Research Article

Applying GG-Convex Function to Hermite-Hadamard Inequalities Involving Hadamard Fractional Integrals

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By virtue of fractional integral identities, incomplete beta function, useful series, and inequalities, we apply the concept of GG-convex function to derive new type Hermite-Hadamard inequalities involving Hadamard fractional integrals. Finally, some applications to special means of real numbers are demonstrated.

1. Introduction

Fractional calculus played an important role in various fields such as electricity, biology, economics, and signal and image processing [1–8]. The fractional Hermite-Hadamard inequality gives a lower and an upper estimation for both right-hand and left-hand integrals average of any convex function defined on a compact interval, involving the midpoint and the endpoints of the domain.

As we know, Set [9] firstly studied fractional Ostrowski inequalities involving Riemann-Liouville fractional integrals. Then, Sarikaya et al. [10] studied Hermite-Hadamard type inequalities involving Riemann-Liouville fractional integrals. Further, our group go on studying fractional version Hermite-Hadamard inequality involving Riemann-Liouville and Hadamard fractional integrals for all kinds of functions [11–19].

Recently, Wang et al. [16, 17] established the following two powerful fractional integral identities involving Hadamard fractional integrals.

Lemma 1 (see [12, Lemma 3.1]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) . If $f' \in L[a, b]$, then the following equality for fractional integrals holds:

$$\begin{aligned} \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \\ = \frac{\ln b - \ln a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] a^t b^{1-t} f'(a^t b^{1-t}) dt, \end{aligned} \quad (1)$$

where the symbols ${}_H J_{a^+}^\alpha f$ and ${}_H J_{b^-}^\alpha f$ are defined by

$$\begin{aligned} ({}_H J_{a^+}^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t} \right)^{\alpha-1} f(t) \frac{dt}{t}, \\ &\quad (0 < a < x \leq b), \\ ({}_H J_{b^-}^\alpha f)(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x} \right)^{\alpha-1} f(t) \frac{dt}{t}, \\ &\quad (0 < a \leq x < b), \end{aligned} \quad (2)$$

where $\Gamma(\cdot)$ is the Gamma function.

Lemma 2 (see [11, Lemma 2.1]). Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $0 < a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:

$$\begin{aligned}
& \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \\
&= \frac{b-a}{2} \int_0^1 k f'(ta + (1-t)b) dt \\
&\quad - \frac{\ln b - \ln a}{2} \int_0^1 [(1-t)^\alpha - t^\alpha] a^t b^{1-t} f'(a^t b^{1-t}) dt,
\end{aligned} \tag{3}$$

where

$$k = \begin{cases} 1, & 0 \leq t < \frac{1}{2}, \\ -1, & \frac{1}{2} \leq t < 1. \end{cases} \tag{4}$$

Remark 3. It is remarkable that Professor Srivastava et al. [20] give some further refinements and extensions of the Hermite-Hadamard inequalities in n variables. In the forthcoming works, we will try to extend to study fractional version Hermite-Hadamard inequalities in n variables based on such fundamental results.

Next, we recall the following basic concepts and results in our previous papers.

Definition 4 (see [21, 22]). Let $f: I \subseteq \mathbb{R}^+ \rightarrow \mathbb{R}^+$. A function f is said to be GG-convex on I if, for every $x, y \in I$ and $\lambda \in [0, 1]$, one has

$$f(x^\lambda y^{1-\lambda}) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}. \tag{5}$$

Remark 5. By the arithmetic-geometric mean inequality, we have

$$[f(x)]^\lambda [f(y)]^{1-\lambda} \leq \lambda f(x) + (1-\lambda) f(y). \tag{6}$$

Linking (5) and (6), we obtain

$$f(x^\lambda y^{1-\lambda}) \leq \lambda f(x) + (1-\lambda) f(y), \tag{7}$$

which appears in the standard definition of GA-convex function [21]. So GG-convex function is GA-convex function.

Lemma 6 (see [19, Lemma 2.5]). For $t \in [0, 1]$, $x, y > 0$, one has

$$tx + (1-t)y \geq y^{1-t} x^t. \tag{8}$$

Lemma 7 (see [13, Lemma 2.1]). For $\alpha > 0$ and $k > 0$, one has

$$I(\alpha) = \int_0^1 t^{\alpha-1} k^t dt = k \sum_{i=1}^{\infty} (-1)^{i-1} \frac{(\ln k)^{i-1}}{(\alpha)_i} < +\infty, \tag{9}$$

where $(\alpha)_i = \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+i-1)$.

Lemma 8 (see [13, Lemma 2.2]). For $\alpha > 0$ and $k > 0$, $z > 0$, one has

$$\begin{aligned}
J(\alpha, k) &= \int_0^1 (1-t)^{\alpha-1} k^t dt = \sum_{i=1}^{\infty} \frac{(\ln k)^{i-1}}{(\alpha)_i} < +\infty, \\
H(\alpha, k, z) &= \int_0^z t^{\alpha-1} k^t dt = z^\alpha k^z \sum_{i=1}^{\infty} \frac{(-z \ln k)^{i-1}}{(\alpha)_i} < +\infty.
\end{aligned} \tag{10}$$

Lemma 9 (see [13, Lemma 2.3]). For $\alpha > 0$ and $k > 0$, $1 > z > 0$, one has

$$\begin{aligned}
R(\alpha, k, z) &= \int_0^z (1-t)^{\alpha-1} k^t dt \\
&= \sum_{i=0}^{\infty} \frac{(\ln k)^{i-1}}{(\alpha)_i} (1 - k^z (1-z)^{\alpha+i-1}).
\end{aligned} \tag{11}$$

In the present paper, we will use the above concepts and lemmas to derive some new fractional Hermite-Hadamard inequalities involving Hadamard fractional integrals.

2. Main Results Based on Lemma 1

Now we are ready to state the following main results in this section.

Theorem 10. Let $f: [0, b] \rightarrow \mathbb{R}^+$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is GG-convex on $[a, b]$ for some fixed $\alpha \in (0, \infty)$ and $t \in [0, 1]$, $0 \leq a < b$, then the following integrals hold:

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\
& \leq \frac{\ln b - \ln a}{2} \\
& \quad \times \left\{ \left(1 - \frac{1}{2^{\alpha+1}} \right) \right. \\
& \quad \times \left[\frac{b |f'(b)|}{\alpha+1} + \frac{b |f'(a)| + a |f'(b)| - 2b |f'(b)|}{\alpha+2} \right] \\
& \quad + \frac{a |f'(a)| - a |f'(b)| - b |f'(a)|}{\alpha+3} \\
& \quad - \frac{a |f'(a)| - b |f'(a)|}{(\alpha+3) 2^{\alpha+3}} + \frac{2a |f'(a)|}{(\alpha+1)(\alpha+2)(\alpha+3)} \\
& \quad + \frac{b |f'(a)| + a |f'(b)| - 2a |f'(a)|}{(\alpha+1)(\alpha+3)} \\
& \quad + \frac{2a (|f'(a)| - |f'(b)|) - 2b |f'(a)|}{2^{\alpha+3} (\alpha+2)(\alpha+3)} \\
& \quad + \frac{b |f'(a)| + a |f'(b)|}{2^{\alpha+2} (\alpha+2)} \\
& \quad \left. + \frac{a |f'(a)|}{(\alpha+1) 2^{\alpha+2}} - \frac{a |f'(a)|}{(\alpha+2) 2^{\alpha+2}} \right\}.
\end{aligned} \tag{12}$$

