Research Article

Four-Point n -Ary Interpolating Subdivision Schemes

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Received 27 July 2013; Revised 27 September 2013; Accepted 27 September 2013

Academic Editor: Palle E. Jorgensen

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We present an efficient and simple algorithm to generate 4-point n -ary interpolating schemes. Our algorithm is based on three simple steps: second divided differences, determination of position of vertices by using second divided differences, and computation of new vertices. It is observed that 4-point n -ary interpolating schemes generated by completely different frameworks (i.e., Lagrange interpolant and wavelet theory) can also be generated by the proposed algorithm. Furthermore, we have discussed continuity, Hölder regularity, degree of polynomial generation, polynomial reproduction, and approximation order of the schemes.

1. Introduction

In general, subdivision schemes can be divided into two categories: approximating and interpolating. For interpolating curve subdivision, new vertices are computed and added to the old polygons for each time of subdivision and the limit curve passes through all the vertices of the original control polygon. Interpolating subdivision schemes are more attractive than approximating schemes in computer aided geometric designs because of their interpolation property. In addition, the interpolation subdivisions are more intuitive to the users.

Initial work on interpolating subdivision schemes was started by Dubuc [1]. Later on, Deslauriers and Dubuc [2] have introduced a family of schemes by using Lagrange polynomials indexed by the size of the mask and the arity. In [3], Dyn et al. have studied a family of interpolating schemes with mask size of four. Consequent to this, the research communities are interested in introducing higher arity schemes (i.e., ternary, quaternary, $\dots$, n -ary) which give better results and less computational cost. Lian [4] has constructed both the $2m$ -point n -ary for any $n \geq 2$ and $(2m + 1)$ -point n -ary for any odd $n \geq 3$ interpolatory subdivision schemes for curve design by using wavelet theory. Mustafa and Rehman [5] have presented general formulae for the mask of $(2b + 4)$ -point n -ary interpolating and approximating schemes for any integer $b \geq 0$ and $n \geq 2$. These formulae

provide mask of higher arity schemes and generalize lower arity schemes. Mustafa et al. [6] have presented an explicit formula for the mask of odd points n -ary, for any odd $n \geq 3$, interpolating subdivision schemes.

In [7], it has been proved that the large support and higher arity schemes may outperform than small support and lower arity schemes. Even though these schemes are not in practice. It has been suggested that the research on large support and higher arity schemes may continue.

The multistage approach is very handy to construct subdivision schemes. This idea is variously used by others. Lane and Riesenfeld [8] have presented two fast subdivision algorithms for the evaluation of B-spline and Bernstein curves and surfaces. They have expressed new idea, in which, after a single duplication stage in which the number of control points is doubled by just taking each point twice, a sequence of smoothing operators is applied. Catmull and Clark [9] have used this technique to present the original description of subdivision in which each refinement is expressed in three stages. Zorin and Schröder [10] have considered the construction of an increasing sequence of alternating primal/dual quadrilateral subdivision schemes based on a simple averaging approach. Oswald and Schröder [11] used the same motif to generate families of subdivision schemes.

Augsdörfer et al. [12] first describe the original 4-point binary subdivision scheme and then apply six variations on the scheme which are obtained by tuning the local stages

in various ways, producing some interesting subdivision schemes all of which are improvements on the original. We generalize the same technique to build the family of four-point n -ary interpolating subdivision schemes. It is observed that 4-point n -ary interpolating schemes introduced by [2, 4] can also be constructed by our generalized technique, even though these schemes have been constructed by different frameworks.

This paper is organized as follows: Section 2 presents some preliminary results. Section 3 consists of multistep algorithm which generates 4-point n -ary interpolating subdivision schemes. Analysis of two schemes is also presented in this section. In Section 4, Hölder regularity, polynomial generation, polynomial reproductions and approximation order of ternary and quaternary subdivision schemes have been discussed. Numerical examples and conclusion are presented in Section 5.

