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Research Article

Partial Actions and Power Sets

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We consider a partial action (X, α) with enveloping action (T, β). In this work we extend α to a partial action on the ring $(P(X), \Delta, \cap)$ and find its enveloping action (E, β). Finally, we introduce the concept of partial action of *finite type* to investigate the relationship between (E, β) and $(P(T), \beta)$.

1. Introduction

Partial actions of groups appeared independently in various areas of mathematics, in particular, in the study of operator algebras. The formal definition of this concept was given by Exel in 1998 [1]. Later in 2003, Abadie [2] introduced the notion of enveloping action and found that any partial action possesses an enveloping action. The study of partial actions on arbitrary rings was initiated by Dokuchaev and Exel in 2005 [3]. Among other results, they prove that there exist partial actions without an enveloping action and give sufficient conditions to guarantee the existence of enveloping actions. Many studies have shown that partial actions are a powerful tool to generalize many well-known results of global actions (see [3, 4] and the literature quoted therein).

The theory of partial actions of groups has taken several directions over the past thirteen years. One way is to consider actions of monoids and groupoids rather than group actions. Another is to consider sets with some additional structure such as rings, topological spaces, ordered sets, or metric spaces. Partial actions on the power set and its compatibility with its ring structure have not been considered. This work is devoted to study some topics related to partial actions on the power set $P(X)$ arising from partial actions on the set X and its enveloping actions. In Section 1, we present some theoretical results of partial actions and enveloping actions. In Section 2, we extend a partial action α on the set X to a partial action on the ring $(P(X), \Delta, \cap)$. In addition, we introduce the concept of partial action *of finite type* to investigate the relationship between the enveloping action of $(P(X), \alpha)$ and $(P(T), \beta)$, the power set of the enveloping action of (X, α).

2. Preliminaries

In this section, we present some results related to the partial actions, which will be used in Section 2. Other details of this theory can be found in [2, 3].

Definition 1. A partial action α of the group G on the set X is a collection of subsets S_g, $g \in G$, of X and bijections $\alpha_g : S_{g^{-1}} \to S_g$ such that for all $g, h \in G$, the following statements hold.

(1) $S_1 = X$ and α_1 is the identity of X.

(2) $S_{(gh)^{-1}} \supseteq \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.

(3) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for all $x \in \alpha_h^{-1}(S_h \cap S_{g^{-1}})$.

The partial action α will be denoted by (X, α) or $\alpha = \{S_g, \alpha_g\}_{g \in G}$. Examples of partial actions can be obtained by restricting a global action to a subset. More exactly, suppose that G acts on Y by bijections $\gamma_g : Y \to Y$ and let X be a subset of Y. Set $S_g = X \cap \gamma_g(X)$ and let α_g be the restriction of γ_g to $S_{g^{-1}}$, for each $g \in G$. Then, it is easy to see that

$\alpha = \{S_g, \alpha_g\}_{g\in G}$ is a partial action of G on X. In this case, α is called the restriction of γ to X. In fact, for any partial action (X, α) there exists a minimal global action (T, β) (enveloping action of (X, α)), such that α is the restriction of β to X [2, Theorem 1.1].

To define a partial action of the group G on the ring R, it is enough to assume in Definition 1 that each S_g, $g \in G$, is an ideal of R and that every map $\alpha_g : S_{g^{-1}} \to S_g$ is an isomorphism of ideals. Natural examples of partial actions on rings can be obtained by restricting a global action to an ideal. In this case, the notion of enveloping action is the following ([3, Definition 4.2]).

Definition 2. A global action β of a group G on the ring E is said to be an enveloping action for the partial action α of G on a ring R, if there exists a ring isomorphism φ of R onto an ideal of E such that for all $g \in G$, the following conditions hold.

(1) $\varphi(S_g) = \varphi(R) \cap \beta_g(\varphi(R))$.

(2) $\varphi \circ \alpha_g(x) = \beta_g \circ \varphi(x)$, for all $x \in S_{g^{-1}}$.

(3) E is generated by $\bigcup_{g\in G} \beta_g(\varphi(R))$.

In general, there exist partial actions on rings which do not have an enveloping action [3, Example 3.5]. The conditions that guarantee the existence of such an enveloping action are given in the following result [3, Theorem 4.5].