Proof. Noting Definition 4 and Lemmas 1 and 6, we have

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\
 &= \left| \frac{\ln b - \ln a}{2} \int_0^1 a^t b^{1-t} [(1-t)^\alpha - t^\alpha] f'(a^t b^{1-t}) dt \right| \\
 &\leq \frac{\ln b - \ln a}{2} \int_0^1 a^t b^{1-t} |(1-t)^\alpha - t^\alpha| |f'(a^t b^{1-t})| dt \\
 &\leq \frac{\ln b - \ln a}{2} \int_{1/2}^1 a^t b^{1-t} [t^\alpha - (1-t)^\alpha] |f'(a^t b^{1-t})| dt \\
 &\quad + \frac{\ln b - \ln a}{2} \int_0^{1/2} a^t b^{1-t} [(1-t)^\alpha - t^\alpha] \\
 &\quad \times |f'(a^t b^{1-t})| dt \\
 &\leq \frac{\ln b - \ln a}{2} \int_{1/2}^1 [at + b(1-t)] [t^\alpha - (1-t)^\alpha] \\
 &\quad \times |f'(a)|^t |f'(b)|^{1-t} dt \\
 &\quad + \frac{\ln b - \ln a}{2} \int_0^{1/2} [at + b(1-t)] [(1-t)^\alpha - t^\alpha] \\
 &\quad \times |f'(a)|^t |f'(b)|^{1-t} dt \\
 &\leq \frac{\ln b - \ln a}{2} \int_{1/2}^1 [at + b(1-t)] [t^\alpha - (1-t)^\alpha] \\
 &\quad \times [|f'(a)|t + |f'(b)|(1-t)] dt \\
 &\quad + \frac{\ln b - \ln a}{2} \int_0^{1/2} [at + b(1-t)] [(1-t)^\alpha - t^\alpha] \\
 &\quad \times [|f'(a)|t + |f'(b)|(1-t)] dt \\
 &\leq \frac{\ln b - \ln a}{2} \left\{ \frac{a|f'(a)|}{\alpha + 3} \int_{1/2}^1 (\alpha + 3)t^{\alpha+2} dt \right. \\
 &\quad + \frac{b|f'(a)|}{\alpha + 2} \int_{1/2}^1 (\alpha + 2)t^{\alpha+1} dt \\
 &\quad - \frac{b|f'(a)|}{\alpha + 3} \int_{1/2}^1 (\alpha + 3)t^{\alpha+2} dt \\
 &\quad + \frac{a|f'(b)|}{\alpha + 2} \int_{1/2}^1 (\alpha + 2)t^{\alpha+1} dt \\
 &\quad - \frac{a|f'(b)|}{\alpha + 3} \int_{1/2}^1 (\alpha + 3)t^{\alpha+2} dt \\
 &\quad + \frac{b|f'(b)|}{\alpha + 1} \int_{1/2}^1 (\alpha + 1)t^\alpha dt \\
 &\quad - \frac{2b|f'(b)|}{\alpha + 2} \int_{1/2}^1 (\alpha + 2)t^{\alpha+1} dt \\
 &\quad \left. - \frac{2b|f'(b)|}{\alpha + 2} \int_{1/2}^1 (\alpha + 2)t^{\alpha+1} dt \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{b|f'(b)|}{\alpha + 3} \int_{1/2}^1 (\alpha + 3)t^{\alpha+2} dt \\
 & - a|f'(a)| \int_{1/2}^1 (1-t)^\alpha t^2 dt \\
 & - b|f'(a)| \int_{1/2}^1 t(1-t)^{\alpha+1} dt \\
 & - a|f'(b)| \int_{1/2}^1 t(1-t)^{\alpha+1} dt \\
 & - \frac{b|f'(b)|}{\alpha + 3} \int_{1/2}^1 (\alpha + 3)(1-t)^{\alpha+2} dt \Big\} \\
 & + \frac{\ln b - \ln a}{2} \left\{ a|f'(a)| \int_0^{1/2} t^2(1-t)^\alpha dt \right. \\
 & + b|f'(a)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
 & + a|f'(b)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
 & + \frac{b|f'(b)|}{\alpha + 3} \int_0^{1/2} (\alpha + 3)(1-t)^{\alpha+2} dt \\
 & - \frac{a|f'(a)|}{\alpha + 3} \int_0^{1/2} (\alpha + 3)t^{\alpha+2} dt \\
 & - \frac{b|f'(a)|}{\alpha + 2} \int_0^{1/2} (\alpha + 2)t^{\alpha+1} dt \\
 & + \frac{b|f'(a)|}{\alpha + 3} \int_0^{1/2} (\alpha + 3)t^{\alpha+2} dt \\
 & - \frac{a|f'(b)|}{\alpha + 2} \int_0^{1/2} (\alpha + 2)t^{\alpha+1} dt \\
 & + \frac{a|f'(b)|}{\alpha + 3} \int_0^{1/2} (\alpha + 3)t^{\alpha+2} dt \\
 & - \frac{b|f'(b)|}{\alpha + 1} \int_0^{1/2} (\alpha + 1)t^\alpha dt \\
 & + \frac{2b|f'(b)|}{\alpha + 2} \int_0^{1/2} (\alpha + 2)t^{\alpha+1} dt \\
 & \left. - \frac{b|f'(b)|}{\alpha + 3} \int_0^{1/2} (\alpha + 3)t^{\alpha+2} dt \right\} \\
 &\leq \frac{\ln b - \ln a}{2} \\
 &\quad \times \left\{ \frac{a|f'(a)|}{\alpha + 3} \left(1 - \frac{1}{2^{\alpha+3}} \right) \right.
 \end{aligned}$$