2. Preliminaries

A general compact form of univariate n -ary subdivision scheme S [13] which maps polygon $f^k = \{f_i^k\}_{i \in \mathbb{Z}}$ to a refined polygon $f^{k+1} = \{f_i^{k+1}\}_{i \in \mathbb{Z}}$ is defined by

$$f_i^{k+1} = \sum_{j \in \mathbb{Z}} a_{nj-i} f_j^k, \quad i \in \mathbb{Z}, \quad (1)$$

where the set $a = \{a_i : i \in \mathbb{Z}\}$ of coefficients is called the mask at k th level of refinement. A necessary condition for the uniform convergence of subdivision scheme (1) is that

$$\sum_{j \in \mathbb{Z}} a_{nj} = \sum_{j \in \mathbb{Z}} a_{nj+1} = \sum_{j \in \mathbb{Z}} a_{nj+2} = \cdots = \sum_{j \in \mathbb{Z}} a_{nj+n-1} = 1. \quad (2)$$

A subdivision scheme is uniformly convergent if for any initial data $f^0 = \{f_i^0 : i \in \mathbb{Z}\}$, there exists a continuous function f such that for any closed interval $I \subset \mathbb{R}$, it satisfies

$$\lim_{k \rightarrow \infty} \sup_{i \in n^k I} |f_i^k - f(n^{-k}i)| = 0. \quad (3)$$

Obviously, $f = S^\infty f^0$.

A symbol called Laurent polynomial

$$a(z) = \sum_{i \in \mathbb{Z}} a_i z^i \quad (4)$$

of the mask $a = \{a_i : i \in \mathbb{Z}\}$ plays an efficient role to analyze the convergence and smoothness of subdivision scheme. From (2) and (4) the Laurent polynomial of convergent subdivision scheme satisfies

$$a(\alpha_n^j) = 0, \quad j = 1, 2, \dots, n-1, \quad a(1) = n, \quad (5)$$

where $\alpha_n^j = \exp((2\pi i/n)j)$ are the n th root of unity. This condition guarantees the existence of a related subdivision scheme for the divided differences of the original control points and the existence of an associated Laurent polynomial

$$a^{(1)}(z) = nz^{n-1} \left(\frac{1-z}{1-z^n} \right) a(z). \quad (6)$$

The subdivision scheme S_1 with Laurent polynomial $a^{(1)}(z)$ is related to the scheme S with Laurent polynomial $a(z)$ by the following theorem.

Theorem 1 (see [14]). *Let S denote a subdivision scheme with Laurent polynomial $a(z)$ satisfying (5). Then there exists a subdivision scheme S_1 with the property*

$$\Delta f^k = S_1 \Delta f^{k-1}, \quad (7)$$

where $f^k = S^k f^0$ and $\Delta f^k = \{(\Delta f^k)_i = n^k(f_{i+1}^k - f_i^k); i \in \mathbb{Z}\}$. Furthermore, S is a uniformly convergent if and only if $(1/n)S_1$ converges uniformly to zero function for all initial data f^0 , in the sense that

$$\lim_{k \rightarrow \infty} \left(\frac{1}{n} S_1 \right)^k f^0 = 0. \quad (8)$$

The above theorem indicates that for any given scheme S , with the mask a satisfying (2), we can prove the uniform convergence of S by deriving the mask of $(1/n)S_1$ and computing $\|((1/n)S_1)^i\|_\infty$ for $i = 1, 2, 3, \dots, L$, where L is the first integer for which $\|((1/n)S_1)^L\|_\infty < 1$. If such an L exists, then S converges uniformly. Since there are “ n ” rules for computing the values at the next refinement level, so we define the norm

$$\|S\|_\infty = \max \left\{ \sum_{j \in \mathbb{Z}} |a_{nj}|, \sum_{j \in \mathbb{Z}} |a_{nj+1}|, \sum_{j \in \mathbb{Z}} |a_{nj+2}|, \dots, \sum_{j \in \mathbb{Z}} |a_{nj+n-1}| \right\}, \quad (9)$$

$$\left\| \left(\frac{1}{n} S_\beta \right)^L \right\|_\infty = \max \left\{ \sum_{j \in \mathbb{Z}} |b_{i+n^L j}^{[\beta, L]}|; i = 0, 1, 2, \dots, n^L - 1 \right\}, \quad (10)$$

