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Research Article

Generalized Derivations of *BCC*-Algebras

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The notion of generalized derivations of *BCC*-algebras is introduced, and some related properties are investigated. Also, we consider regular generalized derivations and the D-invariant on ideals of *BCC*-algebras. We also characterized Ker$D$ by generalized derivations.

1. Introduction

Imai and Iséki [1] defined a class of algebras of type $(2,0)$ called *BCK*-algebras which generalizes on one hand the notion of algebra of sets with the set subtraction as the only fundamental nonnullary operation on the other hand the notion of implication algebra. The class of all *BCK*-algebras is a quasivariety. Iséki posed an interesting problem (solved by Wroński [2]) whether the class of *BCK*-algebras is a variety. In connection with this problem, Komori [3] introduced a notion of *BCC*-algebras, and Dudek [4] redefined the notion of *BCC*-algebras by using a dual form of the ordinary definition in the sense of Komori. Dudek and Zhang [5] introduced a new notion of ideals in *BCC*-algebras and described connections between such ideals and congruences. On the other hand, Jun and Xin [6] applied the notion of derivations in ring and near-ring theories to *BCI*-algebras, and they also introduced a new concept called a regular derivation in *BCI*-algebras and investigated related properties. They defined a d-derivation ideal and gave conditions for an ideal to be d-derivation. Prabayak and Leerawat [7] introduced the notion of derivation in *BCC*-algebras. In this paper, the notion of generalized derivations of *BCC*-algebras is introduced, and some related properties are investigated. Also, we consider regular generalized derivations and the D-invariant on ideals of *BCC*-algebras. We also characterized Ker$D$ by generalized derivations.

2. Preliminaries

We review some definitions and properties that will be useful in our results.

Definition 1. Let X be a set with a binary operation "$*$" and a constant 0. Then $(X,*,0)$ is called a *BCC*-algebra if the following axioms are satisfied for all $x,y,z \in X$:

(i) $((x*y)*(z*y))*(x*z)=0$,

(ii) $0*x=0$,

(iii) $x*0=x$,

(iv) $x*x=0$,

(v) $x*y=0$ and $y*x=0 \Rightarrow x=y$.

Define a binary relation $\leq$ on X by letting $x*y=0$ if and only if $x\leq y$. Then $(X,\leq)$ is a partially ordered set.

In any *BCC*-algebra X for all $x,y \in X$, the following properties hold:

(1) $(x*y)*x=0$,

(2) $x\leq y$ implies $x*z\leq y*z$,

(3) $x\leq y$ implies $z*y\leq z*x$.

Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras. Note that a BCC-algebra is a BCK-algebra if and only if satisfies

(4) $(x * y) * z = (x * z) * y$, or

(5) $(x * (x * y)) * y = 0$.

Definition 2. A nonempty subset S of a BCC-algebra X is called subalgebra of X if $x * y \in S$ whenever $x, y \in S$. For a BCC-algebra X, we denote $x \wedge y = y * (y * x)$ for all $x, y \in X$.

Remark 3. Let X be a BCC-algebra and $x, y \in X$; then, consider

(1) $0 \wedge x = 0$,

(2) $x \wedge y \leq y$.

Definition 4. A BCC-algebra is said to be commutative if and only if satisfies for all $x, y \in X$, $x * (x * y) = y * (y * x)$; that is, $x \wedge y = y \wedge x$.

Definition 5. Let X be a BCC-algebra and $\phi \neq I \subseteq X$. I is called an ideal of X if it satisfies the following conditions:

(i) $0 \in I$,

(ii) $x * y \in I$ and $y \in I$ imply $x \in I$.

Definition 6. Let X be a BCC-algebra. A map $d : X \to X$ is a left-right derivation (briefly, (l, r)-derivation) of X, if it satisfies the identity

$$d(x * y) = (d(x) * y) \wedge (x * d(y)) \quad \forall x, y \in X. \tag{1}$$

If d satisfies the identity

$$d(x * y) = (x * d(y)) \wedge (d(x) * y) \quad \forall x, y \in X, \tag{2}$$

then d is a right-left derivation (briefly, (r, l)-derivation) of X. Moreover, if d is both (l, r)-derivation and (r, l)-derivation, then d is a derivation of X.

