Research Article

The Height of a Class in the Cohomology Ring of Polygon Spaces

Yasuhiko Kamiyama and Kazufumi Kimoto

Department of Mathematics, University of the Ryukyus, Nishihara-Cho, Okinawa 903-0213, Japan

Correspondence should be addressed to Yasuhiko Kamiyama; kamiyama@sci.u-ryukyu.ac.jp

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Let $\overline{M}_{n,r}$ be the configuration space of planar n -gons having side lengths $1, \dots, 1$ and r modulo isometry group. For generic r , the cohomology ring $H^*(\overline{M}_{n,r}; \mathbb{Z}_2)$ has a form $H^*(\overline{M}_{n,r}; \mathbb{Z}_2) = \mathbb{Z}_2[R(n, r), V_1, \dots, V_{n-1}]/\mathcal{J}_{n,r}$, where $R(n, r)$ is the first Stiefel-Whitney class of a certain regular 2-cover $\pi : M_{n,r} \rightarrow \overline{M}_{n,r}$ and the ideal $\mathcal{J}_{n,r}$ is in general big. For generic r , we determine the number $h(n, r)$ such that $R(n, r)^{h(n,r)} \neq 0$ but $R(n, r)^{h(n,r)+1} = 0$.

1. Introduction

Given a string $\mathbf{r} = (r_1, \dots, r_n)$ of n positive real numbers, one considers the configuration space $\overline{M}_{\mathbf{r}}$ of planar polygon linkages having side lengths r_i modulo isometry group. Starting in [1–3], the topology of $\overline{M}_{\mathbf{r}}$ has been considered by many authors. A notable achievement is [4] which determined the mod 2 cohomology ring $H^*(\overline{M}_{\mathbf{r}}; \mathbb{Z}_2)$ for generic $\mathbf{r}$. The study culminated in the proof by [5] of a conjecture by Kevin Walker which states that one can recover relative lengths of edges from the mod 2 cohomology ring of the configuration space.

By [4], the cohomology ring $H^*(\overline{M}_{\mathbf{r}}; \mathbb{Z}_2)$ has a form

$$H^*(\overline{M}_{\mathbf{r}}; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[R_{\mathbf{r}}, V_1, \dots, V_{n-1}]}{\mathcal{J}_{\mathbf{r}}}, \quad (1)$$

where $R_{\mathbf{r}}$ is the first Stiefel-Whitney class of a certain regular 2-cover $\pi : M_{\mathbf{r}} \rightarrow \overline{M}_{\mathbf{r}}$ and the ideal $\mathcal{J}_{\mathbf{r}}$ is in general big. For example, when $\mathbf{r} = (1, \dots, 1)$, $\mathcal{J}_{\mathbf{r}}$ is generated by polynomials whose number is approximately 2^{n-2} . While [5] made clever arguments to distinguish the cohomology rings, it is difficult to extract more concrete information from the ring. The reason is that the polynomials do not form a Gröbner basis.

In this direction, one natural problem is to compute the height of $R_{\mathbf{r}}$, that is, the unique $h(\mathbf{r})$ such that $R_{\mathbf{r}}^{h(\mathbf{r})} \neq 0$ but $R_{\mathbf{r}}^{h(\mathbf{r})+1} = 0$. The purpose of this paper is to determine $h(\mathbf{r})$ for the case $\mathbf{r} = (1, \dots, 1, r)$. Note that such $\mathbf{r}$ is a string which

comes next to the equilateral case. For example, the homology groups $H_*(M_{\mathbf{r}}; \mathbb{Z})$ were determined in [6].

Finally, we note that the height of an element in the cohomology ring of a space X has been studied in order to give nice lower bounds on the Lusternik-Schnirelmann category of X . For example, [7] studied the problem for the case that $X = G_k(\mathbb{R}^{n+k})$ and the element is the first Stiefel-Whitney class of the universal bundle.

This paper is organized as follows. In Section 2 we state our main results. Theorem A determines the height of $R_{\mathbf{r}}$. Theorem B determines which element represents the mod 2 fundamental class of $\overline{M}_{\mathbf{r}}$. In Section 3 we prove auxiliary results. In Section 4 we prove Theorem B and in Section 5 we prove Theorem A.

