(\bar{x}) - f_i(x)/g_i(x)}{f_j(x)/g_j(x) - f_j(\bar{x})/g_j(\bar{x})} > M, \quad (27)$$

for all j such that $f_j(\bar{x})/g_j(\bar{x}) < f_j(x)/g_j(x)$, which contradicts the properly efficient solution of $\bar{x}$ for (MFP). Therefore, we have $\bar{x}$ is a properly efficient solution for (MP) _{$\bar{v}$} . Conversely, if $\bar{x}$ is a properly efficient solution for (MP) _{$\bar{v}$} , along with the same lines of above, we can obtain that $\bar{x}$ is a properly efficient solution for (MFP). Thus, the proof is complete. $\square$

Lemma 7. Let $\bar{x} \in S$ and $\bar{v}_i = f_i(\bar{x})/g_i(\bar{x})$, $1 \leq i \leq m$. If there exists $\bar{\lambda} \in \mathbb{R}_+^m \setminus \{0\}$ such that

$$\sum_{i=1}^m \bar{\lambda}_i (f_i(x) - \bar{v}_i g_i(x)) \geq 0, \quad \text{for any } x \in S, \quad (28)$$

then $\bar{x}$ is a properly efficient solution to (MFP).

Proof. Assume that $\bar{x}$ is not an efficient for (MFP). Then there exists a point $x \in S$ such that

$$\frac{f_i(x)}{g_i(x)} \leq \frac{f_i(\bar{x})}{g_i(\bar{x})}, \quad 1 \leq i \leq m, \quad (29)$$

with at least one strict inequality. Thus, we have

$$f_i(x) - \frac{f_i(\bar{x})}{g_i(\bar{x})} g_i(x) \leq 0, \quad 1 \leq i \leq m, \quad (30)$$

or equivalently,

$$f_i(x) - \bar{v}_i g_i(x) \leq 0, \quad 1 \leq i \leq m, \quad (31)$$

with at least one strict inequality.

Thus, for any $\lambda \in \mathbb{R}_+^m \setminus \{0\}$, we have

$$\sum_{i=1}^m \lambda_i (f_i(x) - \bar{v}_i g_i(x)) < 0, \quad (32)$$

which contradicts the inequality (28).

It remains to show that $\bar{x}$ is properly efficient. Assume that it is an improperly efficient point for the problem (MFP). Then, by Lemma 6, $\bar{x}$ is an improperly efficient solution for (MP) $_{\bar{v}}$. Then every scalar $M > 0$, there is a point $x \in S$ and an i such that $f_i(\bar{x}) - \bar{v}_i g_i(\bar{x}) > f_i(x) - \bar{v}_i g_i(x)$ and

$$\frac{(f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})) - (f_i(x) - \bar{v}_i g_i(x))}{(f_j(x) - \bar{v}_j g_j(x)) - (f_j(\bar{x}) - \bar{v}_j g_j(\bar{x}))} > M, \quad (33)$$

for all j such that $f_j(\bar{x}) - \bar{v}_j g_j(\bar{x}) < f_j(x) - \bar{v}_j g_j(x)$.

For any $\lambda \in \mathbb{R}_+^m \setminus \{0\}$, we choose $M = (m-1) \max_{i,j} \{\lambda_j / \lambda_i\} > 0$, (we may assume that $m \geq 2$) such that

$$\begin{aligned} & [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] - [f_i(x) - \bar{v}_i g_i(x)] \\ & > (m-1) \frac{\lambda_j}{\lambda_i} [(f_j(x) - \bar{v}_j g_j(x)) - (f_j(\bar{x}) - \bar{v}_j g_j(\bar{x}))], \end{aligned} \quad (34)$$

for all $j \neq i$. These imply that

$$\frac{\lambda_i}{m-1} [\bar{v}_i g_i(x) - f_i(x)] > \lambda_j [f_j(x) - \bar{v}_j g_j(x)], \quad \forall j \neq i. \quad (35)$$

Summing over $j \neq i$ yield

$$\lambda_i [\bar{v}_i g_i(x) - f_i(x)] > \sum_{j \neq i} \lambda_j [f_j(x) - \bar{v}_j g_j(x)], \quad (36)$$

or

$$\sum_{i=1}^m \lambda_i [f_i(x) - \bar{v}_i g_i(x)] < 0, \quad (37)$$

which contradicts the inequality (28). Thus, the proof is complete. $\square$

3. Sufficient Optimality Conditions

Applying Lemmas 6 and 7, we can establish the following sufficient optimality conditions for properly efficient solution of (MFP).