$$\begin{aligned}
& + \frac{b|f'(a)|}{\alpha+2} \left(1 - \frac{1}{2^{\alpha+2}}\right) - \frac{b|f'(a)|}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) \\
& + \frac{a|f'(b)|}{\alpha+2} \left(1 - \frac{1}{2^{\alpha+2}}\right) - \frac{a|f'(b)|}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) \\
& + \frac{b|f'(b)|}{\alpha+1} \left(1 - \frac{1}{2^{\alpha+1}}\right) - \frac{2b|f'(b)|}{\alpha+2} \left(1 - \frac{1}{2^{\alpha+2}}\right) \\
& + \frac{b|f'(b)|}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) - a|f'(a)| \int_{1/2}^1 (1-t)^{\alpha} t^2 dt \\
& - b|f'(a)| \int_{1/2}^1 t(1-t)^{\alpha+1} dt \\
& - a|f'(b)| \int_{1/2}^1 t(1-t)^{\alpha+1} dt \\
& - \frac{b|f'(b)|}{\alpha+3} \int_{1/2}^1 (\alpha+3)(1-t)^{\alpha+2} dt \Big\} \\
& + \frac{\ln b - \ln a}{2} \\
& \times \left\{ a|f'(a)| \int_0^{1/2} t^2(1-t)^{\alpha} dt \right. \\
& + b|f'(a)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
& + a|f'(b)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
& + \frac{b|f'(b)|}{\alpha+3} \left(\frac{1}{2^{\alpha+3}} - 1\right) - \frac{a|f'(a)|}{\alpha+3} \frac{1}{2^{\alpha+3}} \\
& - \frac{b|f'(a)|}{\alpha+2} \frac{1}{2^{\alpha+2}} + \frac{b|f'(a)|}{\alpha+3} \frac{1}{2^{\alpha+3}} \\
& - \frac{a|f'(b)|}{\alpha+2} \frac{1}{2^{\alpha+2}} + \frac{a|f'(b)|}{\alpha+3} \frac{1}{2^{\alpha+3}} \\
& - \frac{b|f'(b)|}{\alpha+1} \frac{1}{2^{\alpha+1}} + \frac{2b|f'(b)|}{\alpha+2} \frac{1}{2^{\alpha+2}} \\
& \left. - \frac{b|f'(a)|}{\alpha+3} \frac{1}{2^{\alpha+3}} \right\} \\
& \leq \frac{\ln b - \ln a}{2} \left\{ \frac{(a-b)[|f'(a)| - |f'(b)|]}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) \right. \\
& + \frac{b|f'(b)|}{\alpha+1} \left(1 - \frac{1}{2^{\alpha+1}}\right) \\
& + \frac{b[|f'(a)| - |f'(b)|] + (a-b)|f'(b)|}{\alpha+2} \\
& \times \left(1 - \frac{1}{2^{\alpha+2}}\right) - a|f'(a)| \int_0^1 (1-t)^{\alpha} t^2 dt
\end{aligned}$$

$$\begin{aligned}
& + a|f'(a)| \int_0^{1/2} (1-t)^{\alpha} t^2 dt \\
& - b|f'(a)| \int_0^1 t(1-t)^{\alpha+1} dt \\
& + b|f'(a)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
& - a|f'(b)| \int_0^1 t(1-t)^{\alpha+1} dt \\
& + a|f'(b)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \Big\} \\
& + \frac{\ln b - \ln a}{2} \left\{ a|f'(a)| \int_0^{1/2} t^2(1-t)^{\alpha} dt \right. \\
& + b|f'(a)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
& + a|f'(b)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
& + \frac{b|f'(b)|}{\alpha+3} \left(\frac{1}{2^{\alpha+3}} - 1\right) - \frac{a|f'(b)|}{(\alpha+3)2^{\alpha+3}} \\
& - \frac{b|f'(a)| + a|f'(b)| - 2b|f'(b)|}{(\alpha+2)2^{\alpha+2}} \\
& \left. - \frac{b|f'(b)|}{(\alpha+1)2^{\alpha+1}} \right\} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \times \left\{ \left(1 - \frac{1}{2^{\alpha+1}}\right) \right. \\
& \times \left[\frac{b|f'(b)|}{\alpha+1} \right. \\
& + \frac{b|f'(a)| + a|f'(b)| - 2b|f'(b)|}{\alpha+2} \Big] \\
& + \frac{a|f'(a)| - a|f'(b)| - b|f'(a)|}{\alpha+3} \\
& - \frac{a|f'(a)| - b|f'(a)|}{(\alpha+3)2^{\alpha+3}} \\
& + \frac{2a|f'(a)|}{(\alpha+1)(\alpha+2)(\alpha+3)} \\
& + \frac{b|f'(a)| + a|f'(b)| - 2a|f'(a)|}{(\alpha+1)(\alpha+3)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{2a(|f'(a)| - |f'(b)|) - 2b|f'(a)|}{2^{\alpha+3}(\alpha+2)(\alpha+3)} \\
& + \frac{b|f'(a)| + a|f'(b)|}{2^{\alpha+2}(\alpha+2)} + \frac{a|f'(a)|}{(\alpha+1)2^{\alpha+2}} \\
& - \frac{a|f'(a)|}{(\alpha+2)^2 2^{\alpha+2}} \Bigg\}.
\end{aligned} \tag{13}$$

The proof is done. $\square$

Theorem 11. Let $f: [0, b] \rightarrow \mathbb{R}^+$ be a differentiable mapping. If $|f'|^q$ is measurable and $|f'|^q$, ($q > 1$) is GG-convex on $[a, b]$ for some fixed $\alpha \in (0, \infty)$ and $t \in [0, 1]$, $0 \leq a < b$, then the following integrals hold:

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\
& \leq \frac{\ln b - \ln a}{2} \left(\frac{|f'(a)|^q + 2|f'(b)|^q - |f'(b)|^q}{2} \right)^{1/q} \\
& \times \left(\frac{a^p - 2b^p}{p\alpha + 2} + \frac{b^p}{p\alpha + 1} - \frac{2b^p + a^p}{2^{p\alpha+2}(p\alpha + 1)} \right. \\
& + \frac{2b^p - a^p}{2^{p\alpha+2}(p\alpha + 2)} - \frac{a^p}{p\alpha(p\alpha + 1)} \\
& - \frac{a^p + b^p}{2^{p\alpha+2}(p\alpha + 1)(p\alpha + 2)} \\
& \left. + \frac{2a^p}{(p\alpha + 1)(p\alpha + 2)} \right)^{1/p},
\end{aligned} \tag{14}$$

where $1/p + 1/q = 1$.

Proof. By using Definition 4 and Lemmas 1 and 6, we have

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\
& \leq \frac{\ln b - \ln a}{2} \left(\int_0^1 a^{pt} b^{p(1-t)} |(1-t)^\alpha - t^\alpha|^p dt \right)^{1/p} \\
& \times \left(\int_0^1 |f'(a)|^{qt} |f'(b)|^{q(1-t)} dt \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \times \left(\int_0^1 [ta^p + (1-t)b^p] |(1-t)^\alpha - t^\alpha|^p dt \right)^{1/p} \\
& \times \left(\int_0^1 [t|f'(a)|^q + (1-t)|f'(b)|^q] dt \right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{\ln b - \ln a}{2} \\
& \times \left(\int_{1/2}^1 [ta^p + (1-t)b^p] (t^{p\alpha} - (1-t)^{p\alpha}) dt \right. \\
& \left. + \int_0^{1/2} [ta^p + (1-t)b^p] [(1-t)^{p\alpha} - t^{p\alpha}] dt \right)^{1/p} \\
& \times \left(\frac{|f'(a)|^q}{2} + |f'(b)|^q - \frac{|f'(b)|^q}{2} \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \left(\frac{|f'(a)|^q + 2|f'(b)|^q - |f'(b)|^q}{2} \right)^{1/q} \\
& \times \left(a^p \int_{1/2}^1 t^{p\alpha+1} dt - a^p \int_{1/2}^1 t(1-t)^{p\alpha} dt \right. \\
& + b^p \int_{1/2}^1 t^{p\alpha} (1-t) dt - b^p \int_{1/2}^1 (1-t)^{p\alpha+1} dt \\
& + a^p \int_0^{1/2} t(1-t)^{p\alpha} dt - a^p \int_0^{1/2} t^{p\alpha+1} dt \\
& + b^p \int_0^{1/2} (1-t)^{p\alpha+1} dt \\
& \left. - b^p \int_0^{1/2} t^{p\alpha} (1-t) dt \right)^{1/p} \\
& \leq \frac{\ln b - \ln a}{2} \left(\frac{|f'(a)|^q + 2|f'(b)|^q - |f'(b)|^q}{2} \right)^{1/q} \\
& \times \left(\frac{a^p}{p\alpha + 2} - a^p \int_0^1 t(1-t)^{p\alpha-1} dt \right. \\
& + a^p \int_0^{1/2} t(1-t)^{p\alpha} dt + \frac{b^p}{p\alpha + 1} \\
& - \frac{1}{2^{p\alpha+1}} \frac{b^p}{p\alpha + 1} - \frac{b^p}{p\alpha + 2} + \frac{1}{2^{p\alpha+2}} \frac{b^p}{p\alpha + 2} \\
& + a^p \int_0^{1/2} t(1-t)^{p\alpha} dt - \frac{1}{2^{p\alpha+2}} \frac{a^p}{p\alpha + 2} \\
& + \frac{1}{2^{p\alpha+2}} \frac{b^p}{p\alpha + 2} - \frac{b^p}{p\alpha + 2} \\
& \left. - b^p \int_0^{1/2} t^{p\alpha} (1-t) dt \right)^{1/p} \\
& \leq \frac{\ln b - \ln a}{2} \left(\frac{|f'(a)|^q + 2|f'(b)|^q - |f'(b)|^q}{2} \right)^{1/q} \\
& \times \left(\frac{a^p - 2b^p}{p\alpha + 2} + \frac{b^p}{p\alpha + 1} - \frac{2b^p + a^p}{2^{p\alpha+2}(p\alpha + 1)} \right)
\end{aligned}$$