2. Main Results

For $4 \leq n \in \mathbb{N}$ and $0 < r \in \mathbb{R}$, we define the configuration space of planar n -gons having side lengths $1, \dots, 1$ and r by

$$M_{n,r} = \left\{ (z_1, \dots, z_{n-1}) \in (S^1)^{n-1} \mid \sum_{i=1}^{n-1} z_i = r \right\}. \quad (2)$$

Here $z_i \in S^1 \subset \mathbb{C}$ denotes the unit vectors in the direction of the sides of a polygon. Note that M_n comes with a natural involution

$$\tau : M_{n,r} \rightarrow M_{n,r}, \quad \tau(z_1, \dots, z_{n-1}) = (\overline{z}_1, \dots, \overline{z}_{n-1}). \quad (3)$$

We set

$$\overline{M}_{n,r} = \frac{M_{n,r}}{\tau}. \quad (4)$$

It is clear that $\overline{M}_{n,r} = \emptyset$ if $r > n - 1$ and $\overline{M}_{n,n-1} = \{\text{one point}\}$. In [1–3] it is proved that if r is a natural number which satisfies $r \leq n - 2$ and has the same parity as n , then there is a diffeomorphism

$$\overline{M}_{n,r'} \cong \overline{M}_{n,r}, \quad \forall r' \in (r - 1, r + 1). \quad (5)$$

Moreover, if n is even and $r' \in (0, 1)$, then we have $\overline{M}_{n,r'} \cong M_{n-1,1} \times_{\tau} S^1$, where τ acts on S^1 by complex conjugate. On the other hand, if $n > 4$ and $r \in \mathbb{N}$ has the different parity as n , then $\overline{M}_{n,r}$ has singular points.

Hereafter we assume that r is a natural number which satisfies $r \leq n - 2$ and has the same parity as n . In this case, $\overline{M}_{n,r}$ is a connected closed manifold of dimension $n - 3$. Moreover, $\overline{M}_{n,r}$ is orientable if and only if n is even.

The following examples are well known:

$$\overline{M}_{n,n-4} \cong \# \mathbb{R}P^{n-3}, \quad \overline{M}_{n,n-2} \cong \mathbb{R}P^{n-3}. \quad (6)$$

The following theorem is crucial in this paper.

Theorem 1 (see [4, Corollary 9.2]). *The mod 2 cohomology ring of $\overline{M}_{n,r}$ is*

$$H^*(\overline{M}_{n,r}; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[R(n, r), V_1, \dots, V_{n-1}]}{\mathcal{J}_{n,r}}, \quad (7)$$

where $R(n, r)$ and $V_1, \dots, V_{n-1}$ are of degree 1 and $\mathcal{J}_{n,r}$ is the ideal generated by the three families of elements:

(R1)

$$V_i^2 + R(n, r) V_i, \quad \text{where } i = 1, \dots, n - 1, \quad (8)$$

(R2)

$$\prod_{i \in S} V_i, \quad \text{where } S \subset \{1, \dots, n - 1\} \text{ is such that } |S| \geq \frac{n - r}{2}, \quad (9)$$

(R3)

$$\sum_{S \subset L} R(n, r)^{|L-S|-1} \prod_{i \in S} V_i, \quad \text{where} \quad (10)$$

$$L \subset \{1, \dots, n - 1\} \text{ is such that } |L| \geq \frac{n + r}{2}.$$

The symbol S in (R3) runs over all subsets of L including the empty set. By (R2) a term of the sum in (R3) vanishes if $|S| \geq (n - r)/2$.

The class $R(n, r)$ coincides with the first Stiefel-Whitney class of the regular 2-cover $\pi : M_{n,r} \rightarrow \overline{M}_{n,r}$. We define the height of $R(n, r)$ is the unique $h(n, r)$ such that $R(n, r)^{h(n, r)} \neq 0$ but $R(n, r)^{h(n, r)+1} = 0$.

In order to state our main results, we prepare notations. Throughout this paper, the notation $a \equiv b$ means that $a \equiv b \pmod{2}$. We set

$$D(n) = n - 2, \quad e(n, r) = \frac{n - r}{2} - 1, \quad (11)$$

$$k(n, r) = \max \left\{ i \mid 0 \leq i \leq e(n, r) - 1, \right. \\ \left. \binom{D(n) - e(n, r) + i}{i} \equiv 1 \right\}.$$

Moreover, we set

$$\phi(n, r) = \begin{cases} n - 3, & \text{if } \binom{D(n)}{e(n, r)} \equiv 1, \\ \frac{n + r}{2} + k(n, r) - 2, & \text{if } \binom{D(n)}{e(n, r)} \equiv 0. \end{cases} \quad (12)$$

Now our first result is the following.