Let

$$P = \{1, 2, \dots, p\}, \quad P(x) = \{j \in P \mid h_j(x) = 0\}. \quad (38)$$

Theorem 8 (sufficient optimality conditions). *Let $\bar{x} \in S$, and let there exist $\bar{\lambda}_i > 0$, $1 \leq i \leq m$, $\bar{v} \in \mathbb{R}_+^m$, and $\bar{\beta} \in \mathbb{R}_+^p$ such that (18), (19), and (20). Let*

$$A(x) = \sum_{i=1}^m \bar{\lambda}_i [f_i(x) + \bar{v}_i (-g_i)(x)] + \sum_{j=1}^p \bar{\beta}_j h_j(x),$$

$$B(x) = \sum_{i=1}^m \bar{\lambda}_i [f_i(x) + \bar{v}_i (-g_i)(x)], \quad (39)$$

$$C(x) = \sum_{j=1}^p \bar{\beta}_j h_j(x).$$

If any one of the following conditions holds:

(a)

- (i) f_i is $(\mathcal{F}, b, \phi_1, \rho_{1i}, \theta)$ -univex at $\bar{x}$, $-g_i$ is $(\mathcal{F}, b, \phi_1, \rho_{2i}, \theta)$ -univex at $\bar{x}$, $1 \leq i \leq m$, ϕ_1 is superlinear, and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
- (ii) for each $j \in P(\bar{x})$, h_j is $(\mathcal{F}, b, \phi_2, \rho_{3j}, \theta)$ -quasi-univex at $\bar{x}$, ϕ_2 is increasing, and $\phi_2(0) = 0$;
- (iii) $\sum_{i=1}^m \bar{\lambda}_i \rho_{1i} + \sum_{i=1}^m \bar{\lambda}_i \bar{v}_i \rho_{2i} + \sum_{j \in P(\bar{x})} \bar{\beta}_j \rho_{3j} \geq 0$;

(b)

- (i) f_i is $(\mathcal{F}, b, \phi_1, \rho_{1i}, \theta)$ -univex at $\bar{x}$, $-g_i$ is $(\mathcal{F}, b, \phi_1, \rho_{2i}, \theta)$ -univex at $\bar{x}$, $1 \leq i \leq m$, ϕ_1 is superlinear, and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
- (ii) C is $(\mathcal{F}, b, \phi_2, \rho_3, \theta)$ -quasiunivex at $\bar{x}$, ϕ_2 is increasing, and $\phi_2(0) = 0$;
- (iii) $\sum_{i=1}^m \bar{\lambda}_i \rho_{1i} + \sum_{i=1}^m \bar{\lambda}_i \bar{v}_i \rho_{2i} + \rho_3 \geq 0$;

(c)

- (i) A is $(\mathcal{F}, b, \phi, 0, \theta)$ -pseudounivex at $\bar{x}$ and $\phi(a) \geq 0 \Rightarrow a \geq 0$;

(d)

- (i) B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -pseudounivex at $\bar{x}$, and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
- (ii) C is $(\mathcal{F}, b, \phi_2, \rho_2, \theta)$ -quasiunivex at $\bar{x}$, ϕ_2 is increasing, and $\phi_2(0) = 0$;
- (iii) $\rho_1 + \rho_2 \geq 0$;

(e)

- (i) B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -prestrictly-quasiunivex at $\bar{x}$, and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
- (ii) C is $(\mathcal{F}, b, \phi_2, \rho_2, \theta)$ -quasiunivex at $\bar{x}$, ϕ_2 is increasing, and $\phi_2(0) = 0$;
- (iii) $\rho_1 + \rho_2 > 0$;

(f)

- (i) B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -prestrictly-quasiunivex at $\bar{x}$, and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
- (ii) C is $(\mathcal{F}, b, \phi_2, \rho_2, \theta)$ -strictly-pseudounivex at $\bar{x}$, ϕ_2 is increasing, and $\phi_2(0) = 0$;
- (iii) $\rho_1 + \rho_2 \geq 0$;

(g)

- (i) B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -pseudounivex at $\bar{x}$, and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
- (ii) for each $j \in P(\bar{x})$, h_j is $(\mathcal{F}, b, \phi_{2j}, \rho_{2j}, \theta)$ -quasiunivex at $\bar{x}$, ϕ_{2j} is increasing, and $\phi_{2j}(0) = 0$;
- (iii) $\rho_1 + \sum_{j \in P(\bar{x})} \bar{\beta}_j \rho_{2j} \geq 0$.

Then $\bar{x}$ is a properly efficient solution for (MFP).