$$\begin{aligned}
& + \frac{2b^p - a^p}{2^{p\alpha+2}(p\alpha+2)} - \frac{a^p}{p\alpha(p\alpha+1)} \\
& - \frac{a^p + b^p}{2^{p\alpha+2}(p\alpha+1)(p\alpha+2)} \\
& + \frac{2a^p}{(p\alpha+1)(p\alpha+2)} \Big)^{1/p}.
\end{aligned} \tag{15}$$

The proof is done. $\square$

Theorem 12. Let $f : [0, b] \rightarrow \mathbb{R}^+$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is GG-convex on $[a, b]$ for some fixed $\alpha \in (0, \infty)$, $t \in [0, 1]$ and $k > 0$, $0 \leq a < b$, then the following integrals hold:

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\
& \leq b |f'(b)| \frac{\ln b - \ln a}{2} \\
& \times \left\{ \frac{a |f'(a)|}{b |f'(b)|} \right. \\
& \times \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \\
& + \sum_{i=1}^{\infty} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \\
& - \frac{(1/2)^{\alpha-1} (a |f'(a)| / b |f'(b)|)^{1/2}}{\alpha+1} \\
& \left. + 2 \frac{1 - (1/2)^\alpha (a |f'(a)| / b |f'(b)|)^{1/2}}{\ln(a |f'(a)| / b |f'(b)|)} \right\}. \tag{16}
\end{aligned}$$

Proof. By using Definition 4 and Lemmas 1, 7, 8, and 9, we have

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\
& \leq \frac{\ln b - \ln a}{2} \int_{1/2}^1 a^t b^{1-t} [t^\alpha - (1-t)^\alpha] |f'(a^t b^{1-t})| dt \\
& + \frac{\ln b - \ln a}{2} \int_0^{1/2} a^t b^{1-t} [(1-t)^\alpha - t^\alpha] \\
& \times |f'(a^t b^{1-t})| dt
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{\ln b - \ln a}{2} \int_{1/2}^1 a^t b^{1-t} [t^\alpha - (1-t)^\alpha] \\
& \times |f'(a)|^t |f'(b)|^{1-t} dt \\
& + \frac{\ln b - \ln a}{2} \int_0^{1/2} a^t b^{1-t} [(1-t)^\alpha - t^\alpha] \\
& \times |f'(a)|^t |f'(b)|^{1-t} dt \\
& \leq b |f'(b)| \frac{\ln b - \ln a}{2} \\
& \times \int_{1/2}^1 \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t [t^\alpha - (1-t)^\alpha] dt \\
& + b |f'(b)| \frac{\ln b - \ln a}{2} \\
& \times \int_0^{1/2} \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t [(1-t)^\alpha - t^\alpha] dt \\
& \leq b |f'(b)| \frac{\ln b - \ln a}{2} \\
& \times \left\{ \int_0^1 \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t t^\alpha dt \right. \\
& - 2 \int_0^{1/2} \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t t^\alpha dt \\
& - \int_0^1 \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t (1-t)^\alpha dt \\
& \left. + 2 \int_0^{1/2} \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t (1-t)^\alpha dt \right\} \\
& \leq b |f'(b)| \frac{\ln b - \ln a}{2} \\
& \times \left\{ \frac{a |f'(a)|}{b |f'(b)|} \right.
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \\
& - \sum_{i=1}^{\infty} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \\
& - \left(\frac{1}{2} \right)^\alpha \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^{1/2} \\
& \times \sum_{i=1}^{\infty} \frac{[-(1/2) \ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i}
\end{aligned}$$

$$\begin{aligned} & + 2 \sum_{i=0}^{\infty} \frac{[\ln(a|f'(a)|/b|f'(b)|)]^{i-1}}{(\alpha+1)_i} \\ & \times \left[1 - \left(\frac{1}{2}\right)^{\alpha+i} \left(\frac{a|f'(a)|}{b|f'(b)|}\right)^{1/2} \right] \Bigg\} \\ & \leq b|f'(b)| \frac{\ln b - \ln a}{2} \\ & \times \left\{ \frac{a|f'(a)|}{b|f'(b)|} \right. \\ & \times \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a|f'(a)|/b|f'(b)|)]^{i-1}}{(\alpha+1)_i} \\ & - \sum_{i=1}^{\infty} \frac{[\ln(a|f'(a)|/b|f'(b)|)]^{i-1}}{(\alpha+1)_i} \\ & - \left(\frac{1}{2}\right)^{\alpha} \left(\frac{a|f'(a)|}{b|f'(b)|}\right)^{1/2} \\ & \times \sum_{i=1}^{\infty} \frac{[-(1/2)\ln(a|f'(a)|/b|f'(b)|)]^{i-1}}{(\alpha+1)_i} \\ & + 2 \frac{1 - (1/2)^{\alpha} (a|f'(a)|/b|f'(b)|)^{1/2}}{\ln(a|f'(a)|/b|f'(b)|)} \\ & + 2 \sum_{i=1}^{\infty} \frac{[\ln(a|f'(a)|/b|f'(b)|)]^{i-1}}{(\alpha+1)_i} \\ & - \left(\frac{1}{2}\right)^{\alpha} \left(\frac{a|f'(a)|}{b|f'(b)|}\right)^{1/2} \\ & \times \sum_{i=1}^{\infty} \frac{[(1/2)\ln(a|f'(a)|/b|f'(b)|)]^{i-1}}{(\alpha+1)_i} \Bigg\} \\ & \leq b|f'(b)| \frac{\ln b - \ln a}{2} \\ & \times \left\{ \frac{a|f'(a)|}{b|f'(b)|} \right. \\ & \times \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a|f'(a)|/b|f'(b)|)]^{i-1}}{(\alpha+1)_i} \\ & + \sum_{i=1}^{\infty} \frac{[\ln(a|f'(a)|/b|f'(b)|)]^{i-1}}{(\alpha+1)_i} \end{aligned}$$

$$- \frac{(1/2)^{\alpha-1} (a |f'(a)|/b |f'(b)|)^{1/2}}{\alpha + 1} + 2 \frac{1 - (1/2)^{\alpha} (a |f'(a)|/b |f'(b)|)^{1/2}}{\ln (a |f'(a)|/b |f'(b)|)} \Bigg\} . \quad (17)$$