Theorem A. *For all $n \geq 4$ and $r \in \mathbb{N}$ which satisfies $r \leq n - 2$ and has the same parity as n , are has $h(n, r) = \phi(n, r)$.*

We study the generator of $H^{n-3}(\overline{M}_{n,r}; \mathbb{Z}_2) = \mathbb{Z}_2$. Note that if

$$1 \leq \nu_1 < \dots < \nu_i \leq n - 1, \quad (13)$$

then we have in $H^{n-3}(\overline{M}_{n,r}; \mathbb{Z}_2)$ that

$$R(n, r)^{n-3-i} V_1 \cdots V_i = R(n, r)^{n-3-i} V_{\nu_1} \cdots V_{\nu_i}. \quad (14)$$

We set

$$p(n, r, i) = R(n, r)^{n-3-i} V_1 \cdots V_i. \quad (15)$$

Our second result is the following.

Theorem B. *Let n and r be as in Theorem A. Then for $0 \leq i \leq e(n, r)$, the following equality holds in $H^{n-3}(\overline{M}_{n,r}; \mathbb{Z}_2)$:*

$$p(n, r, i) = \binom{D(n) - i}{e(n, r) - i}. \quad (16)$$

Theorem A implies that $R(n, r)^{n-3}$ is a generator if and only if $\binom{D(n)}{e(n, r)} \equiv 1$. This is in agreement with Theorem B for $i = 0$.

For the case of almost equilateral polygons, that is, the case for $r = 1$ or 2 , we can write Theorem A more explicitly.

Proposition C. (i) *About $\overline{M}_{n,1}$ for odd n , one has the following.*

(a) *One has $\binom{D(n)}{e(n, r)} \equiv 1$ if and only if n is of the form $n = 2^s + 1$.*

(b) *If n satisfies that*

$$2^s + 1 \leq n \leq 2^{s+1} - 1, \quad (17)$$

then $h(n, 1) = 2^s - 2$.

(c) One has $p(2^s + 1, 1, i) \neq 0$ for all i .

(ii) About $\overline{M}_{n,2}$ for even n , one has the following.

(a) One has $\binom{D(n)}{e(n,r)} \equiv 1$ if and only if n is of the form $n = 2^s$.

(b) If $n = 2^s$, then $h(n, 2) = 2^s - 3$. On the other hand, if n satisfies that

$$2^s + 2 \leq n \leq 2^{s+1} - 2, \quad (18)$$

then $h(n, 2) = 2^s - 2$.

(c) One has $p(2^s, 2, i) \neq 0$ for all i .

One deduces the proposition from Theorem A in Section 5.

We give two examples of Theorem A. The first example is about the case when n is small.

Example 2. We consider Table 1 for the case $h(n, r) = n - 3$. Then $p(n, r, i) = 0$ (where $p(n, r, i)$ is defined in (15)) in $H^{n-3}(\overline{M}_{n,r}; \mathbb{Z}_2)$ if and only if (n, r, i) satisfies the case which is given in Table 2.

Our second example is about the case when r is large.

Example 3. (i) We have

$$h(n, n-4) = \begin{cases} n-3 & n \text{ is odd,} \\ n-4 & n \text{ is even.} \end{cases} \quad (19)$$

(ii) We have $h(n, n-2) = n-3$.

In fact, (i) for odd n and (ii) correspond to the case $\binom{D(n)}{e(n,r)} \equiv 1$ in Theorem A. But we know a stronger result as in (6).

3. Auxiliary Results

Proposition 4. Let x be indeterminate and $e \in \mathbb{N}$. Let $A_e(x)$ be an $e \times e$ matrix over $\mathbb{Z}_2[x]$ defined as follows:

$$A_e(x) = \left(\binom{x+i}{j} \right)_{\substack{1 \leq i \leq e \\ 1 \leq j \leq e}}. \quad (20)$$

Then one has the following results.

(i) One has $\text{rank } A_e(x) \geq e-1$.

(ii) One has $\text{rank } A_e(x) = e$ if and only if $\binom{x+e}{e} \neq 0$ in $\mathbb{Z}_2[x]$.

(iii) When $\text{rank } A_e(x) = e-1$, the following results hold.