Proof. Let x be any feasible solution of (MFP). By (18), there exist $\xi_i \in \partial f_i(\bar{x})$, $\varsigma_i \in \partial(-g_i)(\bar{x})$, for $1 \leq i \leq m$, and $\zeta_j \in \partial h_j(\bar{x})$, for $1 \leq j \leq p$ such that

$$\sum_{i=1}^m \bar{\lambda}_i (\xi_i + \bar{v}_i \varsigma_i) + \sum_{j=1}^p \bar{\beta}_j \zeta_j = 0. \quad (40)$$

From here it results

$$\mathcal{F} \left(x, \bar{x}; \sum_{i=1}^m \bar{\lambda}_i (\xi_i + \bar{v}_i \varsigma_i) + \sum_{j=1}^p \bar{\beta}_j \zeta_j \right) \geq 0. \quad (41)$$

If hypothesis (a) holds,

$$\begin{aligned} \phi_1(f_i(x) - f_i(\bar{x})) &\geq \mathcal{F}(x, \bar{x}; b(x, \bar{x}) \xi_i) \\ &\quad + \rho_{1i} d^2(\theta(x, \bar{x})), \quad 1 \leq i \leq m, \end{aligned} \quad (42)$$

$$\begin{aligned} \phi_1(-g_i(x) + g_i(\bar{x})) &\geq \mathcal{F}(x, \bar{x}; b(x, \bar{x}) \varsigma_i) \\ &\quad + \rho_{2i} d^2(\theta(x, \bar{x})), \quad 1 \leq i \leq m. \end{aligned} \quad (43)$$

Now, multiplying (42) by $\bar{\lambda}_i$, (43) by $\bar{\lambda}_i \bar{v}_i$, and adding the resulting inequalities, and then using the superlinearity of ϕ_1 and sublinearity of $\mathcal{F}(x, \bar{x}; \cdot)$, we obtain

$$\begin{aligned} &\phi_1 \left(\sum_{i=1}^m \bar{\lambda}_i [f_i(x) - \bar{v}_i g_i(x)] - \sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] \right) \\ &\geq \mathcal{F} \left(x, \bar{x}; b(x, \bar{x}) \sum_{i=1}^m \bar{\lambda}_i (\xi_i + \bar{v}_i \varsigma_i) \right) \\ &\quad + \left(\sum_{i=1}^m \bar{\lambda}_i \rho_{1i} + \sum_{i=1}^m \bar{\lambda}_i \bar{v}_i \rho_{2i} \right) d^2(\theta(x, \bar{x})). \end{aligned} \quad (44)$$

Since $x \in S$, it is clear that

$$h_j(x) - h_j(\bar{x}) \leq 0, \quad \forall j \in P(\bar{x}). \quad (45)$$

Using the increasing property of ϕ_{2j} and $\phi_{2j}(0) = 0$, we have

$$\phi_{2j}(h_j(x) - h_j(\bar{x})) \leq 0, \quad \forall j \in P(\bar{x}). \quad (46)$$

Using the $(\mathcal{F}, b, \phi_{2j}, \rho_{3j}, \theta)$ -quasiunivexity of h_j at $\bar{x}$, we get from (46)

$$\mathcal{F}(x, \bar{x}; b(x, \bar{x}) \zeta_j) \leq -\rho_{3j} d^2(\theta(x, \bar{x})), \quad \forall j \in P(\bar{x}). \quad (47)$$

Because $\bar{\beta} \in \mathbb{R}_+^p$ and (20) holds, $\bar{\beta}_j = 0$ for each $j \in P \setminus P(\bar{x})$, and $\mathcal{F}(x, \bar{x}; \cdot)$ is sublinear, the above inequality yields

$$\mathcal{F} \left(x, \bar{x}; b(x, \bar{x}) \sum_{j=1}^p \bar{\beta}_j \zeta_j \right) \leq - \left(\sum_{j \in P(\bar{x})} \bar{\beta}_j \rho_{3j} \right) d^2(\theta(x, \bar{x})). \quad (48)$$

From the nonnegativity of $b(x, \bar{x})$, sublinearity of $\mathcal{F}(x, \bar{x}; \cdot)$, and (41), it follows that

$$\begin{aligned} &\mathcal{F} \left(x, \bar{x}; b(x, \bar{x}) \sum_{i=1}^m \bar{\lambda}_i (\xi_i + \bar{v}_i \varsigma_i) \right) \\ &\quad + \mathcal{F} \left(x, \bar{x}; b(x, \bar{x}) \sum_{j=1}^p \bar{\beta}_j \zeta_j \right) \geq 0. \end{aligned} \quad (49)$$

Now adding (44) and (48), and then using (49) and $\sum_{i=1}^m \bar{\lambda}_i \rho_{1i} + \sum_{i=1}^m \bar{\lambda}_i \bar{v}_i \rho_{2i} + \sum_{j \in P(\bar{x})} \bar{\beta}_j \rho_{3j} \geq 0$, we obtain

$$\phi_1 \left(\sum_{i=1}^m \bar{\lambda}_i [f_i(x) - \bar{v}_i g_i(x)] - \sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] \right) \geq 0. \quad (50)$$

But $\phi_1(a) \geq 0 \Rightarrow a \geq 0$, and so (50) yields

$$\sum_{i=1}^m \bar{\lambda}_i [f_i(x) - \bar{v}_i g_i(x)] - \sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] \geq 0. \quad (51)$$

Thus, we have

$$\sum_{i=1}^m \bar{\lambda}_i [f_i(x) + \bar{v}_i (-g_i(x))] \geq \sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}) + \bar{v}_i (-g_i(\bar{x}))] = 0. \quad (52)$$

Therefore, by Lemma 7, $\bar{x}$ is a properly efficient solution for (MFP).