The proof is done. $\square$

Theorem 13. Let $f: [0, b] \rightarrow \mathbb{R}^+$ be a differentiable mapping. If $|f'|^q$ is measurable and $|f'|^q$, ($q > 1$) is GG-convex on $[a, b]$, for some fixed $\alpha \in (0, \infty)$, $t \in [0, 1]$, and $k > 0$, $0 \leq a < b$, then the following integrals hold:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{\ln b - \ln a}{2} \\ & \quad \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha + 1)_i} \right. \\ & \quad \left. + b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha + 1)_i} - \frac{(1/2)^{p\alpha-1} (ab^{p/2})}{p\alpha + 1} \right. \\ & \quad \left. - \left[\ln\left(\frac{a}{b}\right)^p \right]^{-1} \left[\left(\frac{1}{2}\right)^{p\alpha-1} (ab)^{p/2} + 2b^p \right] \right)^{1/p} \\ & \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)| / |f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q}, \end{aligned} \quad (18)$$

where $1/p + 1/q = 1$.

Proof. By using Definition 4 and Lemmas 1, 7, 8, and 9, we have

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] \right| \\ & \leq \frac{\ln b - \ln a}{2} \left(\int_0^1 (a^t b^{1-t} |(1-t)^\alpha - t^\alpha|)^p dt \right)^{1/p} \\ & \quad \times \left(\int_0^1 |f'(a^t b^{1-t})|^q dt \right)^{1/q} \\ & \leq \frac{\ln b - \ln a}{2} \left(b^p \int_{1/2}^1 \left(\frac{a}{b} \right)^{pt} t^{p\alpha} dt \right. \\ & \quad \left. - b^p \int_{1/2}^1 \left(\frac{a}{b} \right)^{pt} (1-t)^{p\alpha} dt \right. \\ & \quad \left. + b^p \int_0^{1/2} \left(\frac{a}{b} \right)^{pt} (1-t)^{p\alpha} dt \right. \\ & \quad \left. - b^p \int_0^{1/2} \left(\frac{a}{b} \right)^{pt} t^{p\alpha} dt \right)^{1/p} \end{aligned}$$

$$\begin{aligned}
& \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \right. \\
& \quad \times \int_0^1 \ln \left(\frac{|f'(a)|}{|f'(b)|} \right)^q \left[\frac{|f'(a)|}{|f'(b)|} \right]^{qt} dt \Bigg)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \left(b^p \left(\frac{a}{b} \right)^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad - b^p \sum_{i=0}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad - 2b^p \int_0^{1/2} \left(\frac{a}{b} \right)^{pt} t^{p\alpha} dt \\
& \quad \left. + 2b^p \int_0^{1/2} \left(\frac{a}{b} \right)^{pt} (1-t)^{p\alpha} dt \right)^{1/p} \\
& \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \quad \times \left(b^p \left(\frac{a}{b} \right)^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad - b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad - \left(\frac{1}{2} \right)^{p\alpha} b^p \left(\frac{a}{b} \right)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad \left. + 2b^p \sum_{i=0}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad \left. \times \left[1 - \left(\frac{a}{b} \right)^{p/2} \left(1 - \frac{1}{2} \right)^{p\alpha+i} \right] \right)^{1/p} \\
& \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \quad \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad - b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad - \left(\frac{1}{2} \right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad \left. + 2b^p \sum_{i=0}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad \left. \times \left[1 - \left(\frac{a}{b} \right)^{p/2} \left(1 - \frac{1}{2} \right)^{p\alpha+i} \right] \right)^{1/p} \\
& \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \quad \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad - b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad - \left(\frac{1}{2} \right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad \left. + 2b^p \sum_{i=0}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad \left. \times \left[1 - \left(\frac{a}{b} \right)^{p/2} \left(1 - \frac{1}{2} \right)^{p\alpha+i} \right] \right)^{1/p} \\
& \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
& + 2b^p \sum_{i=0}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& - 2b^p \left(\frac{a}{b} \right)^{p/2} \sum_{i=0}^{\infty} \left(1 - \frac{1}{2} \right)^{p\alpha+i} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \Bigg)^{1/p} \\
& \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \quad \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} - b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad - \left(\frac{1}{2} \right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad + 2b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} + 2b^p [\ln(a/b)^p]^{-1} \\
& \quad - 2b^p \left(\frac{a}{b} \right)^{p/2} \sum_{i=0}^{\infty} \left(1 - \frac{1}{2} \right)^{p\alpha+i} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \Bigg)^{1/p} \\
& \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \quad \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} + b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad - \left(\frac{1}{2} \right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad - \left(\frac{1}{2} \right)^{p\alpha} (ab)^{p/2} \sum_{i=0}^{\infty} \left(\frac{1}{2} \right)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad \left. + 2b^p \left[\ln \left(\frac{a}{b} \right)^p \right]^{-1} \right)^{1/p} \\
& \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \quad \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} + b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad - \left(\frac{1}{2} \right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad \left. + 2b^p \left[\ln \left(\frac{a}{b} \right)^p \right]^{-1} \right)^{1/p} \\
& \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
& - \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& - \left(\frac{1}{2}\right)^{p\alpha-1} (ab)^{p/2} \left[\ln\left(\frac{a}{b}\right)^p \right]^{-1} \\
& + 2b^p \left[\ln\left(\frac{a}{b}\right)^p \right]^{-1/p} \\
& \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} + b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad \left. - 2 \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \frac{1}{p\alpha+1} \right. \\
& \quad \left. - \left[\ln\left(\frac{a}{b}\right)^p \right]^{-1} \left[\left(\frac{1}{2}\right)^{p\alpha-1} (ab)^{p/2} + 2b^p \right] \right)^{1/p} \\
& \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\
& \leq \frac{\ln b - \ln a}{2} \\
& \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\
& \quad \left. + b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} - \frac{(1/2)^{p\alpha-1} (ab)^{p/2}}{p\alpha+1} \right. \\
& \quad \left. - \left[\ln\left(\frac{a}{b}\right)^p \right]^{-1} \left[\left(\frac{1}{2}\right)^{p\alpha-1} (ab)^{p/2} + 2b^p \right] \right)^{1/p} \\
& \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q}.
\end{aligned} \tag{19}$$