Definition 7. A self map d of a BCC-algebra X is said to be regular if

$$d(0) = 0. \tag{3}$$

Corollary 8. *A derivation d of a BCC-algebra X is regular.*

Definition 9. Let d be a derivation of a BCC-algebra X. An ideal A of X is said to be d-invariant if $d(A) \subseteq A$, where $d(A) = \{d(x) \mid x \in A\}$.

Proposition 10. *Let X be a BCC-algebra with partial order $\leq$, and let d be a derivation of X. Then the following hold for all $x, y \in X$:*

(i) $d(x) \leq x$,

(ii) $d(x * y) \leq d(x) * y$,

(iii) $d(x * y) \leq x * d(y)$,

(iv) $d(d(x)) \leq x$,

(v) $d(x * d(x)) = 0$,

(vi) $d^{-1}(0) = \{x \in X : d(x) = 0\}$ *is a subalgebra of X.*

Table 1

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	3	1	0

3. Generalized Derivations of *BCC*-Algebras

Definition 11. Let X be a BCC-algebra. A mapping $D : X \to X$ is called a generalized (l, r)-derivation if there exists an (l, r)-derivation $d : X \to X$ such that $D(x * y) = (D(x) * y) \wedge (x * d(y))$ for all $x, y \in X$; if there exists an (r, l)-derivation $d : X \to X$ such that $D(x * y) = (x * D(y)) \wedge (d(x) * y)$ for all $x, y \in X$, the mapping $D : X \to X$ is called a generalized (r, l)-derivation. Moreover, if D is both a generalized (l, r)- and (r, l)-derivation, we say that D is a generalized derivation.

Example 12. Let $X = \{0, 1, 2, 3\}$ be a BCC-algebra with Cayley table (see Table 1).

Define a map $d : X \to X$ by

$$d(x) = \begin{cases} 0 & \text{if } x = 0, 1, 3 \\ 2 & \text{if } x = 2. \end{cases} \tag{4}$$

Then d is a derivation of X. Now we define a map $D : X \to X$ by

$$D(x) = \begin{cases} 0 & \text{if } x = 0, 2 \\ x & \text{if } x = 1, 3. \end{cases} \tag{5}$$

It is easy to verify that D is a generalized (r, l)-derivation of X.

Proposition 13. *A generalized derivation D of a BCC-algebra X is regular.*

Proof. Directly from Corollary 8, we have that $D(0) = 0$. □

Proposition 14. *Let D be a self-map of a BCC-algebra X. Then,*

(i) *if D is a generalized (l, r)-derivation of X, then $D(x) = D(x) \wedge x$ for all $x \in X$;*

(ii) *if D is a generalized (r, l)-derivation of X, then $D(x) = x \wedge d(x)$ for all $x \in X$.*

Proof. (i) If D is a generalized (l, r)-derivation of X, then there exists an (l, r)-derivation d such that $D(x * y) = (D(x) * y) \wedge (x * d(y))$ for all $x, y \in X$. Hence, we get

$$\begin{aligned} D(x) &= D(x * 0) = (D(x) * 0) \wedge (x * d(0)) \\ &= D(x) \wedge (x * d(0)) \\ &= (x * d(0)) * ((x * d(0)) * D(x)) \\ &= (x * 0) * ((x * 0) * D(x)) \\ &= x * (x * D(x)) = D(x) \wedge x. \end{aligned} \tag{6}$$

(ii) If D is a generalized (r,l)-derivation of X, then there exists an (r,l)-derivation d such that $D(x*y)=(x*D(y))\wedge(d(x)*y)$ for all $x,y\in X$. Hence, we get

$$\begin{aligned} D(x) &= D(x*0)=(x*D(0))\wedge(d(x)*0) \\ &= (x*0)\wedge d(x)=x\wedge d(x). \end{aligned} \tag{7}$$

□

Proposition 15. *Let X be a BCC-algebra with partial order $\leq$, and let D be a generalized derivation of X. Then the following hold for all $x,y\in X$:*

(1) $D(x)\leq d(x)\leq x$,

(2) $D(x*y)\leq x*d(y)$,

(3) $D(x*y)\leq d(x)*y$,

(4) $D(x*D(x))=0$,

(5) $D(D(x)*x)=0$,

(6) $D(d(x)*x)=0$,

(7) $D(x*d(x))=0$,

(8) $D(D(x))\leq x$.