If hypothesis (b) holds, from (20), $x \in S$, and $\bar{\beta} \in \mathbb{R}_+^p$, we have

$$\sum_{j=1}^p \bar{\beta}_j h_j(x) \leq 0 = \sum_{j=1}^p \bar{\beta}_j h_j(\bar{x}). \quad (53)$$

and so using the properties of ϕ_2 , we obtain

$$\phi_2 \left(\sum_{j=1}^p \bar{\beta}_j [h_j(x) - h_j(\bar{x})] \right) \leq 0. \quad (54)$$

Using the $(\mathcal{F}, b, \phi_2, \rho_3, \theta)$ -quasiunivexity of C at $\bar{x}$, we get from (54)

$$\mathcal{F}\left(x, \bar{x}; b(x, \bar{x}) \sum_{j=1}^p \bar{\beta}_j \zeta_j\right) \leq -\rho_3 d^2(\theta(x, \bar{x})). \quad (55)$$

Now combining (44), (49), and (55), and using (iii), we obtain (52). Therefore, by Lemma 7, $\bar{x}$ is a properly efficient solution for (MFP).

If hypothesis (c) holds, using the $(\mathcal{F}, b, \phi, 0, \theta)$ -pseudounivexity of A , we get from (41)

$$\begin{aligned} & \phi\left(\sum_{i=1}^m \bar{\lambda}_i [f_i(x) - \bar{v}_i g_i(x)] + \sum_{j=1}^p \bar{\beta}_j h_j(x) \right. \\ & \quad \left. - \left(\sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] + \sum_{j=1}^p \bar{\beta}_j h_j(\bar{x})\right)\right) \geq 0. \end{aligned} \quad (56)$$

In view of the properties of ϕ , we deduce from this inequality that

$$\begin{aligned} & \sum_{i=1}^m \bar{\lambda}_i [f_i(x) - \bar{v}_i g_i(x)] + \sum_{j=1}^p \bar{\beta}_j h_j(x) \\ & \quad - \left(\sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})] + \sum_{j=1}^p \bar{\beta}_j h_j(\bar{x})\right) \geq 0. \end{aligned} \quad (57)$$

Equation (53) along with (57) yields

$$\sum_{i=1}^m \bar{\lambda}_i [f_i(x) + \bar{v}_i (-g_i(x))] \geq \sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}) + \bar{v}_i (-g_i(\bar{x}))] = 0. \quad (58)$$

Therefore, by Lemma 7, $\bar{x}$ is a properly efficient solution for (MFP).

If hypothesis (d) holds, using the properties of ϕ_2 and (53), we obtain

$$\phi_2\left(\sum_{j=1}^p \bar{\beta}_j [h_j(x) - h_j(\bar{x})]\right) \leq 0, \quad (59)$$

which in view of (ii) implies that

$$\mathcal{F}\left(x, \bar{x}; b(x, \bar{x}) \sum_{j=1}^p \bar{\beta}_j \zeta_j\right) \leq -\rho_2 d^2(\theta(x, \bar{x})). \quad (60)$$

Now combining (49), (60), and $\rho_1 + \rho_2 \geq 0$, we obtain

$$\begin{aligned} & \mathcal{F}\left(x, \bar{x}; b(x, \bar{x}) \sum_{i=1}^m \bar{\lambda}_i (\xi_i + \bar{v}_i \zeta_i)\right) \\ & \geq -\mathcal{F}\left(x, \bar{x}; b(x, \bar{x}) \sum_{j=1}^p \bar{\beta}_j \zeta_j\right) \\ & \geq \rho_2 d^2(\theta(x, \bar{x})) \geq -\rho_1 d^2(\theta(x, \bar{x})). \end{aligned} \quad (61)$$

Along with the fact that the B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -pseudounivex at $\bar{x}$, we get from (61) that

$$\phi_1\left(\sum_{i=1}^m \bar{\lambda}_i [f_i(x) - \bar{v}_i g_i(x)] - \sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}) - \bar{v}_i g_i(\bar{x})]\right) \geq 0. \quad (62)$$

In view of the properties of ϕ_1 , we deduce from this inequality that

$$\sum_{i=1}^m \bar{\lambda}_i [f_i(x) + \bar{v}_i (-g_i(x))] \geq \sum_{i=1}^m \bar{\lambda}_i [f_i(\bar{x}) + \bar{v}_i (-g_i(\bar{x}))] = 0. \quad (63)$$

Therefore, by Lemma 7, $\bar{x}$ is a properly efficient solution for (MFP). (e), (f), and (g) follows along with the same lines of (d) and (a). Thus, the proof is complete. $\square$