The proof is done. $\square$

3. Main Results Based on Lemma 2

Theorem 14. Let $f: [0, b] \rightarrow \mathbb{R}^+$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is GG-convex on $[a, b]$, for some

fixed $\alpha \in (0, \infty)$ and $t \in [0, 1]$, $0 \leq a < b$, then the following integrals hold:

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{|f'(b)| - |f'(a)|}{2} + \frac{\ln b - \ln a}{2} \\
& \times \left\{ \left(1 - \frac{1}{2^{\alpha+1}}\right) \right. \\
& \quad \times \left[\frac{b|f'(b)|}{\alpha+1} + \frac{b|f'(a)| + a|f'(b)| - 2b|f'(b)|}{\alpha+2} \right] \\
& \quad + \frac{a|f'(a)| - a|f'(b)| - b|f'(a)|}{\alpha+3} \\
& \quad - \frac{a|f'(a)| - b|f'(a)|}{(\alpha+3)2^{\alpha+3}} + \frac{2a|f'(a)|}{(\alpha+1)(\alpha+2)(\alpha+3)} \\
& \quad + \frac{b|f'(a)| + a|f'(b)| - 2a|f'(a)|}{(\alpha+1)(\alpha+3)} \\
& \quad + \frac{2a(|f'(a)| - |f'(b)|) - 2b|f'(a)|}{2^{\alpha+3}(\alpha+2)(\alpha+3)} \\
& \quad + \frac{b|f'(a)| + a|f'(b)|}{2^{\alpha+2}(\alpha+2)} + \frac{a|f'(a)|}{(\alpha+1)2^{\alpha+2}} \\
& \quad \left. - \frac{a|f'(a)|}{(\alpha+2)^2 2^{\alpha+2}} \right\}.
\end{aligned} \tag{20}$$

Proof. By using Definition 4 and Lemmas 2 and 6, we have

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{b-a}{2} \int_0^1 |kf'(ta + (1-t)b)| dt \\
& \quad + \frac{\ln b - \ln a}{2} \int_0^1 a^t b^{1-t} |(1-t)^\alpha - t^\alpha| \\
& \quad \times |f'(a^t b^{1-t})| dt \\
& \leq \frac{-1}{2} (|f(a)| - |f(b)|) \\
& \quad + \frac{\ln b - \ln a}{2} \\
& \quad \times \left\{ \int_{1/2}^1 a^t b^{1-t} [t^\alpha - (1-t)^\alpha] |f'(a^t b^{1-t})| dt \right. \\
& \quad \left. + \int_0^{1/2} a^t b^{1-t} [(1-t)^\alpha - t^\alpha] |f'(a^t b^{1-t})| dt \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{-1}{2} (|f(a)| - |f(b)|) \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \int_{1/2}^1 t^\alpha [at + b(1-t)] \right. \\
&\quad \quad \times [t|f'(a)| + (1-t)|f'(b)|] dt \\
&\quad \quad - \int_{1/2}^1 (1-t)^\alpha [at + b(1-t)] \\
&\quad \quad \quad \times [t|f'(a)| + (1-t)|f'(b)|] dt \Big\} \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \int_0^{1/2} (1-t)^\alpha [at + b(1-t)] \right. \\
&\quad \quad \times [t|f'(a)| + (1-t)|f'(b)|] dt \\
&\quad \quad - \int_0^{1/2} t^\alpha [at + b(1-t)] \\
&\quad \quad \quad \times [t|f'(a)| + (1-t)|f'(b)|] dt \Big\} \\
&\leq \frac{-1}{2} (|f(a)| - |f(b)|) \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ |f'(a)| \int_{1/2}^1 t^{\alpha+1} [at + b(1-t)] dt \right. \\
&\quad \quad + |f'(b)| \int_{1/2}^1 (1-t) t^\alpha [at + b(1-t)] dt \\
&\quad \quad - |f'(a)| \int_{1/2}^1 (1-t)^\alpha t [at + b(1-t)] dt \\
&\quad \quad \left. - |f'(b)| \int_{1/2}^1 (1-t)^{\alpha+1} [at + b(1-t)] dt \right\} \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ |f'(a)| \int_0^{1/2} t(1-t)^\alpha [at + b(1-t)] dt \right. \\
&\quad \quad + |f'(b)| \int_0^{1/2} (1-t)^{\alpha+1} [at + b(1-t)] dt \\
&\quad \quad - |f'(a)| \int_0^{1/2} t^{\alpha+1} [at + b(1-t)] dt \\
&\quad \quad \left. - |f'(b)| \int_0^{1/2} (1-t) t^\alpha [at + b(1-t)] dt \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|f'(b)| - |f'(a)|}{2} \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \frac{a|f'(a)|}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) + \frac{b|f'(a)|}{\alpha+2} \left(1 - \frac{1}{2^{\alpha+2}}\right) \right. \\
&\quad \quad - \frac{b|f'(a)|}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) + \frac{a|f'(b)|}{\alpha+2} \left(1 - \frac{1}{2^{\alpha+2}}\right) \\
&\quad \quad - \frac{a|f'(b)|}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) + \frac{b|f'(b)|}{\alpha+1} \left(1 - \frac{1}{2^{\alpha+1}}\right) \\
&\quad \quad - \frac{2b|f'(b)|}{\alpha+2} \left(1 - \frac{1}{2^{\alpha+2}}\right) + \frac{b|f'(b)|}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) \\
&\quad \quad - a|f'(a)| \int_{1/2}^1 (1-t)^\alpha t^2 dt \\
&\quad \quad - b|f'(a)| \int_{1/2}^1 t(1-t)^{\alpha+1} dt \\
&\quad \quad - a|f'(b)| \int_{1/2}^1 t(1-t)^{\alpha+1} dt \\
&\quad \quad \left. - \frac{b|f'(b)|}{\alpha+3} \int_{1/2}^1 (\alpha+3)(1-t)^{\alpha+2} dt \right\} \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ a|f'(a)| \int_0^{1/2} t^2(1-t)^\alpha dt \right. \\
&\quad \quad + b|f'(a)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
&\quad \quad + a|f'(b)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
&\quad \quad + \frac{b|f'(b)|}{\alpha+3} \left(\frac{1}{2^{\alpha+3}} - 1\right) - \frac{a|f'(a)|}{\alpha+3} \frac{1}{2^{\alpha+3}} \\
&\quad \quad - \frac{b|f'(a)|}{\alpha+2} \frac{1}{2^{\alpha+2}} + \frac{b|f'(a)|}{\alpha+3} \frac{1}{2^{\alpha+3}} \\
&\quad \quad - \frac{a|f'(b)|}{\alpha+2} \frac{1}{2^{\alpha+2}} + \frac{a|f'(b)|}{\alpha+3} \frac{1}{2^{\alpha+3}} - \frac{b|f'(b)|}{\alpha+1} \\
&\quad \quad \left. \times \frac{1}{2^{\alpha+1}} + \frac{2b|f'(b)|}{\alpha+2} \frac{1}{2^{\alpha+2}} - \frac{b|f'(a)|}{\alpha+3} \frac{1}{2^{\alpha+3}} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|f'(b)| - |f'(a)|}{2} \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \frac{(a-b)[|f'(a)| - |f'(b)|]}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) \right. \\
&\quad + \frac{b|f'(b)|}{\alpha+1} \left(1 - \frac{1}{2^{\alpha+1}}\right) \\
&\quad + \frac{b[|f'(a)| - |f'(b)|] + (a-b)|f'(b)|}{\alpha+2} \\
&\quad \times \left(1 - \frac{1}{2^{\alpha+2}}\right) - a|f'(a)| \int_0^1 (1-t)^\alpha t^2 dt \\
&\quad + a|f'(a)| \int_0^{1/2} (1-t)^\alpha t^2 dt \\
&\quad - b|f'(a)| \int_0^1 t(1-t)^{\alpha+1} dt \\
&\quad + b|f'(a)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
&\quad - a|f'(b)| \int_0^1 t(1-t)^{\alpha+1} dt \\
&\quad \left. + a|f'(b)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \right\} \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ a|f'(a)| \int_0^{1/2} t^2(1-t)^\alpha dt \right. \\
&\quad + b|f'(a)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
&\quad + a|f'(b)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
&\quad + \frac{b|f'(b)|}{\alpha+3} \left(\frac{1}{2^{\alpha+3}} - 1\right) - \frac{a|f'(b)|}{(\alpha+3)2^{\alpha+3}} \\
&\quad - \frac{b|f'(a)| + a|f'(b)| - 2b|f'(b)|}{(\alpha+2)2^{\alpha+2}} \\
&\quad \left. - \frac{b|f'(b)|}{(\alpha+1)2^{\alpha+1}} \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{|f'(b)| - |f'(a)|}{2} \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \frac{(a-b)[|f'(a)| - |f'(b)|]}{\alpha+3} \left(1 - \frac{1}{2^{\alpha+3}}\right) \right. \\
&\quad + \frac{b|f'(b)|}{\alpha+1} \left(1 - \frac{1}{2^{\alpha+1}}\right) \\
&\quad + \frac{b[|f'(a)| - |f'(b)|] + (a-b)|f'(b)|}{\alpha+2} \\
&\quad \times \left(1 - \frac{1}{2^{\alpha+2}}\right) - a|f'(a)| \int_0^1 t^2(1-t)^\alpha dt \\
&\quad + a|f'(a)| \int_0^{1/2} t^2(1-t)^\alpha dt \\
&\quad - b|f'(a)| \int_0^1 t(1-t)^{\alpha+1} dt \\
&\quad + b|f'(a)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
&\quad - a|f'(b)| \int_0^1 t(1-t)^{\alpha+1} dt \\
&\quad \left. + a|f'(b)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \right\} \\
&\quad + \frac{\ln b - \ln a}{2} \left\{ a|f'(a)| \int_0^{1/2} t^2(1-t)^\alpha dt \right. \\
&\quad + b|f'(a)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
&\quad + a|f'(b)| \int_0^{1/2} t(1-t)^{\alpha+1} dt \\
&\quad + \frac{b|f'(b)|}{\alpha+3} \left(\frac{1}{2^{\alpha+3}} - 1\right) - \frac{a|f'(b)|}{(\alpha+3)2^{\alpha+3}} \\
&\quad - \frac{b|f'(a)| + a|f'(b)| - 2b|f'(b)|}{(\alpha+2)2^{\alpha+2}} \\
&\quad \left. - \frac{b|f'(b)|}{(\alpha+1)2^{\alpha+1}} \right\} \\
&\leq \frac{|f'(b)| - |f'(a)|}{2} \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \left(1 - \frac{1}{2^{\alpha+1}}\right) \right.
\end{aligned}$$