(a) Let ${}^t\mathbf{y} = (y_1, \dots, y_e)$ be the unique nontrivial element of $\text{Ker } A_e(x)$. Then one has

$$y_j = \binom{x+e-j}{e-j}. \quad (21)$$

(b) Let ${}^t\mathbf{y} = (y_1, \dots, y_e)$ be as in (a). Then one has

$$\sum_{j=1}^e \binom{x+e+1}{j} y_j = 1. \quad (22)$$

Proof. We define 4 matrices as follows:

$$P_e = \left(\binom{i}{j} \right)_{\substack{1 \leq i \leq e \\ 0 \leq j \leq e}}, \quad Q_e(x) = \left(\binom{x}{j-i} \right)_{\substack{0 \leq i \leq e \\ 1 \leq j \leq e}}, \quad (23)$$

$$\widehat{P}_e = \left(\binom{i}{j} \right)_{1 \leq i, j \leq e}, \quad \widehat{Q}_e(x) = \left(\binom{x}{j-i} \right)_{1 \leq i, j \leq e}.$$

Since $\det \widehat{P}_e = \det \widehat{Q}_e(x) = 1$, they are regular matrices.

We write P_e and $Q_e(x)$ as follows:

$$P_e = (\mathbf{1} \ \widehat{P}_e), \quad Q_e(x) = \begin{pmatrix} \mathbf{q} \\ \widehat{Q}_e(x) \end{pmatrix}. \quad (24)$$

Using the Chu-Vandermonde identity

$$\binom{x+i}{j} = \sum_{k=0}^j \binom{i}{k} \binom{x}{j-k}, \quad (25)$$

we have

$$A_e(x) = P_e Q_e(x) = \mathbf{1q} + \widehat{P}_e \widehat{Q}_e(x). \quad (26)$$

(i) Since $\text{rank } \mathbf{1q} = 1$ and $\widehat{P}_e \widehat{Q}_e(x)$ is a regular matrix, we have $\text{rank } A_e(x) \geq e-1$.

(ii) Note that

$$\det A_e(x) = \binom{x+e}{e}. \quad (27)$$

In fact, both the sides coincide for $x = 0$ and vanish for $x \in \mathbb{Z}$ such that $-e \leq x \leq -1$.

Now (27) tells us that $\text{rank } A_e(x) = e$ if and only if $\binom{x+e}{e} \neq 0$.

(iii)(a) Since $y_e = 1$, we have ${}^t\mathbf{y} \neq 0$. We must prove that

$$\sum_{j=1}^e \binom{x+i}{j} \binom{x+e-j}{e-j} = 0 \quad (28)$$

for $1 \leq i \leq e$. Since we are assuming that $\text{rank } A_e(x) = e-1$, it will suffice to prove that

$$\sum_{j=0}^e \binom{x+i}{j} \binom{x+e-j}{x} = 0. \quad (29)$$

We use the formula

$$\sum_{j=0}^e (-1)^j \binom{a}{j} \binom{b-j}{b-c} = \binom{b-a}{c} \quad (30)$$

which holds over $\mathbb{Z}$. Setting $a = x+i$, $b = x+e$, and $c = e$, we have in $\mathbb{Z}_2$ that the left-hand side of (29) is $\binom{e-i}{e}$. Since $1 \leq i \leq e$, this is 0.

(b) The left-hand side of (b) is that of (29) for $i = e+1$. Then this is $\binom{e-i}{e} = \binom{-1}{e} = 1$. $\square$

TABLE 1: $h(n, r)$ for $4 \leq n \leq 20$.

r	n																	
	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	
1	—	2	—	2	—	6	—	6	—	6	—	6	—	14	—	14	—	
2	1	—	2	—	5	—	6	—	6	—	6	—	13	—	14	—	14	
3	—	2	—	4	—	6	—	6	—	6	—	12	—	14	—	14	—	
4	—	—	3	—	4	—	6	—	6	—	11	—	12	—	14	—	14	
5	—	—	—	4	—	6	—	6	—	10	—	12	—	14	—	14	—	
6	—	—	—	—	5	—	6	—	9	—	10	—	13	—	14	—	14	
7	—	—	—	—	—	6	—	8	—	10	—	10	—	14	—	14	—	
8	—	—	—	—	—	—	7	—	8	—	10	—	10	—	14	—	14	
9	—	—	—	—	—	—	—	8	—	10	—	10	—	14	—	14	—	
10	—	—	—	—	—	—	—	—	9	—	10	—	13	—	14	—	14	
11	—	—	—	—	—	—	—	—	—	10	—	12	—	14	—	14	—	
12	—	—	—	—	—	—	—	—	—	—	11	—	12	—	14	—	14	
13	—	—	—	—	—	—	—	—	—	—	—	12	—	14	—	14	—	
14	—	—	—	—	—	—	—	—	—	—	—	—	13	—	14	—	17	
15	—	—	—	—	—	—	—	—	—	—	—	—	—	14	—	16	—	
16	—	—	—	—	—	—	—	—	—	—	—	—	—	—	15	—	16	
17	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	16	—	
18	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	17	

TABLE 2: The cases for $p(n, r, i) = 0$.