4. Duality Theorem

From the optimality of properly efficient for the problem (MFP), we can formulate the following parametric dual problem:

$$\begin{aligned} (D) \quad & \text{maximize} \quad v = (v_1, v_2, \dots, v_m) \\ & \text{subject to} \quad 0 \in \sum_{i=1}^m \lambda_i [\partial f_i(u) + v_i \partial (-g_i)(u)] \\ & \quad + \sum_{j=1}^p \beta_j \partial h_j(u), \end{aligned} \quad (64)$$

$$f_i(u) - v_i g_i(u) \geq 0, \quad 1 \leq i \leq m, \quad (65)$$

$$\beta_j h_j(u) \geq 0, \quad 1 \leq j \leq p, \quad (66)$$

$$\begin{aligned} u \in \mathbb{R}^n, \quad v \in \mathbb{R}_+^m, \quad \beta \in \mathbb{R}_+^p, \\ \lambda_i > 0, \quad 1 \leq i \leq m. \end{aligned} \quad (67)$$

We denote by F_D the set of all feasible solutions (u, v, λ, β) of problem (D). The point $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\beta}) \in F_D$ is called an efficient for (D) if there is no $(u, v, \lambda, \beta) \in F_D$ such that

$$v_i \geq \bar{v}_i, \quad 1 \leq i \leq m, \quad (68)$$

with at least one strict inequality.

The point $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\beta}) \in F_D$ is said to be a properly efficient for (D) if and only if

- (a) $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\beta})$ is an efficient solution for (D),
- (b) there exists a scalar $M > 0$ such that for each i , we have

$$\frac{v_i - \bar{v}_i}{\bar{v}_j - v_j} \leq M, \quad (69)$$

for some j such that $v_j < \bar{v}_j$, whenever $(u, v, \lambda, \beta) \in F_D$ and $v_i > \bar{v}_i$.

If $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{\beta})$ was not properly efficient solution for (D) , for each fixed real number $M > 0$, there would exist a feasible solution $(u, v, \lambda, \beta) \in F_D$ and an index i such that $v_i > \bar{v}_i$ and

$$\frac{v_i - \bar{v}_i}{\bar{v}_j - v_j} > M, \quad (70)$$

for all j such that $v_j < \bar{v}_j$.

Theorem 9 (duality theorem). Let $\bar{x} \in S$, $(u, v, \lambda, \beta) \in F_D$, and let

$$\begin{aligned} A(\cdot) &= \sum_{i=1}^m \lambda_i [f_i(\cdot) + v_i(-g_i)(\cdot)] + \sum_{j=1}^p \beta_j h_j(\cdot), \\ B(\cdot) &= \sum_{i=1}^m \lambda_i [f_i(\cdot) + v_i(-g_i)(\cdot)], \\ C(\cdot) &= \sum_{j=1}^p \beta_j h_j(\cdot). \end{aligned} \quad (71)$$

If any one of the following conditions holds:

- (a)
- (i) f_i is $(\mathcal{F}, b, \phi_1, \rho_{1i}, \theta)$ -univex at u , $-g_i$ is $(\mathcal{F}, b, \phi_1, \rho_{2i}, \theta)$ -univex at u , $1 \leq i \leq m$, ϕ_1 is superlinear, and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
 - (ii) $\beta_j h_j$ is $(\mathcal{F}, b, \phi_{2j}, \rho_{3j}, \theta)$ -quasiunivex at u , $1 \leq j \leq p$, ϕ_{2j} is increasing, and $\phi_{2j}(0) = 0$;
 - (iii) $\sum_{i=1}^m \lambda_i \rho_{1i} + \sum_{i=1}^m \lambda_i v_i \rho_{2i} + \sum_{j=1}^p \rho_{3j} \geq 0$;
- (b)
- (i) f_i is $(\mathcal{F}, b, \phi_1, \rho_{1i}, \theta)$ -univex at u , $-g_i$ is $(\mathcal{F}, b, \phi_1, \rho_{2i}, \theta)$ -univex at u , $1 \leq i \leq m$, ϕ_1 is superlinear, and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
 - (ii) C is $(\mathcal{F}, b, \phi_2, \rho_3, \theta)$ -quasiunivex at u , ϕ_2 is increasing, and $\phi_2(0) = 0$;
 - (iii) $\sum_{i=1}^m \lambda_i \rho_{1i} + \sum_{i=1}^m \lambda_i v_i \rho_{2i} + \rho_3 \geq 0$;
- (c)
- (i) A is $(\mathcal{F}, b, \phi, \theta)$ -pseudounivex at u and $\phi(a) \geq 0 \Rightarrow a \geq 0$.
- (d)
- (i) B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -pseudounivex at u , and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
 - (ii) C is $(\mathcal{F}, b, \phi_2, \rho_2, \theta)$ -quasiunivex at u , ϕ_2 is increasing, and $\phi_2(0) = 0$;
 - (iii) $\rho_1 + \rho_2 \geq 0$;
- (e)
- (i) B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -prestrictly-quasiunivex at u , and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
 - (ii) C is $(\mathcal{F}, b, \phi_2, \rho_2, \theta)$ -quasiunivex at u , ϕ_2 is increasing, and $\phi_2(0) = 0$;
 - (iii) $\rho_1 + \rho_2 > 0$;