$$\begin{aligned}
& \times \left[\frac{b|f'(b)|}{\alpha+1} + \frac{b|f'(a)| + a|f'(b)| - 2b|f'(b)|}{\alpha+2} \right] \\
& + \frac{a|f'(a)| - a|f'(b)| - b|f'(a)|}{\alpha+3} \\
& - \frac{a|f'(a)| - b|f'(a)|}{(\alpha+3)2^{\alpha+3}} + \frac{2a|f'(a)|}{(\alpha+1)(\alpha+2)(\alpha+3)} \\
& + \frac{b|f'(a)| + a|f'(b)| - 2a|f'(a)|}{(\alpha+1)(\alpha+3)} \\
& + \frac{2a(|f'(a)| - |f'(b)|) - 2b|f'(a)|}{2^{\alpha+3}(\alpha+2)(\alpha+3)} \\
& + \frac{b|f'(a)| + a|f'(b)|}{2^{\alpha+2}(\alpha+2)} \\
& + \frac{a|f'(a)|}{(\alpha+1)2^{\alpha+2}} - \frac{a|f'(a)|}{(\alpha+2)2^{\alpha+2}} \Big\}.
\end{aligned} \tag{21}$$

The proof is done. $\square$

Theorem 15. Let $f: [0, b] \rightarrow \mathbb{R}^+$ be a differentiable mapping. If $|f'|^q$ is measurable and $|f'|^q$, ($q > 1$) is GG-convex on $[a, b]$, for some fixed $\alpha \in (0, \infty)$ and $t \in [0, 1]$, $0 \leq a < b$, then the following integrals hold:

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{|f'(b)| - |f'(a)|}{2} + \frac{\ln b - \ln a}{2} \\
& \times \left(\frac{|f'(a)|^q + 2|f'(a)|^q - |f'(a)|^q}{2} \right)^{1/q} \\
& \times \left(\frac{a^p - 2b^p}{p\alpha + 2} + \frac{b^p}{p\alpha + 1} - \frac{2b^p + a^p}{2^{p\alpha+2}(p\alpha + 1)} \right. \\
& + \frac{2b^p - a^p}{2^{p\alpha+2}(p\alpha + 2)} - \frac{a^p}{p\alpha(p\alpha + 1)} \\
& - \frac{a^p + b^p}{2^{p\alpha+2}(p\alpha + 1)(p\alpha + 2)} \\
& \left. + \frac{2a^p}{(p\alpha + 1)(p\alpha + 2)} \right)^{1/p},
\end{aligned} \tag{22}$$

where $1/p + 1/q = 1$.