where

$$\begin{aligned} b^{[\beta, L]}(z) &= \frac{1}{n^L} \prod_{j=0}^{L-1} a_\beta(z^{n^j}), \\ a_\beta(z) &= \left(nz^{n-1} \left(\frac{1-z}{1-z^n} \right) \right) a_{\beta-1}(z) \\ &= \left(nz^{n-1} \left(\frac{1-z}{1-z^n} \right) \right)^\beta a(z), \quad \beta \geq 1. \end{aligned} \quad (11)$$

Theorem 2 (see [13]). *Let S be the subdivision scheme with a characteristic $\mathbb{U}$ -polynomial $a(z) = ((z^n - 1)/(nz^{n-1}(z - 1)))^m q(z)$, $q \in \mathbb{U}$. If the subdivision scheme S_m , corresponding to the $\mathbb{U}$ -polynomial $q(z)$, converges uniformly, then $S^\infty f^0 \in C^m(\mathbb{R})$ for any initial control polygon f^0 .*

Corollary 3 (see [13]). *If S is a subdivision scheme of the form above and $(1/n)S_{m+1}$ converges uniformly to the zero function for all initial data f^0 , then $S^\infty f^0 \in C^m(\mathbb{R})$ for any initial control polygon f^0 .*

FIGURE 1: Labeling a sample control polygon. The newly inserted points between old vertices b and c are referred to as $p_1, p_2, \dots, p_{n-1}$, respectively.

The above Corollary 3 indicates that for any given n -ary subdivision scheme S , we can prove $S^\infty f^0 \in C^m$ by first deriving the mask of $(1/n)S_{m+1}$ and then computing $\|((1/n)S_{m+1})^i\|_\infty$ for $i = 1, 2, 3, \dots, L$ (where L is the first integer for which $\|((1/n)S_{m+1})^L\|_\infty < 1$). If such an L exists, then $S^\infty f^0 \in C^m$.

Definition 4 (see [15]). For any subdivision scheme S_a we denote by $\tau = a'(1)/n$ the corresponding parametric shift and attach the data f_i^l for $i \in \mathbb{Z}, l \in \mathbb{N}$ to the parameter values

$$t_i^l = t_0^l + \frac{i}{n^l} \quad \text{with } t_0^l = t_0^{l-1} - \frac{\tau}{n^l}. \quad (12)$$

Theorem 5 (see [15]). A convergent subdivision scheme S_a reproduces polynomials of degree d with respect to the parameterizations (12) if and only if

$$a^{(k)}(1) = n \prod_{l=0}^{k-1} (\tau - l), \quad a^{(k)}(\alpha_n^j) = 0, \quad (13)$$

$$j = 1, 2, \dots, n-1 \quad \text{for } k = 0, \dots, d,$$

where $\alpha_n^j = \exp((2\pi i/n)j)$, $j = 1, 2, \dots, n-1$.

Theorem 6 (see [16]). A convergent subdivision scheme S_a that reproduces polynomial π_n (set of polynomials at most degree n) has an approximation order of $n+1$.

3. Multistep Algorithm

We construct 4-point n -ary interpolating subdivision schemes by using three-step algorithm instead of using Lagrange polynomial and wavelets theory, and so forth. These three steps are as follows.

(i) Calculate second divided differences.

At each old vertex compute the second divided difference D ; that is D_b is the second divided difference at point b and D_c is the second divided difference at point c (see Figure 1):

$$D_b = \frac{c - 2b + a}{n^2}, \quad (14)$$

$$D_c = \frac{d - 2c + b}{n^2},$$

where $n = 3, 4, \dots$

FIGURE 2: Labeling a sample control polygon. The newly inserted points between old vertices b and c are referred to as p_1 and p_2 , respectively.

(ii) Determine the position of vertices by using divided differences.