Theorem 3. *Let R be a unital ring. Then a partial action α of a group G on R admits an enveloping action β if and only if each ideal S_g, $g \in G$, is a unital ring. Moreover, if such an enveloping action exists, it is unique up equivalence.*

3. Results

In this section, we consider a nonempty set X and a partial action $\alpha = \{X_g, \alpha_g\}_{g\in G}$ be of the group G on X. By [2, Theorem 1.1] there exists an enveloping action (T, β) for (X, α). That is, there exist a set T and a global action $\beta = \{\beta_g \mid g \in G\}$ of G on T, where each β_g is a bijection of T, such that the partial action α is given by restriction. Thus, we can assume that $X \subseteq T$, T is the orbit of X, $S_g = X \cap \beta_g(X)$ for each $g \in G$ and $\alpha_g(x) = \beta_g(x)$ for all $g \in G$ and all $x \in S_{g^{-1}}$.

The action β on T can be extended to an action on $P(T)$. Moreover, since β_g, $g \in G$, is a bijective function, we have that $\beta_g(A\Delta B) = \beta_g(A)\Delta\beta_g(B)$ and $\beta_g(A\cap B) = \beta_g(A)\cap \beta_g(B)$ for all $A, B \in P(T)$ and all $g \in G$. Therefore, the group G acts on the ring $(P(T), \Delta, \cap)$. This action will also be denoted by β.

Proposition 4. *If G acts partially on X then G acts partially on the ring $(P(X), \Delta, \cap)$.*

Proof. Let $\alpha = \{X_g, \alpha_g\}_{g\in G}$ a partial action of G on X and consider the collection $\alpha' = \{S_g, \alpha'_g\}_{g\in G}$, where $S_g = P(X_g)$, $g \in G$, and $\alpha'_g : S_{g^{-1}} \to S_g$ is defined by $\alpha'_g(A) = \{\alpha_g(a) : a \in A\}$ for all $g \in G$ and all $A \in P(X_g)$. It is clear that α'_g, $g \in G$, is a well-defined function, and it is a bijection. Now, we must prove that α' defines a partial action of G on the ring $P(X)$. We verify 2 and 3 of Definition 1, since 1 is evident.

(2) If $A \in \alpha'^{-1}_h(S_h \cap S_{g^{-1}})$, then $\alpha'_h(A) \in P(X_h \cap X_{g^{-1}})$. Thus, $a \in \alpha_h^{-1}(X_h \cap X_{g^{-1}}) \subseteq X_{(gh)^{-1}}$ for each $a \in A$. Hence, $A \in P(X_{(gh)^{-1}}) = S_{(gh)^{-1}}$, and we conclude that $\alpha'^{-1}_h(S_h \cap S_{g^{-1}}) \subseteq S_{(gh)^{-1}}$.

(3) For all $A \in \alpha'^{-1}_h(S_h \cap S_{g^{-1}})$, we have that $(\alpha'_g \circ \alpha'_h)(A) = \{(\alpha_g \circ \alpha_h)(a) : a \in A\}$. Since $A \in S_{(gh)^{-1}}$ (item 2), then $(\alpha'_g \circ \alpha'_h)(A) = \{\alpha_{gh}(a) : a \in A\} = \alpha'_{gh}(A)$. In conclusion, $(\alpha'_g \circ \alpha'_h)(A) = \alpha'_{gh}(A)$ for all $A \in \alpha'^{-1}_h(S_h \cap S_{g^{-1}})$.

Finally, for all $A, B \in P(X_{g^{-1}})$, we have that $\alpha'_g(A\Delta B) = \alpha'_g(A)\Delta\alpha'_g(B)$ and $\alpha'_g(A \cap B) = \alpha'_g(A) \cap \alpha'_g(B)$, because each α'_g, $g \in G$, is a bijection. Therefore, G acts partially on the ring $(P(X), \Delta, \cap)$. □

The partial action of G on $(P(X), \Delta, \cap)$ will also be denoted by α.

In the previous proposition, note that each ideal $S_g = P(X_g)$, $g \in G$, has the identity element X_g. Thus, by Theorem 3, we conclude that there exists an enveloping action for the partial action $(P(X), \alpha)$. In the following result, we find this enveloping action and show its relationship with $(P(T), \beta)$.

Proposition 5. *Let α be a partial action of G on the nonempty set X. The following statements hold.*

(1) *$P(X)$ is an ideal of $P(T)$.*

(2) *$E = \sum_{g\in G} \beta_g(P(X))$ is a β-invariant ideal of $P(T)$.*

(3) *The enveloping action of $(P(X), \alpha)$ is (E, β), where each β_g, $g \in G$, acts on E by restriction.*

Proof. (1) It is a direct consequence of the inclusion $X \subseteq T$.