(f)

- (i) B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -prestrictly-quasiunivex at u , and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
- (ii) C is $(\mathcal{F}, b, \phi_2, \rho_2, \theta)$ -strictly-pseudounivex at u , ϕ_2 is increasing, and $\phi_2(0) = 0$;
- (iii) $\rho_1 + \rho_2 \geq 0$;

(g)

- (i) B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -pseudounivex at u , and $\phi_1(a) \geq 0 \Rightarrow a \geq 0$;
- (ii) $\beta_j h_j$ is $(\mathcal{F}, b, \phi_{2j}, \rho_{2j}, \theta)$ -quasiunivex at u , $1 \leq j \leq p$, ϕ_{2j} is increasing, and $\phi_{2j}(0) = 0$;
- (iii) $\rho_1 + \sum_{j=1}^p \rho_{2j} \geq 0$.

Assume that $\bar{x}$ is a properly efficient solution for (MFP) and satisfies an appropriate constraint qualification [21] for (SFP) _{i} , $i = 1, 2, \dots, m$. Then there exist $\bar{\lambda}_i > 0$, $1 \leq i \leq m$, $\bar{v} \in \mathbb{R}_+^m$, and $\bar{\beta} \in \mathbb{R}_+^p$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\beta})$ is a properly efficient solution for (D) .

Proof. With Theorem 5, there exist $\bar{\lambda}_i > 0$, $1 \leq i \leq m$, $\bar{v} \in \mathbb{R}_+^m$, and $\bar{\beta} \in \mathbb{R}_+^p$ such that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\beta})$ is a feasible solution of (D) . By (64), there exist $\xi_i \in \partial f_i(u)$, $\varsigma_i \in \partial(-g_i)(u)$, for $1 \leq i \leq m$, and $\zeta_j \in \partial h_j(u)$, for $1 \leq j \leq p$ such that

$$\sum_{i=1}^m \lambda_i (\xi_i + v_i \varsigma_i) + \sum_{j=1}^p \beta_j \zeta_j = 0. \quad (72)$$

From here it results

$$\mathcal{F} \left(\bar{x}, u; \sum_{i=1}^m \lambda_i (\xi_i + v_i \varsigma_i) + \sum_{j=1}^p \beta_j \zeta_j \right) \geq 0. \quad (73)$$

If hypothesis (a) holds, then

$$\begin{aligned} \phi_1(f_i(\bar{x}) - f_i(u)) &\geq \mathcal{F}(\bar{x}, u; b(\bar{x}, u) \xi_i) \\ &\quad + \rho_{1i} d^2(\theta(\bar{x}, u)), \quad 1 \leq i \leq m, \end{aligned} \quad (74)$$

$$\begin{aligned} \phi_1(-g_i(\bar{x}) + g_i(u)) &\geq \mathcal{F}(\bar{x}, u; b(\bar{x}, u) \varsigma_i) \\ &\quad + \rho_{2i} d^2(\theta(\bar{x}, u)), \quad 1 \leq i \leq m. \end{aligned} \quad (75)$$

Now, multiplying (74) by λ_i , (75) by $\lambda_i v_i$, and adding the resulting inequalities, and then using the superlinearity of ϕ_1 and sublinearity of $\mathcal{F}(\bar{x}, u; \cdot)$, we obtain

$$\begin{aligned} \phi_1 \left(\sum_{i=1}^m \lambda_i [f_i(\bar{x}) - v_i g_i(\bar{x})] - \sum_{i=1}^m \lambda_i [f_i(u) - v_i g_i(u)] \right) \\ \geq \mathcal{F} \left(\bar{x}, u; b(\bar{x}, u) \sum_{i=1}^m \lambda_i (\xi_i + v_i \varsigma_i) \right) \\ + \left(\sum_{i=1}^m \lambda_i \rho_{1i} + \sum_{i=1}^m \lambda_i v_i \rho_{2i} \right) d^2(\theta(\bar{x}, u)). \end{aligned} \quad (76)$$

Since $\bar{x} \in S$, $\beta \in \mathbb{R}_+^p$, and (66), it is clear that

$$\beta_j h_j(\bar{x}) - \beta_j h_j(u) \leq 0, \quad 1 \leq j \leq p. \quad (77)$$

Using the increasing property of ϕ_{2j} and $\phi_{2j}(0) = 0$, we have

$$\phi_{2j}(\beta_j h_j(\bar{x}) - \beta_j h_j(u)) \leq 0, \quad 1 \leq j \leq p. \quad (78)$$