In n -ary subdivision scheme each segment is divided into n subsegments at each refinement level. First point is inserted at the position $1/n$ and second point at the position $2/n$ and proceeding in the same way the $(n-1)$ -th point at the position $(n-1)/n$. By using divided differences D_b and D_c , we calculate the position of $(n-1)$ -th newly inserted points between two old vertices b and c by

$$D_{p_j} = \left(\frac{n-j}{n}\right) D_b + \left(\frac{j}{n}\right) D_c, \quad j = 1, 2, 3, \dots, n-1. \quad (15)$$

(iii) Computation new vertices.

Finally, we calculate positions of new vertices $p_1, p_2, \dots, p_{n-1}$ by using $D_{p_1}, D_{p_2}, \dots, D_{p_{n-1}}$, respectively, by

$$D_{p_1} = p_2 - 2p_1 + b,$$

$$D_{p_i} = p_{i+1} - 2p_i + p_{i-1}, \quad (16)$$

$$D_{p_{n-1}} = c - 2p_{n-1} + p_{n-2},$$

where $i = 2, 3, \dots, n-2$. By solving, above set of equations, we get the position of new vertices $p_1, p_2, \dots, p_{n-1}$.

3.1. Examples. A 4-Point Ternary Interpolating Scheme. In ternary subdivision scheme each segment is divided into three subsegments at each refinement level. One point is inserted at the position $1/3$ and another point at the position $2/3$ (see Figure 2). For $n = 3$ in (14), we get second divided differences D_b and D_c at points b and c

$$D_b = \frac{c - 2b + a}{9}, \quad (17)$$

$$D_c = \frac{d - 2c + b}{9}.$$

For $n = 3$ in (15), we get

$$D_{p_j} = \frac{3-j}{3} D_b + \frac{j}{3} D_c, \quad j = 1, 2. \quad (18)$$

FIGURE 3: Labeling a sample control polygon. The newly inserted points between old vertices b and c are referred to as p_1 , p_2 , and p_3 respectively.

By using (17), we get

$$\begin{aligned} D_{p_1} &= \frac{2a - 3b + d}{27}, \\ D_{p_2} &= \frac{a - 2b + 2d}{27}. \end{aligned} \quad (19)$$

For $n = 3$ in (16), we have

$$\begin{aligned} D_{p_1} &= p_2 - 2p_1 + b, \\ D_{p_2} &= c - 2p_2 + p_1. \end{aligned} \quad (20)$$

This implies

$$\begin{aligned} p_1 &= \frac{2b + c - 2D_{p_1} - D_{p_2}}{3}, \\ p_2 &= \frac{b + 2c - D_{p_1} - 2D_{p_2}}{3}. \end{aligned} \quad (21)$$

By using (19), we get

$$\begin{aligned} p_1 &= \frac{-5a + 60b + 30c - 4d}{81}, \\ p_2 &= \frac{-4a + 30b + 60c - 5d}{81}. \end{aligned} \quad (22)$$

Now 4-point ternary scheme can be written as

$$\begin{aligned} f_{3i}^{k+1} &= f_i^k, \\ f_{3i+1}^{k+1} &= -\frac{5}{81}f_{i-1}^k + \frac{60}{81}f_i^k + \frac{30}{81}f_{i+1}^k - \frac{4}{81}f_{i+2}^k, \\ f_{3i+2}^{k+1} &= -\frac{4}{81}f_{i-1}^k + \frac{30}{81}f_i^k + \frac{60}{81}f_{i+1}^k - \frac{5}{81}f_{i+2}^k. \end{aligned} \quad (23)$$

The above scheme was introduced by Deslauriers and Dubuc [2] in 1989 by using Lagrange interpolant. Later on, this scheme was also reconstructed by [4] by using wavelet theory.