(2) Since $P(X)$ is an ideal of $P(T)$, we have that $E = \sum_{g\in G} \beta_g(P(X))$ is an ideal of $P(T)$, and it is clear that E is β-invariant.

(3) We must prove 1, 2, and 3 of Definition 2. Note that by item 2, the action β on E is global. Moreover, we can identify $P(X)$ with $\varphi(P(X))$ because $P(X)$ is an ideal of E. The item 3 is consequence of 2.

To prove 2, let $A \in P(S_{g^{-1}})$. Then, $\alpha'_g(A) = \{\alpha_g(x) \mid x \in A\}$. Since $A \subseteq S_{g^{-1}} \subseteq X$ and (T, β) is the enveloping action of (X, α), we have that $\alpha_g(x) = \beta_g(x)$ for all $x \in A$ and all $g \in G$. Thus, $\alpha'_g(A) = \{\beta_g(x)x \in A\} = \beta_g(A)$, and we conclude that $\alpha'_g(A) = \beta_g(A)$ for all $A \in P(S_{g^{-1}})$.

To prove 1, let $A \in P(S_g)$. Then, $\beta_{g^{-1}}(A) = \alpha'_{g^{-1}}(A)$ and thus $A = \beta_g(\alpha'_{g^{-1}}(A)) \in \beta_g(P(X))$. Hence, $P(S_g) \subseteq P(X) \cap \beta_g(P(X))$ for all $g \in G$.

For the other inclusion, let $A, B \in P(X)$ such that $A = \beta_g(B)$. Then, $A = X \cap \beta_g(X \cap B) = X \cap \beta_g(X) \cap \beta_g(B)$. Since (T, β) is the enveloping action of (X, α), we have $S_g = X \cap \beta_g(X)$ for all $g \in G$. Hence, $A = S_g \cap \beta_g(B) \subseteq S_g$ and thus $A \in P(S_g)$.

We conclude that (E, β) is the enveloping action of $(P(X), \alpha)$. □

The final result shows that (E, β), the enveloping action of $(P(X), \alpha)$, is a subaction of $(P(T), \beta)$. Thus it is natural to ask in which case $E = P(T)$ or equivalently when $(P(T), \beta)$ is the enveloping action of $(P(X), \alpha)$. To solve this problem, we first define the concept of partial action *of finite type*.

Definition 6. Let (X, α) be a partial action of G on the set X with enveloping action (T, β). (X, α) is said to be of finite type if there exist $g_1, \ldots, g_n \in G$ such that $T = \bigcup_{i=1}^{n} \beta_{g_i}(X)$.

A partial action $\alpha' = \{D_g, \alpha'_g\}_{g \in G}$ of G on the ring R is called *of finite type* [5, Definition 1.1] if there exists a finite subset $\{g_1, \ldots, g_n\}$ of G, such that, $\sum_{i=1}^{n} D_{gg_i} = R$ for any $g \in G$. If the partial action has an enveloping action, then it can be characterized as follows [5, Proposition 1.2].

Proposition 7. *Let α' be a partial action of G on the ring R with enveloping action (E', β'). The following statements are equivalent.*

(1) *α' is of finite type.*

(2) *There exist $g_1, \ldots, g_n \in G$ such that $E' = \sum_{i=1}^{n} \beta'_{g_i}(R)$.*

(3) *E' has an identity element.*

The following theorem is the main result of this work. Without loss of generality, we can assume that $g_1 = 1$ in Definition 6 and Proposition 7. First, we prove the following specialization of [5, Proposition 1.10].

Proposition 8. *Under the previous assumptions, if $S = \sum_{i=1}^{n} \beta_{g_i}(P(X))$ with $g_1 = 1, \ldots, g_n \in G$, then $1_S = \bigcup_{i=1}^{n} \beta_{g_i}(X)$ is the identity element of S.*

Proof. By induction on n, it is enough to consider the case with two summands, that is, $S = P(X) + \beta_{g_2}(P(X))$. Then, $1_S = 1_{P(X)} + 1_{\beta_{g_2}(P(X))} - 1_{P(X)} 1_{\beta_{g_2}(P(X))}$. Since the addition is the symmetric difference and the product is the intersection, we obtain that $1_S = X \,\Delta\, \beta_{g_2}(X) \,\Delta\, (X \cap \beta_{g_2}(X)) = X \cup \beta_{g_2}(X)$. □