(n, r)	i
(9, 3)	1
(13, 7)	1
(15, 5)	1, 3
(17, 3)	1, 3, 5
(17, 5)	2, 3
(17, 7)	1, 2, 3
(17, 11)	1

Corollary 5. Assume that $\text{rank } A_e(x) = e - 1$ and let ${}^t\mathbf{y}$ be the element in Proposition 4(iii)(a). One sets

$$\ell(x, e) = \min \{j \mid 1 \leq j \leq e, y_j \equiv 1\}. \quad (31)$$

If one has also that $\text{rank } A_e(x - 1) = e - 1$, then one has

$$\ell(x - 1, e) = \ell(x, e) - 1. \quad (32)$$

Proof. The unique nontrivial element of $\text{Ker } A_e(x - 1)$ is given by

$${}^t(y_1 + y_2, y_2 + y_3, \dots, y_{e-1} + y_e, y_e). \quad (33)$$

The corollary is clear from this. $\square$

The following property of $\phi(n, r)$ will be used in Section 5.

Lemma 6. (i) If $r \geq 2$ and $\binom{D(n)}{e(n, r)} \equiv 0$, one has

$$\phi(n - 1, r - 1) = \phi(n, r). \quad (34)$$

(ii) If n is odd and satisfies that

$$2^s + 1 \leq n \leq 2^{s+1} - 1, \quad (35)$$

then the following results hold.

(a) One has $\binom{D(n)}{e(n, r)} \equiv 1$ if and only if n is of the form $n = 2^s + 1$.

(b) $\phi(n, 1) = 2^s - 2$.

Proof. (i) We need to consider two cases according to $\binom{D(n-1)}{e(n-1, r-1)} \equiv 0$ or 1.

Case (a) ($\binom{D(n-1)}{e(n-1, r-1)} \equiv 0$).

Note that

$$D(n - 1) = D(n) - 1 = n - 3, \quad (36)$$

$$e(n - 1, r - 1) = e(n, r) = \frac{n - r}{2} - 1. \quad (37)$$

We shall prove that

$$k(n - 1, r - 1) = k(n, r) + 1. \quad (38)$$

We consider Proposition 4 for $x = D(n) - e(n, r)$ and $e = e(n, r)$. For $\mathbf{y}$ in the proposition, we set $\tilde{y}_j = y_{e-j}$ and put $\tilde{\mathbf{y}} = (\tilde{y}_0, \dots, \tilde{y}_{e-1})$. Note that $\tilde{\mathbf{y}}$ is the sequence which appears in the definition of $k(n, r)$. This and the definition of $\ell(x, e)$ in Corollary 5 tell us that

$$k(n, r) = e(n, r) - \ell(D(n) - e(n, r), e(n, r)). \quad (39)$$

Using Corollary 5, (36), and (37), we have

$$\begin{aligned} \ell(D(n-1) - e(n-1, r-1), e(n-1, r-1)) \\ = \ell(D(n) - e(n, r), e(n, r)) - 1. \end{aligned} \quad (40)$$

Combining (37), (39), and (40), we obtain (38).

Now (i) follows from the following computation:

$$\begin{aligned} \phi(n-1, r-1) &= \frac{n-1+r-1}{2} + (k(n, r) + 1) - 2 \\ &= \phi(n, r). \end{aligned} \quad (41)$$

Case (b) $\left(\binom{D(n-1)}{e(n-1, r-1)}\right) \equiv 1$.

By definition, we have $\phi(n-1, r-1) = n-4$. Hence it will suffice to prove $\phi(n, r) = n-4$. This is equivalent to the assertion that $k(n, r) = e(n, r) - 1$ and also equivalent to the assertion that $\left(\binom{D(n)-1}{e(n, r)-1}\right) \equiv 1$. But the last assertion is clear if we apply (36) and (37) to our assumptions that

$$\left(\binom{D(n)}{e(n, r)}\right) \equiv 0, \quad \left(\binom{D(n-1)}{e(n-1, r-1)}\right) \equiv 1. \quad (42)$$