A 4-Point Quaternary Interpolating Scheme. In quaternary subdivision scheme each segment is divided into four subsegments at each refinement level. First, second and third points are inserted at the positions $1/4$, $2/4$, and $3/4$, respectively (see Figure 3). For $n = 4$ in (14)–(16), we get the following

4-point quaternary interpolating scheme of Deslauriers and Dubuc

$$\begin{aligned} f_{4i}^{k+1} &= f_i^k, \\ f_{4i+1}^{k+1} &= -\frac{7}{128}f_{i-1}^k + \frac{105}{128}f_i^k + \frac{35}{128}f_{i+1}^k - \frac{5}{128}f_{i+2}^k, \\ f_{4i+2}^{k+1} &= -\frac{1}{16}f_{i-1}^k + \frac{9}{16}f_i^k + \frac{9}{16}f_{i+1}^k - \frac{1}{16}f_{i+2}^k, \\ f_{4i+3}^{k+1} &= -\frac{5}{128}f_{i-1}^k + \frac{35}{128}f_i^k + \frac{105}{128}f_{i+1}^k - \frac{7}{128}f_{i+2}^k. \end{aligned} \quad (24)$$

This scheme was also reconstructed by [4] in 2009.

Remark 7. By substituting $n \geq 3$ in (14)–(16), we get the mask of 4-point n -ary interpolating scheme generated by two different frameworks, that is, Lagrange interpolation [2] and wavelet theory [4]. In this paper, we propose alternative approach completely different from these approaches. In coming section, we discuss two existing schemes also produced by our framework.

3.2. Analysis of Subdivision Schemes. Here we present the analysis of 4-point ternary and quaternary interpolating subdivision schemes. Analysis of other schemes can be done in a similar way.

3.2.1. Analysis of 4-Point Ternary Subdivision Scheme. The Laurent polynomial $a(z)$ for the scheme (23) is

$$\begin{aligned} a(z) &= \frac{1}{81} \{ -4z^5 - 5z^4 + 30z^2 \\ &\quad + 60z^1 + 81 + 60z^{-1} + 30z^{-2} \\ &\quad - 5z^{-4} - 4z^{-5} \}. \end{aligned} \quad (25)$$

Using (11) for $n = 3$, $\beta = 1, 2$ and $L = 1$, we get

$$\begin{aligned} b^{[1,1]}(z) &= \frac{1}{3}a_1(z) = -\frac{4}{81}z^5 - \frac{1}{81}z^4 \\ &\quad + \frac{5}{81}z^3 + \frac{26}{81}z^2 + \frac{29}{81}z^1 + \frac{26}{81} + \frac{5}{81}z^{-1} \\ &\quad - \frac{1}{81}z^{-2} - \frac{4}{81}z^{-3}, \end{aligned} \quad (26)$$

$$\begin{aligned} b^{[2,1]}(z) &= \frac{1}{3}a_2(z) = -\frac{4}{27}z^5 + \frac{1}{9}z^4 + \frac{2}{9}z^3 \\ &\quad + \frac{17}{27}z^2 + \frac{2}{9}z^1 + \frac{1}{9} - \frac{4}{27}z^{-1}. \end{aligned} \quad (27)$$

If S_β is the scheme corresponding to $a_\beta(z)$, then by (10)

$$\left\| \frac{1}{3}S_\beta \right\|_\infty = \max \left\{ \sum_{j \in \mathbb{Z}} |b_{i+3j}^{[\beta,1]}| : i = 0, 1, 2 \right\}, \quad \beta = 1, 2. \quad (28)$$

Using (9), (26), and (27), we get

$$\begin{aligned}\left\|\frac{1}{3}S_1\right\|_{\infty} &= \max\left\{\left|\frac{-4}{81}\right| + \left|\frac{26}{81}\right| + \left|\frac{5}{81}\right|, \left|\frac{-1}{81}\right| + \left|\frac{29}{81}\right| + \left|\frac{-1}{81}\right|\right\}, \\ \left\|\frac{1}{3}S_2\right\|_{\infty} &= \max\left\{\left|\frac{-4}{27}\right| + \left|\frac{17}{27}\right| + \left|\frac{-4}{27}\right|, \left|\frac{1}{9}\right| + \left|\frac{2}{9}\right|\right\}.\end{aligned}\quad (29)$$

As we see $\|(1/3)S_1\|_{\infty} < 1$, then by Theorem 1 the scheme is C^0 . Similarly, $\|(1/3)S_2\|_{\infty} < 1$ then by Corollary 3 the scheme is C^1 .