Using the $(\mathcal{F}, b, \phi_{2j}, \rho_{3j}, \theta)$ -quasiunivexity of $\beta_j h_j$ at u , we get from (78)

$$\mathcal{F}(\bar{x}, u; b(\bar{x}, u) \beta_j \zeta_j) \leq -\rho_{3j} d^2(\theta(\bar{x}, u)), \quad 1 \leq j \leq p. \quad (79)$$

Because $\beta \in \mathbb{R}_+^p$ the nonnegativity of $b(\bar{x}, u)$, and $\mathcal{F}(\bar{x}, u; \cdot)$ is sublinear, the above inequality yields

$$\mathcal{F}\left(\bar{x}, u; b(\bar{x}, u) \sum_{j=1}^p \beta_j \zeta_j\right) \leq -\left(\sum_{j=1}^p \rho_{3j}\right) d^2(\theta(\bar{x}, u)). \quad (80)$$

From the nonnegativity of $b(\bar{x}, u)$, sublinearity of $\mathcal{F}(\bar{x}, u; \cdot)$, and (73), it follows that

$$\begin{aligned} & \mathcal{F}\left(\bar{x}, u; b(\bar{x}, u) \sum_{i=1}^m \lambda_i (\xi_i + v_i \zeta_i)\right) \\ & + \mathcal{F}\left(\bar{x}, u; b(\bar{x}, u) \sum_{j=1}^p \beta_j \zeta_j\right) \geq 0. \end{aligned} \quad (81)$$

Now adding (76) and (80), and then using (81) and $\sum_{i=1}^m \lambda_i \rho_{1i} + \sum_{i=1}^m \lambda_i v_i \rho_{2i} + \sum_{j=1}^p \rho_{3j} \geq 0$, we obtain

$$\phi_1\left(\sum_{i=1}^m \lambda_i [f_i(\bar{x}) - v_i g_i(\bar{x})] - \sum_{i=1}^m \lambda_i [f_i(u) - v_i g_i(u)]\right) \geq 0. \quad (82)$$

But $\phi_1(a) \geq 0 \Rightarrow a \geq 0$, and so (82) yields

$$\sum_{i=1}^m \lambda_i [f_i(\bar{x}) - v_i g_i(\bar{x})] - \sum_{i=1}^m \lambda_i [f_i(u) - v_i g_i(u)] \geq 0. \quad (83)$$

Equation (65) along with (83) yields

$$\sum_{i=1}^m \lambda_i [f_i(\bar{x}) + v_i (-g_i(\bar{x}))] \geq \sum_{i=1}^m \lambda_i [f_i(u) + v_i (-g_i(u))] \geq 0. \quad (84)$$

Assume that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\beta})$ is not an efficient for (D) . Then there exists a point $(u, v, \lambda, \beta) \in \mathcal{F}_D$ such that

$$v_i \geq \bar{v}_i = \frac{f_i(\bar{x})}{g_i(\bar{x})}, \quad 1 \leq i \leq m, \quad (85)$$

with at least one strict inequality.

Thus, for any $\lambda \in \mathbb{R}_+^m \setminus \{0\}$, we have

$$\sum_{i=1}^m \lambda_i [f_i(\bar{x}) - v_i g_i(\bar{x})] < 0, \quad (86)$$

which contradicts inequality (84).

It remains to show that $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\beta})$ is properly efficient. Assume that it is an improperly efficient point for the problem (D) and choose $M = (m-1)(\lambda_j/\lambda_i)(g_j(\bar{x})/g_i(\bar{x})) > 0$ (we may assume that $m \geq 2$), there would exist a feasible solution $(u, v, \lambda, \beta) \in F_D$ and an index i such that $v_i > \bar{v}_i$ and

$$v_i - \bar{v}_i > M [\bar{v}_j - v_j] = (m-1) \frac{\lambda_j}{\lambda_i} \frac{g_j(\bar{x})}{g_i(\bar{x})} [\bar{v}_j - v_j], \quad (87)$$

for all j such that $v_j < \bar{v}_j$.

These imply that

$$\begin{aligned} & g_i(\bar{x}) \left[v_i - \frac{f_i(\bar{x})}{g_i(\bar{x})} \right] \\ & > (m-1) \frac{\lambda_j}{\lambda_i} g_j(\bar{x}) \left[\frac{f_j(\bar{x})}{g_j(\bar{x})} - v_j \right], \quad \forall j \neq i. \end{aligned} \quad (88)$$

Thus, we have

$$\frac{\lambda_i}{m-1} [v_i g_i(\bar{x}) - f_i(\bar{x})] > \lambda_j [f_j(\bar{x}) - v_j g_j(\bar{x})], \quad \forall j \neq i. \quad (89)$$

Summing over $j \neq i$ yield

$$\lambda_i [v_i g_i(\bar{x}) - f_i(\bar{x})] > \sum_{j \neq i} \lambda_j [f_j(\bar{x}) - v_j g_j(\bar{x})], \quad (90)$$

or

$$\sum_{i=1}^m \lambda_i [f_i(\bar{x}) - v_i g_i(\bar{x})] < 0, \quad (91)$$

which contradicts inequality (84). Thus, the proof of (a) is complete.