Theorem 9. *Let (X, α) be a partial action of G on the set X with enveloping action (T, β). The following statements are equivalent.*

(1) *(X, α) is of finite type.*

(2) *$E = P(T)$.*

(3) *$(P(X), \alpha)$ is of finite type.*

Proof. $1 \Rightarrow 2$. Suppose that there exist $g_1 = 1, \ldots, g_n \in G$ such that $T = \bigcup_{i=1}^{n} \beta_{g_i}(X)$. By Proposition 8, the identity element of the ring $S = \sum_{i=1}^{n} \beta_{g_i}(P(X)) \subseteq E$ is $1_S = \bigcup_{i=1}^{n} \beta_{g_i}(X) = T$. So, $T \in E$, and since E is an ideal of $P(T)$, we conclude that $E = P(T)$.

$2 \Rightarrow 3$. If $E = P(T)$, then $T \in E$. So, E is a ring with identity, and by Proposition 7 the result follows.

$3 \Rightarrow 1$. If $(P(X), \alpha)$ is of finite type, then there exist $g_1 = 1, \ldots, g_n \in G$ such that $E = \sum_{i=1}^{n} \beta_{g_i}(P(X))$. Thus, for each $g \in G$, there exist $A_1^g, \ldots, A_n^g \subseteq X$ such that $\beta_g(X) = \sum_{i=1}^{n} \beta_{g_i}(A_i^g)$, which implies that $\beta_g(X) \subseteq \bigcup_{i=1}^{n} \beta_{g_i}(X)$ for each $g \in G$. Hence, $T = \bigcup_{g \in G} \beta_g(X) \subseteq \bigcup_{i=1}^{n} \beta_{g_i}(X)$. In conclusion, (X, α) is of finite type. □

To illustrate the results obtained, we include the following examples.

Example 10. Let X be the set of even integers and G the group $\mathbb{Z}$. We define a partial action α of G on X as follows: $S_n = X$ if n is even and $S_n = \emptyset$ if n is odd; for n, an even integer $\alpha_n : S_{-n} \to S_n$ is defined by $\alpha_n(k) = k + n$ for all $k \in S_{-n}$, and for n, an odd integer α_n is the empty function. The enveloping action of (X, α) is (T, β) where $T = \mathbb{Z}$ and β is the action of $\mathbb{Z}$ on T, defined by $\beta_n(k) = k + n$ for all $k \in T$. Since $X \in \beta_0(P(X))$ and the set of odd integers $Y \in \beta_1(P(X))$, we have $X + Y = T \in E$. Hence, $E = P(T)$, and thus $(P(T), \beta)$ is the enveloping action of $(P(X), \alpha)$.

Example 11. Let $X = \{n_0\}$ where n_0 is a fixed integer and G is the group $\mathbb{Z}$. We define a partial action α of G on X as follows: $S_0 = X$ and $S_n = \emptyset$ for $n \neq 0$; $\alpha_0 : S_0 \to S_0$ is the identity and α_n is the empty function in other case. The enveloping action of (X, α) is (T, β), that of the previous example.

Note that each singleton of T is an element of $\beta_n(P(X))$ for some integer n. So, $E = \sum_{n \in \mathbb{Z}} \beta_g(P(X))$ the enveloping action of $(P(X), \alpha)$ coincides with the collection of all finite subsets of T. Hence, $T \notin E$ because T is an infinite set, and we conclude that $E \subsetneqq P(T)$.

In [5] it was proved that if R is a ring and α is a partial action of a group G on R with enveloping action (T, β), then T is right (left) Noetherian (Artinian), if and only if R is right (left) Noetherian (Artinian) and α is of finite type (Corollary 1.3). Under the same assumptions, they also proved that T is semisimple if and only if R is semisimple and α is of finite type (Corollary 1.8).

By using these results and Theorem 9 we obtain the following result.

Proposition 12. *Under the previous assumptions, the following statements are equivalent.*

(1) *X is finite, and (X, α) is of finite type.*

(2) *The ring $P(X)$ is Noetherian (Artinian, semisimple), and $(P(X), \alpha)$ is of finite type.*

(3) *The ring E is Noetherian (Artinian, semisimple).*

Proof. It is enough to observe that X is finite if and only if the ring $P(X)$ is noetherian (Artinian, semisimple) and apply Theorem 9. □

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