Remark 8. Similarly, we can prove that quaternary 4-point subdivision scheme is C^1 .

4. Properties of Subdivision Schemes

In this section, we show that how limit curve of 4-point ternary and 4-point quaternary subdivision schemes give response to initial polynomial data. For this we discuss Hölder regularity, degree of polynomial generation, polynomial reproduction and approximation order of the schemes (23) and (24).

4.1. Hölder Regularity. Hölder regularity is an extension of the notion of continuity which gives more information about any scheme. A function $\phi : R \rightarrow R$ is defined to be regular of order $y + \alpha$ (for $y \in N_0$ and $0 < \psi \leq 1$) if it is y time continuously differentiable and ϕ^y is Lipschitz of order α

$$|\phi^{(y)}(x+h) - \phi^{(y)}(x)| \leq c|h|^{\psi} \quad (30)$$

for all x and h in R and some constant c .

According to Dyn and Levin [17] and Rioul [18], the Hölder regularity of subdivision scheme with symbol $a(z)$ can be computed in the following way. Let $a(z) = ((1+z+\dots+z^{n-1})/n)^k b(z)$, without loss of generality we can assume $b_0, \dots, b_m$ to be the nonzero coefficients of $b(z)$ and let $B_0, B_1, \dots, B_m$ be the $m \times m$ matrices with elements

$$(B_q)_{ij} = b_{m+i-nj+q}, \quad i, j = 1, \dots, m, \quad q = 0, 1, \dots, m. \quad (31)$$

Then the Hölder regularity is given by $r = k - \log_n(\mu)$, where μ is the joint spectral radius of the matrices $B_0, B_1, \dots, B_m$, that is,

$$\begin{aligned}\mu &= \rho(B_0, B_1, \dots, B_m) \\ &= \limsup_{l \rightarrow \infty} \left(\max \left\{ \|B_{i_1} \dots B_{i_l} B_{i_l}\|_{\infty}^{1/l} : i_l \in \{0, 1\} \right\} \right), \\ \max \{\rho(B_0), \dots, \rho(B_m)\} &\leq \rho(B_0, \dots, B_m) \\ &\leq \max \{\|B_0\|_{\infty}, \dots, \|B_m\|_{\infty}\}.\end{aligned}\quad (32)$$

Since μ is bounded from below by the spectral radii and from above by the norm of the metrics $B_0, B_1, \dots, B_m$, then

$$\max \{\rho(B_0), \dots, \rho(B_m)\} \leq \mu \leq \max \{\|B_0\|_{\infty}, \dots, \|B_m\|_{\infty}\}. \quad (33)$$

Theorem 9. The Hölder regularity of scheme (23) is $r = 4 - \log_3(11) = 1.8173$.

Proof. The Laurent polynomial (25) of the scheme (23) can be written as

$$a(z) = \left(\frac{1+z+z^2}{3} \right)^4 b(z), \quad (34)$$

where

$$b(z) = \frac{1}{z^5} (-4 + 11z - 4z^2). \quad (35)$$

From (31) and (34), $b_0 = -4$, $b_1 = 11$, $b_2 = -4$, $k = 4$, $m = 2$ and $n = 3$, thus $q = 0, 1, 2$, and then B_0, B_1 , and B_2 are the matrices with elements

$$\begin{aligned}(B_0)_{ij} &= b_{2+i-3j}, \\ (B_1)_{ij} &= b_{2+i-3j+1}, \\ (B_2)_{ij} &= b_{2+i-3j+2},\end{aligned}\quad (36)$$

where $i, j = 1, 2$. This implies

$$B_0 = \begin{pmatrix} -4 & 0 \\ 11 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 11 & 0 \\ -4 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}. \quad (37)$$

From (33) and (37) we have

$$\max \{4, 11, 4\} \leq \mu \leq \max \{11, 11, 4\}. \quad (38)$$

Since the largest eigenvalue and the max-norm of the metrics are 11, so

$$r = 4 - \log_3(11) = 1.8173. \quad (39)$$

□

Theorem 10. The Hölder regularity of scheme (24) is $r = 4 - \log_4(24) = 1.7077$.