If hypothesis (b) holds, from (66), $\bar{x} \in S$, and $\beta \in \mathbb{R}_+^p$, then we have

$$\sum_{j=1}^p \beta_j h_j(\bar{x}) \leq 0 \leq \sum_{j=1}^p \beta_j h_j(u). \quad (92)$$

and so using the properties of ϕ_2 , we obtain

$$\phi_2\left(\sum_{j=1}^p \beta_j [h_j(\bar{x}) - h_j(u)]\right) \leq 0. \quad (93)$$

Using the $(\mathcal{F}, b, \phi_2, \rho_3, \theta)$ -quasiunivexity of C at u , then we get from (93)

$$\mathcal{F}\left(\bar{x}, u; b(\bar{x}, u) \sum_{j=1}^p \beta_j \zeta_j\right) \leq -\rho_3 d^2(\theta(\bar{x}, u)). \quad (94)$$

Now combining (76), (81), and (94), and using (iii), we obtain (84). Therefore, along with the same lines of (a), we obtain $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\beta})$ is a properly efficient solution for (D) .

If hypothesis (c) holds, using the $(\mathcal{F}, b, \phi, 0, \theta)$ -pseudoinvexity of A , we get from (73)

$$\phi \left(\sum_{i=1}^m \lambda_i [f_i(\bar{x}) - v_i g_i(\bar{x})] + \sum_{j=1}^p \beta_j h_j(\bar{x}) - \left(\sum_{i=1}^m \lambda_i [f_i(u) - v_i g_i(u)] + \sum_{j=1}^p \beta_j h_j(u) \right) \right) \geq 0. \quad (95)$$

In view of the properties of ϕ , we deduce from this inequality that

$$\sum_{i=1}^m \lambda_i [f_i(\bar{x}) - v_i g_i(\bar{x})] + \sum_{j=1}^p \beta_j h_j(\bar{x}) - \left(\sum_{i=1}^m \lambda_i [f_i(u) - v_i g_i(u)] + \sum_{j=1}^p \beta_j h_j(u) \right) \geq 0. \quad (96)$$

Equation (92) and (65) along with (96) yields

$$\sum_{i=1}^m \lambda_i [f_i(\bar{x}) + v_i (-g_i(\bar{x}))] \geq \sum_{i=1}^m \lambda_i [f_i(u) + v_i (-g_i(u))] \geq 0. \quad (97)$$

Thus, we obtain $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\beta})$ is a properly efficient solution for (D) .

If hypothesis (d) holds, using the properties of ϕ_2 and (92), we obtain

$$\phi_2 \left(\sum_{j=1}^p \beta_j [h_j(\bar{x}) - h_j(u)] \right) \leq 0, \quad (98)$$

which in view of (ii) implies that

$$\mathcal{F} \left(\bar{x}, u; b(\bar{x}, u) \sum_{j=1}^p \beta_j \zeta_j \right) \leq -\rho_2 d^2(\theta(\bar{x}, u)). \quad (99)$$

Now combining (81), (99), and $\rho_1 + \rho_2 \geq 0$, we obtain

$$\begin{aligned} & \mathcal{F} \left(\bar{x}, u; b(\bar{x}, u) \sum_{i=1}^m \lambda_i (\xi_i + v_i \zeta_i) \right) \\ & \geq -\mathcal{F} \left(\bar{x}, u; b(\bar{x}, u) \sum_{j=1}^p \beta_j \zeta_j \right) \\ & \geq \rho_2 d^2(\theta(\bar{x}, u)) \geq -\rho_1 d^2(\theta(\bar{x}, u)). \end{aligned} \quad (100)$$

Along with the fact that the B is $(\mathcal{F}, b, \phi_1, \rho_1, \theta)$ -pseudoinvex at u , we get from (100)

$$\phi_1 \left(\sum_{i=1}^m \lambda_i [f_i(\bar{x}) - v_i g_i(\bar{x})] - \sum_{i=1}^m \lambda_i [f_i(u) - v_i g_i(u)] \right) \geq 0. \quad (101)$$

In view of the properties of ϕ_1 , we deduce from this inequality that

$$\sum_{i=1}^m \lambda_i [f_i(\bar{x}) + v_i (-g_i(\bar{x}))] \geq \sum_{i=1}^m \lambda_i [f_i(u) + v_i (-g_i(u))] \geq 0. \quad (102)$$

Therefore, $(\bar{x}, \bar{v}, \bar{\lambda}, \bar{\beta})$ is a properly efficient solution for (D) . (e), (f), and (g) follows along with the same lines of (d) and (a). Thus, the proof is complete. $\square$

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