(ii) If we prove (b), then (a) is clear from this. Since (b) clearly holds for $n = 2^s + 1$, we assume that $2^s + 3 \leq n \leq 2^{s+1} - 1$. We set $n = 2m + 1$. By definition,

$$k(n, r) = \max \left\{ i \mid 0 \leq i \leq m-2, \binom{m+i}{i} \equiv 1 \right\}. \quad (43)$$

By Kummer, $\binom{a+b}{b}$ is odd if and only if there are no carries when adding a and b in base 2. Hence if we write $m = 2^{s-1} + u$ (where $1 \leq u \leq 2^{s-1}$), the greatest i (where $i \leq m-2$) such that $\binom{m+i}{i} \equiv 1$ is

$$i = 2^{s-2} + 2^{s-3} + \cdots + 1 - u = 2^{s-1} - 1 - u. \quad (44)$$

Hence $k(n, r) = 2^{s-1} - 1 - u$ and

$$\begin{aligned} \phi(n, 1) &= (m+1) + k(n, r) - 2 \\ &= (2^{s-1} + u + 1) + (2^{s-1} - 1 - u) - 2 = 2^s - 2. \end{aligned} \quad (45)$$

□

Modifying Proposition 4 slightly, we have the following.

Lemma 7. Let $\check{A}_e(x)$ be an $e \times (e+1)$ matrix over $\mathbb{Z}_2[x]$ defined as follows:

$$\check{A}_e(x) = \left(\binom{x+i}{j} \right)_{\substack{1 \leq i \leq e \\ 0 \leq j \leq e}}. \quad (46)$$

Then one has the following results.

- (i) One has $\text{rank } \check{A}_e(x) = e$.
- (ii) Let ${}^t\check{\mathbf{y}} = ({}^t\check{y}_0, \dots, {}^t\check{y}_e)$ be the unique nontrivial element of $\text{Ker } \check{A}_e(x)$. Then one has

$${}^t\check{y}_j = \binom{x+e-j}{e-j}. \quad (47)$$

Proof. (i) Let P_e be the matrix in the proof of Proposition 4 and we define

$$\check{Q}_e(x) = \left(\binom{x}{j-i} \right)_{\substack{0 \leq i \leq e \\ 0 \leq j \leq e}}. \quad (48)$$

Since $\check{A}_e(x) = P_e \check{Q}_e(x)$, $\text{rank } P_e = e$, and $\det \check{Q}_e(x) = 1$, the result follows;

(ii) follows from (29). □

4. Proof of Theorem B

For $(n+r)/2 \leq i \leq n-2$, let $f_i(R(n, r), V_1, \dots, V_{n-1})$ be the polynomial in Theorem 1 (R3) for $L = \{1, \dots, i\}$. Note that the coefficient of $p(n, r, j)$ in

$$R(n, r)^{n-2-i} f_i(R(n, r), V_1, \dots, V_{n-1}) \quad (49)$$

is $\binom{i}{j}$, where $p(n, r, j)$ is defined in (15). We list the coefficient in the following $e(n, r) \times (e(n, r) + 1)$ matrix $B_{n, r}$:

$$B_{n, r} = \left(\binom{i}{j} \right)_{\substack{(n+r)/2 \leq i \leq n-2 \\ 0 \leq j \leq e(n, r)}}. \quad (50)$$

Theorem 1 tells us that

$$\begin{aligned} B_{n, r} {}^t(p(n, r, 0), \dots, p(n, r, j), \dots, p(n, r, e(n, r))) \\ = {}^t(0, \dots, 0, \dots, 0). \end{aligned} \quad (51)$$

In the notation of Lemma 7, we have

$$B_{n, r} = \check{A}_{e(n, r)} \left(\frac{n+r}{2} - 1 \right). \quad (52)$$

Since $p(n, r, j)$ ($0 \leq j \leq e(n, r)$) generate $H^{n-3}(\overline{M}_{n, r}; \mathbb{Z}_2)$, (51) tells us that

$$p(n, r, j) = \check{y}_j. \quad (53)$$

We compute $\check{y}_j$ as follows:

$$\begin{aligned} \check{y}_j &= \binom{\left(\frac{n+r}{2} - 1\right) + \left(\frac{n-r}{2} - 1\right) - j}{e(n, r) - j} \\ &= \binom{n-2-j}{e(n, r) - j} = \binom{D(n) - j}{e(n, r) - j}. \end{aligned} \quad (54)$$

5. Proof of Theorem A

Theorem A is proved by showing the upper bound for those λ with $R(n, r)^\lambda \neq 0$ coinciding with the lower bound. About the upper bound, we have the following.