Proof. The Laurent polynomial $a(z)$ of scheme (24) can be written as

$$a(z) = \left(\frac{1+z+z^2+z^3}{4} \right)^4 b(z), \quad (40)$$

where

$$b(z) = \frac{1}{z^7} (-10 + 24z - 10z^2). \quad (41)$$

From (31) and (40), $b_0 = -10$, $b_1 = 24$, $b_2 = -10$, $k = 4$, $m = 2$ and $n = 4$, thus $q = 0, 1, 2$, and then B_0, B_1 , and B_2 are the matrices with elements

$$\begin{aligned}(B_0)_{ij} &= b_{2+i-4j}, \\ (B_1)_{ij} &= b_{2+i-4j+1}, \\ (B_2)_{ij} &= b_{2+i-4j+2},\end{aligned}\quad (42)$$

where $i, j = 1, 2$. This implies

$$B_0 = \begin{pmatrix} -10 & 0 \\ 24 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 24 & 0 \\ -10 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix}. \quad (43)$$

From (33) and (43) we have

$$\max \{10, 24, 10\} \leq \mu \leq \max \{24, 24, 10\}. \quad (44)$$

Thus the largest eigenvalue and the max-norm of the metrics are 24, so

$$r = 4 - \log_4(24) = 1.7077. \quad (45)$$

□

Remark 11. It is generally observed that as we decrease the arity of the scheme the Hölder exponent increases. For example, Cashman et al. [19] have proved that the Hölder exponent for binary scheme is 2 while from above theorems we see that Hölder exponents for ternary and quaternary schemes are 1.8173 and 1.7077, respectively. Trivially, the Hölder exponent approaches to 1 for large arity scheme.

4.2. Polynomial Generation. The generation degree of a subdivision scheme is the maximum degree of polynomials that can potentially be generated by the scheme, provided that the initial data is chosen correctly. Suppose p^0 is polynomial of degree d of initial data f_i^0 and symbol of the scheme is

$$a(z) = (1 + z + \dots + z^{n-1})^{d+1} b(z); \quad (46)$$

then the limit curve of the refined data f_i^k at any level k is polynomial of degree d . So the condition is necessary and sufficient for the scheme being able to generate polynomial of degree d .

Theorem 12. The degree of polynomial generation of scheme (23) is 3.

Proof. Since the Laurent polynomial $a(z)$ of the scheme (23) is

$$a(z) = (1 + z + z^2)^{(3+1)} b(z), \quad (47)$$

where

$$b(z) = \frac{1}{(3)^4 z^5} (-4 + 11z - 4z^2), \quad (48)$$

then degree of polynomial generation is 3. □

Theorem 13. The degree of polynomial generation of scheme (24) is 3.

Proof. Since the Laurent polynomial of (24) can be written as

$$a(z) = (1 + z + z^2 + z^3)^{(3+1)} b(z), \quad (49)$$

where

$$b(z) = \frac{1}{(4)^4 z^7} (-10 + 24z - 10z^2), \quad (50)$$

then degree of polynomial generation of scheme is 3. □

4.3. Polynomial Reproduction and Approximation Order. The polynomial reproduction property has its own importance, as the reproduction property of the polynomials up to a certain degree d implies that the scheme has $d + 1$ approximation order. Polynomial reproduction of degree d requires polynomial generation of degree d . For this, polynomial reproduction can be made from initial data which has been sampled from some polynomial function. In the view [15], the polynomial reproduction property of the proposed scheme can be obtained after having the parameterizations τ given in (12).

Theorem 14. A convergent subdivision scheme (23) reproduces polynomials of degree 3 with respect to the parameterizations (12) if and only if

$$a^{(k)}(1) = 3 \prod_{l=0}^{k-1} (\tau - l), \quad a^{(k)}(\alpha_3^j) = 0, \quad j = 1, 2, \quad (51)$$

for $k = 0, \dots, 3$, $\alpha_3^j = \exp((2\pi i/3)j)$ and $\tau = a'(1)/3$.