Proof. Consider (1) $D(x)=D(x*0)=(x*D(0))\wedge(d(x)*0)=(x*0)\wedge d(x)=x\wedge d(x)\leq d(x)$.

From Proposition 10, $d(x)\leq x$.

(2) $D(x*y)=(D(x)*y)\wedge(x*d(y))\leq x*d(y)$.

(3) $D(x*y)=(x*D(y))\wedge(d(x)*y)\leq d(x)*y$.

(4) $D(x*D(x))=(D(x)*D(x))\wedge(x*d(D(x)))=0\wedge(x*d(D(x)))=0$.

(5) $D(D(x)*x)=(D(x)*D(x))\wedge(d(D(x))*x)=0\wedge(d(D(x))*x)=0$.

(6) $D(d(x)*x)=(D(d(x))*x)\wedge(d(x)*d(x))=(D(d(x))*x)\wedge 0=0*(0*(D(d(x))*x))=0$.

(7) $D(x*d(x))=(x*D(d(x)))\wedge(d(x)*d(x))=(x*D(d(x)))\wedge 0=0*(0*(x*D(d(x))))=0$.

(8) From Proposition 10, we have

$$\begin{aligned} D(D(x)) &= D(x\wedge d(x))=D(d(x)*(d(x)*x)) \\ &= (d(x)*D(d(x)*x)) \\ &\quad \wedge d(d(x)*(d(x)*x)) \\ &= (d(x)*D(0))\wedge d(d(x)*0) \\ &= (d(x)*0)\wedge d(d(x))=d(x)\wedge d(d(x)) \\ &\leq d(d(x))\leq x. \end{aligned} \tag{8}$$

□

Proposition 16. *Let X be a BCC-algebra. Then $D_n(D_{n-1}(\ldots(D_2(D_1(x)))\ldots))\leq x$ for $n\in N$, where $D_1,D_2,\ldots,D_n$ are generalized derivations of X.*

Proof. Trivial. □

Definition 17. Let D be a generalized derivation of a *BCC*-algebra X. An ideal A of X is said to be D-invariant if $D(A)\subseteq A$, where

$$D(A)=\{D(x)\mid x\in A\}. \tag{9}$$

Theorem 18. *Let D be a generalized derivation of a BCC-algebra X. Then every ideal A of X is D-invariant.*

Proof. Let A be an ideal of a *BCC*-algebra X. Let $y\in D(A)$. Then $y=D(x)$ for some $x\in A$. It follows that $y*x=D(x)*x=0\in A$, which implies that $y\in A$. Thus $D(A)\subseteq A$. Hence, A is D-invariant. □

Definition 19. Let X be a *BCC*-algebra and let D be a generalized derivation. Define a Ker$D$ by Ker$D=\{x\in X: D(x)=0\}$.

Theorem 20. *Let X be a BCC-algebra and let D be generalized derivation. If $y\in\mathrm{Ker}(D)$ and $x\in X$, then $x\wedge y\in\mathrm{Ker}(D)$.*

Proof. Let $y\in\mathrm{Ker}(D)$. Then, we get $D(y)=0$, and so

$$\begin{aligned} D(x\wedge y) &= D(y*(y*x)) \\ &= (D(y)*(y*x))\wedge(y*d(y*x)) \\ &= (0*(y*x))\wedge(y*d(y*x)) \\ &= 0\wedge(y*d(y*x))=0. \end{aligned} \tag{10}$$

Hence, we have $x\wedge y\in\mathrm{Ker}(D)$. □

Theorem 21. *Let X be a commutative BCC-algebra and D be a generalized derivation. If $x\leq y$ and $y\in\mathrm{Ker}(D)$, then $x\in\mathrm{Ker}(D)$.*

Proof. Let $x\leq y$ and $y\in\mathrm{Ker}(D)$. Then we get $x*y=0$ and $D(y)=0$, and so

$$\begin{aligned} D(x) &= D(x*0)=D(x*(x*y))=D(y*(y*x)) \\ &= (D(y)*(y*x))\wedge(y*d(y*x)) \\ &= (0*(y*x))\wedge(y*d(y*x)) \\ &= 0\wedge(y*d(y*x))=0. \end{aligned} \tag{11}$$