Proposition 8. In $H^*(\overline{M}_{n, r}; \mathbb{Z}_2)$, one has $R(n, r)^{\phi(n, r)+1} = 0$.

Proof. Since $\dim \overline{M}_{n, r} = n - 3$, the case for $\binom{D(n)}{e(n, r)} \equiv 1$ is clear. Hence we may assume that $\binom{D(n)}{e(n, r)} \equiv 0$. For p in

$((n+r)/2) \leq p \leq n-2$, we denote by $g_p(R(n, r), V_1, \dots, V_{n-1})$ the sum of polynomials in Theorem 1 (R3) for all L with $|L| = p$. Note that $\deg g_p(R(n, r), V_1, \dots, V_{n-1}) = p-1$. We use the notation $\tilde{y}$ in the proof of Lemma 6. We claim the following:

$$\sum_{j=0}^{k(n,r)} R(n, r)^{k(n,r)-j} g_{((n+r)/2)+j}(R(n, r), V_1, \dots, V_{n-1}) \tilde{y}_j = R(n, r)^{\phi(n,r)+1}. \quad (55)$$

In order to prove (55), we define

$$C_{n,r} = \left(\binom{D(n)+1-i}{e(n,r)-j} \right)_{\substack{0 \leq i \leq e(n,r) \\ 0 \leq j \leq e(n,r)-1}}, \quad (56)$$

$$\bar{C}_{n,r} = \left(\binom{D(n)+1-i}{e(n,r)-j} \right)_{\substack{0 \leq i \leq e(n,r) \\ 0 \leq j \leq k(n,r)}}.$$

Then the following equality holds:

$$\bar{C}_{n,r} {}^t(\tilde{y}_0, \dots, \tilde{y}_{k(n,r)}) = C_{n,r} {}^t\tilde{y} = {}^t(1, 0, \dots, 0). \quad (57)$$

In fact, the left equality follows from the fact that $\tilde{y}_i = 0$ for $k(n, r) + 1 \leq i \leq e(n, r) - 1$. On the other hand, the right equality is proved as follows. Note that we can write $C_{n,r}$ as

$$C_{n,r} = \left(\bar{A}_{e(n,r)} \begin{smallmatrix} \mathbf{c} \\ (D(n) - e(n, r)) \end{smallmatrix} \right), \quad (58)$$

where $\mathbf{c}$ is the first row of $C_{n,r}$ and $\bar{A}_{e(n,r)}(D(n) - e(n, r))$ is the matrix defined from $A_{e(n,r)}(D(n) - e(n, r))$ by shifting the (i, j) -element to the $(e(n, r) + 1 - i, e(n, r) + 1 - j)$ -element. Since ${}^t\mathbf{y} \in \text{Ker } A_{e(n,r)}(D(n) - e(n, r))$, we have ${}^t\tilde{y} \in \text{Ker } \bar{A}_{e(n,r)}(D(n) - e(n, r))$. Now the right equality of (57) follows from Proposition 4(iii)(b).

Let σ_i be the i th symmetric polynomial in variables $V_1, \dots, V_{n-1}$. Then we have

$$g_{((n+r)/2)+j} = \sum_{i=0}^{e(n,r)} \binom{D(n)+1-i}{e(n,r)-j} R(n, r)^{((n+r)/2)+j-i-1} \sigma_i. \quad (59)$$

In fact, the binomial coefficient is computed from $\binom{n-1}{p} \binom{p}{i} / \binom{n-1}{i}$ for $p = ((n+r)/2) + j$. Now we complete the proof of (55) as follows:

the left hand side of (55)

$$\begin{aligned} &= (R(n, r)^{((n+r)/2)+k(n,r)-1} \sigma_0, \dots, \\ &\quad R(n, r)^{((n+r)/2)+k(n,r)-e(n,r)-1} \sigma_{e(n,r)}) \\ &\quad \times \bar{C}_{n,r} {}^t(\tilde{y}_0, \dots, \tilde{y}_{k(n,r)}) \quad (\text{by (59)}) \\ &= (R(n, r)^{((n+r)/2)+k(n,r)-1} \sigma_0, \dots, \\ &\quad R(n, r)^{((n+r)/2)+k(n,r)-e(n,r)-1} \sigma_{e(n,r)}) \end{aligned}$$

$$\times {}^t(1, 0, \dots, 0) \quad (\text{by (57)})$$

$$= R(n, r)^{((n+r)/2)k(n,r)-1} = R(n, r)^{\phi(n,r)+1}. \quad (60)$$

Hence (55) holds.