Proof. By taking first derivative of (25) and substituting $z = 1$ in it, we get

$$a^{(1)}(1) = 0. \quad (52)$$

This implies that

$$\tau = \frac{a^{(1)}(1)}{3} = 0. \quad (53)$$

So from (12), the scheme (23) has primal parametrization. For $k = 0$, $j = 1$, and from (25), we get

$$a^{(0)}(\alpha_3^1) = a(e^{2\pi i/3}) = 0. \quad (54)$$

Similarly, for $j = 1, 2$ and $k = 0, 1, 2, 3$ (k denotes the order of derivative)

$$a^{(k)}(\alpha_3^j) = 0. \quad (55)$$

By (25), we get $a(1) = 3$. Also $3 \prod_{l=0}^{-1} (0 - l) = 3$, which implies that $a(1) = 3 \prod_{l=0}^{-1} (\tau - l)$. Similarly for $k = 1, 2, 3$, we can easily show that

$$a^{(k)}(1) = 3 \prod_{l=0}^{k-1} (\tau - l), \quad (56)$$

which completes the proof. □

Since scheme (23) reproduces polynomial of degree 3, so by using Theorem 6, we get the following theorem.

Theorem 15. A 4-point ternary interpolating scheme (23) has an approximation order of 4.

Proof of the following theorem is similar to the proof of Theorem 14.

FIGURE 4: Comparison of the limit curves generated by proposed 4-point 2-ary, 3-ary, 4-ary, 5-ary, 6-ary, and 7-ary interpolating subdivision schemes at 1st subdivision level.

Theorem 16. A convergent subdivision scheme (24) reproduces polynomials of degree 3 with respect to the parameterizations (12) if and only if

$$a^{(k)}(1) = 4 \prod_{l=0}^{k-1} (\tau - l), \quad a^{(k)}(\alpha_4^j) = 0, \quad j = 1, 2, 3 \quad (57)$$

for $k = 0, \dots, 3$, $\alpha_4^j = \exp((2\pi i/4)j)$ and $\tau = a'(1)/4$.

Again by Theorem 6, we get the following theorem.

Theorem 17. A 4-point quaternary interpolating scheme (24) has an approximation order of 4.

Remark 18. The considered schemes (i.e., 3-point ternary and 4-point quaternary) are exactly the same as obtained

by using imputation from Lagrange interpolation at four consecutive points [2]; therefore, 3-point ternary and 4-point quaternary schemes have polynomial reproduction of degree 3 and approximation properties are obvious (said by the referee). These schemes also have been generated by [4] using wavelet framework and by our algorithm, so by construction these schemes do not inherit these properties and that is only the reason to include the above theorems.

5. Numerical Examples and Conclusion

Six examples are depicted to show the usefulness of 4-point 2-ary, 3-ary, 4-ary, 5-ary, 6-ary, and 7-ary interpolating subdivision schemes at 1st subdivision level in Figure 4. In this figure the control polygons are drawn by dotted lines

while the subdivision curves are drawn by solid lines. From Figure 4, it is clear that the initial polygon converges rapidly to limit curve as we increase the arity of the subdivision scheme.

For many subdivision levels with any of these schemes the limit curves of the n -ary scheme with large n may exhibit sharper singularities at the initial control points compared to the schemes with small n (also mentioned by the referee). But if the initial control points come from noisy source, then n -ary scheme with large n is the better choice. The scheme with small n exhibit overfitting (see [7]). The main purpose to give comparison at first level is to provide the clear visual differences among the refined polygons produced by different schemes.

In this paper, we have presented a multistep algorithm which generates 4-point n -ary interpolating subdivision schemes. We have also observed that the 4-point n -ary schemes generated by Lagrange polynomials and wavelet theory can also be generated by proposed multistep algorithm. Some significant properties like Hölder regularity, degree of polynomial generation, degree of polynomial reproduction, and approximation order have been also discussed in this paper.

Acknowledgments

This work was supported by Indigenous Ph.D. Scholarship Scheme of HEC Pakistan. The authors would like to thank the referees for their helpful suggestions and comments which show the way to improve this work.

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