Hence we have $x\in\mathrm{Ker}(D)$. □

Theorem 22. *Let X be a BCC-algebra and let D be a generalized derivation. If $x\in\mathrm{Ker}(D)$, one has $x*y\in\mathrm{Ker}(D)$ for all $y\in X$.*

Proof. Let $x\in\mathrm{Ker}(D)$. Then, $D(x)=0$. Thus, we have

$$\begin{aligned} D(x*y) &= (D(x)*y)\wedge(x*d(y)) \\ &= (0*y)\wedge(x*d(y))=0\wedge(x*d(y))=0. \end{aligned} \tag{12}$$

Hence, $x*y\in\mathrm{Ker}(D)$. □

Theorem 23. *Let X be a BCC-algebra and let D be a generalized derivation. Then* $\mathrm{Ker}(D)$ *is a subalgebra of X.*

Proof. Directly from Theorem 22. □

Definition 24. Let X be a BCC-algebra and let D be a generalized derivation on X. Denote $\mathrm{Fix}_D(X) = \{x \in X : D(x) = x\}$.

Proposition 25. *Let X be a BCC-algebra and let D be a generalized derivation on X. If $x \in \mathrm{Fix}_D(X)$, then $d(x) = x$.*

Proof. Since $x \in \mathrm{Fix}_D(X)$, then $D(x) = x$. From Proposition 15, we have $D(x) \leq d(x) \leq x$ that implies that $x \leq d(x) \leq x$. Hence, $d(x) = x$. □

Proposition 26. *Let X be a BCC-algebra and let D be a generalized derivation on X. Then* $\mathrm{Fix}_D(X)$ *is a subalgebra of X.*

Proof. If $x, y \in \mathrm{Fix}_D(X)$, we get $D(x) = x$ and $d(y) = y$, and so $D(x * y) = (D(x) * y) \wedge (x * d(y)) = (x * y) \wedge (x * y) = x * y$. Hence, $x * y \in \mathrm{Fix}_D(X)$. □

Proposition 27. *Let X be a BCC-algebra and let D be a generalized derivation on X. If $x, y \in \mathrm{Fix}_D(X)$, we have $x \wedge y \in \mathrm{Fix}_D(X)$.*

Proof. Let $x, y \in \mathrm{Fix}_D(X)$, Then $D(x) = x$ and $D(y) = y$. From Proposition 26, we have $y * x \in \mathrm{Fix}_D(X)$, and so $d(y * x) = y * x$. Hence, we have

$$\begin{aligned} D(x \wedge y) &= D(y * (y * x)) \\ &= (D(y) * (y * x)) \wedge (y * d(y * x)) \\ &= (y * (y * x)) \wedge (y * (y * x)) \\ &= y * (y * x) = x \wedge y. \end{aligned} \tag{13}$$

Hence, $x \wedge y \in \mathrm{Fix}_D(X)$. □

References

[1] Y. Imai and K. Iséki, "On axiom systems of propositional calculi. XIV," *Proceedings of the Japan Academy*, vol. 42, pp. 19–22, 1966.

[2] A. Wroński, "*BCK*-algebras do not form a variety," *Mathematica Japonica*, vol. 28, no. 2, pp. 211–213, 1983.

[3] Y. Komori, "The class of *BCC*-algebras is not a variety," *Mathematica Japonica*, vol. 29, no. 3, pp. 391–394, 1984.

[4] W. A. Dudek, "The number of subalgebras of finite *BCC*-algebras," *Bulletin of the Institute of Mathematics*, vol. 20, no. 2, pp. 129–135, 1992.

[5] W. A. Dudek and X. Zhang, "On ideals and congruences in *BCC*-algebras," *Czechoslovak Mathematical Journal*, vol. 48, no. 123, pp. 21–29, 1998.

[6] Y. B. Jun and X. L. Xin, "On derivations of *BCI*-algebras," *Information Sciences*, vol. 159, no. 3-4, pp. 167–176, 2004.

[7] C. Prabayak and U. Leerawat, "On dervations of *BCC*-algebra," *Kasetsart Journal*, vol. 43, no. 398, 401 pages, 2009.