Now by Theorem 1 (R3), we have

$$g_p(R(n, r), V_1, \dots, V_{n-1}) = 0 \quad (61)$$

in $H^{p-1}(\bar{M}_{n,r}; \mathbb{Z}_2)$. Since the left-hand side of (55) vanishes, so does the right-hand side. $\square$

Proof of Theorem A. We prove the theorem by induction on n and for all r .

(I) The case for $n = 4$.

Since $\bar{M}_{4,2} = S^1$, Theorem A holds.

(II) Fix n and assume that Theorem A holds for all k with $k < n$.

We shall prove Theorem A for $\bar{M}_{n,r}$.

Case (i) ($2 \leq r \leq n-2$).

We need to consider the cases according to $\binom{D(n)}{e(n,r)} \equiv 0$ or 1. Since the case of 1 is proved in Theorem B, we may assume that $\binom{D(n)}{e(n,r)} \equiv 0$. We define an inclusion $i : M_{n-1,r-1} \rightarrow M_{n,r}$ by

$$i(z_1, \dots, z_{n-2}) = (z_1, \dots, z_{n-2}, 1). \quad (62)$$

Note that the map i naturally induces a map $\bar{i} : \bar{M}_{n-1,r-1} \rightarrow \bar{M}_{n,r}$ and we have the following diagram of regular 2-covers:

$$\begin{array}{ccc} \mathbb{Z}_2 & \xlongequal{\quad} & \mathbb{Z}_2 \\ \downarrow & & \downarrow \\ M_{n-1,r-1} & \xrightarrow{\quad i \quad} & M_{n,r} \\ \downarrow \pi & & \downarrow \pi \\ \bar{M}_{n-1,r-1} & \xrightarrow{\quad \bar{i} \quad} & \bar{M}_{n,r} \end{array} \quad (63)$$

Since $R(n, r)$ is the first Stiefel-Whitney class of the regular 2-cover $\pi : M_{n,r} \rightarrow \bar{M}_{n,r}$, we have

$$\bar{i}^*(R(n, r)) = R(n-1, r-1). \quad (64)$$

By inductive hypothesis, we have in $H^*(\bar{M}_{n-1,r-1}; \mathbb{Z}_2)$ that

$$R(n-1, n-1)^{\phi(n-1,r-1)} \neq 0. \quad (65)$$

Using (64), we have in $H^*(\bar{M}_{n,r}; \mathbb{Z}_2)$ that

$$R(n, r)^{\phi(n-1,r-1)} \neq 0. \quad (66)$$

Using Lemma 6(i), we have $R(n, r)^{\phi(n,r)} \neq 0$.

On the other hand, we have by Proposition 8 that $R(n, r)^{\phi(n, r)+1} = 0$. Hence we have proved that $h(n, r) = \phi(n, r)$.

Case (ii) ($r = 1$).

As in (i), we may assume that $\binom{D(n)}{e(n, r)} \equiv 0$. Lemma 6(ii) tells us that if $2^s + 3 \leq n \leq 2^{s+1} - 1$, then $\phi(n, 1) = 2^s - 2$. Moreover, Proposition 8 tells us that $R(n, 1)^{2^s-1} = 0$. Hence it will suffice to prove that $R(n, 1)^{2^s-2} \neq 0$ in $H^*(\overline{M}_{n,1}; \mathbb{Z}_2)$. We define an inclusion $j : M_{2^s+1,1} \rightarrow M_{n,1}$ by

$$i(z_1, \dots, z_{2^s}) = \left(z_1, \dots, z_{2^s}, \underbrace{1, \dots, 1}_d, \underbrace{-1, \dots, -1}_d \right). \quad (67)$$

The map induces a map $\bar{j} : \overline{M}_{2^s+1,1} \rightarrow \overline{M}_{n,1}$. Using a similar diagram to (63) and the fact that $R(2^s + 1, 1)^{2^s-2} \neq 0$ in $H^*(\overline{M}_{2^s+1,1}; \mathbb{Z}_2)$ (see Theorem B), we have in $H^*(\overline{M}_{n,1}; \mathbb{Z}_2)$ that $R(n, 1)^{2^s-2} \neq 0$. $\square$

Proof of Proposition C. Proposition C(i)(a) and (b) follow from Theorem A and Lemma 6(ii), and (i)(c) follows from Theorem B. If we modify Lemma 6(ii) slightly for even n , then we can prove Proposition C(ii) similarly. $\square$

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