Proof. By using Definition 4 and Lemmas 2 and 6, we have

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{b-a}{2} \int_0^1 |kf'(ta + (1-t)b)| dt \\
& + \frac{\ln b - \ln a}{2} \int_0^1 a^t b^{1-t} |(1-t)^\alpha - t^\alpha| |f'(a^t b^{1-t})| dt \\
& \leq \frac{b-a}{2} \int_0^1 |k| |f'(at + b(1-t))| dt \\
& + \frac{\ln b - \ln a}{2} \int_0^1 a^t b^{1-t} |(1-t)^\alpha - t^\alpha| |f'(a^t b^{1-t})| dt \\
& \leq \frac{(b-a)}{2} \int_0^1 |f'(at + b(1-t))| dt \\
& + \frac{\ln b - \ln a}{2} \int_0^1 a^t b^{1-t} |(1-t)^\alpha - t^\alpha| |f'(a^t b^{1-t})| dt \\
& \leq \frac{-1}{2} (|f'(a)| - |f'(b)|) \\
& + \frac{\ln b - \ln a}{2} \left(\int_0^1 (a^t b^{1-t} |(1-t)^\alpha - t^\alpha|)^p dt \right)^{1/p} \\
& \times \left(\int_0^1 |f'(a^t b^{1-t})|^q dt \right)^{1/q} \\
& \leq \frac{-1}{2} (|f'(a)| - |f'(b)|) \\
& + \frac{\ln b - \ln a}{2} \left(\int_0^1 a^{pt} b^{p(1-t)} |(1-t)^\alpha - t^\alpha|^p dt \right)^{1/p} \\
& \times \left(\int_0^1 |f'(a)|^{qt} |f'(b)|^{q(1-t)} dt \right)^{1/q} \\
& \leq \frac{-1}{2} (|f'(a)| - |f'(b)|) \\
& + \frac{\ln b - \ln a}{2} \\
& \times \left(\int_0^1 [ta^p + (1-t)b^p] |(1-t)^\alpha - t^\alpha|^p dt \right)^{1/p} \\
& \times \left(\int_0^1 [t|f'(a)|^q + (1-t)|f'(b)|^q] dt \right)^{1/q} \\
& \leq \frac{-1}{2} (|f'(a)| - |f'(b)|) \\
& + \frac{\ln b - \ln a}{2} \\
& \times \left(\int_{1/2}^1 [ta^p + (1-t)b^p] (t^{pa} - (1-t)^{pa}) dt \right)
\end{aligned}$$

$$\begin{aligned}
& + \int_0^{1/2} [ta^p + (1-t)b^p] [(1-t)^{p\alpha} - t^{p\alpha}] dt \Big)^{1/p} \\
& \times \left(\frac{|f'(a)|^q}{2} + |f'(b)|^q - \frac{|f'(b)|^q}{2} \right)^{1/q} \\
& \leq \frac{-1}{2} (|f'(a)| - |f'(b)|) \\
& + \frac{\ln b - \ln a}{2} \left(\frac{|f'(a)|^q + 2|f'(b)|^q - |f'(b)|^q}{2} \right)^{1/q} \\
& \times \left(a^p \int_{1/2}^1 t^{p\alpha+1} dt - a^p \int_{1/2}^1 t(1-t)^{p\alpha} dt \right. \\
& \quad + b^p \int_{1/2}^1 t^{p\alpha} (1-t) dt - b^p \int_{1/2}^1 (1-t)^{p\alpha+1} dt \\
& \quad + a^p \int_0^{1/2} t(1-t)^{p\alpha} dt - a^p \int_0^{1/2} t^{p\alpha+1} dt \\
& \quad \left. + b^p \int_0^{1/2} (1-t)^{p\alpha+1} - b^p \int_0^{1/2} t^{p\alpha} (1-t) dt \right)^{1/p} \\
& \leq \frac{-1}{2} (|f'(a)| - |f'(b)|) \\
& + \frac{\ln b - \ln a}{2} \left(\frac{|f'(a)|^q + 2|f'(b)|^q - |f'(b)|^q}{2} \right)^{1/q} \\
& \times \left(\frac{a^p}{p\alpha+2} - a^p \int_0^1 t(1-t)^{p\alpha-1} dt \right. \\
& \quad + a^p \int_0^{1/2} t(1-t)^{p\alpha} dt + \frac{b^p}{p\alpha+1} - \frac{1}{2^{p\alpha+1}} \frac{b^p}{p\alpha+1} \\
& \quad - \frac{b^p}{p\alpha+2} + \frac{1}{2^{p\alpha+2}} \frac{b^p}{p\alpha+2} + a^p \int_0^{1/2} t(1-t)^{p\alpha} dt \\
& \quad - \frac{1}{2^{p\alpha+2}} \frac{a^p}{p\alpha+2} + \frac{1}{2^{p\alpha+2}} \frac{b^p}{p\alpha+2} - \frac{b^p}{p\alpha+2} \\
& \quad \left. - b^p \int_0^{1/2} t^{p\alpha} (1-t) dt \right)^{1/p} \\
& \leq \frac{|f'(b)| - |f'(a)|}{2} \\
& + \frac{\ln b - \ln a}{2} \left(\frac{|f'(a)|^q + 2|f'(b)|^q - |f'(b)|^q}{2} \right)^{1/q} \\
& \times \left(\frac{a^p - 2b^p}{p\alpha+2} + \frac{b^p}{p\alpha+1} - \frac{2b^p + a^p}{2^{p\alpha+2} (p\alpha+1)} \right. \\
& \quad + \frac{2b^p - a^p}{2^{p\alpha+2} (p\alpha+2)} - \frac{a^p}{p\alpha (p\alpha+1)} \\
& \quad \left. - \frac{a^p + b^p}{2^{p\alpha+2} (p\alpha+1) (p\alpha+2)} \right. \\
& \quad \left. + \frac{2a^p}{(p\alpha+1) (p\alpha+2)} \right)^{1/p}. \tag{23}
\end{aligned}$$

The proof is done. $\square$

Theorem 16. Let $f : [0, b] \rightarrow \mathbb{R}^+$ be a differentiable mapping. If $|f'|$ is measurable and $|f'|$ is GG-convex on $[a, b]$, for some fixed $\alpha \in (0, \infty)$ and $t \in [0, 1]$, $0 \leq a < b$, then the following integrals hold:

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{1}{2} (|f'(b)| - |f'(a)|) \\
& + b |f'(b)| \frac{\ln b - \ln a}{2} \\
& \times \left\{ \frac{a |f'(a)|}{b |f'(b)|} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \right. \\
& \quad + \sum_{i=1}^{\infty} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \\
& \quad - \frac{(1/2)^{\alpha-1} (a |f'(a)| / b |f'(b)|)^{1/2}}{\alpha+1} \\
& \quad \left. + 2 \frac{1 - (1/2)^\alpha (a |f'(a)| / b |f'(b)|)^{1/2}}{\ln(a |f'(a)| / b |f'(b)|)} \right\}. \tag{24}
\end{aligned}$$

Proof. By using Definition 4 and Lemmas 2, 7, 8, and 9, we have

$$\begin{aligned}
& \left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \right| \\
& \leq \left| \frac{b-a}{2} \int_0^1 k f'(ta + (1-t)b) dt \right| \\
& + \frac{\ln b - \ln a}{2} \int_0^1 a^t b^{1-t} |(1-t)^\alpha - t^\alpha| \\
& \quad \times |f'(a^t b^{1-t})| dt
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{-1}{2} (|f(a)| - |f(b)|) \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \int_{1/2}^1 a^t b^{1-t} [t^\alpha - (1-t)^\alpha] |f'(a^t b^{1-t})| dt \right. \\
&\quad \left. + \int_0^{1/2} a^t b^{1-t} [(1-t)^\alpha - t^\alpha] |f'(a^t b^{1-t})| dt \right\} \\
&\leq \frac{-1}{2} (|f(a)| - |f(b)|) \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \int_{1/2}^1 a^t b^{1-t} [t^\alpha - (1-t)^\alpha] |f'(a)|^t |f'(b)|^{1-t} dt \\
&\quad + \frac{\ln b - \ln a}{2} \\
&\quad \times \int_0^{1/2} a^t b^{1-t} [(1-t)^\alpha - t^\alpha] |f'(a)|^t |f'(b)|^{1-t} dt \\
&\leq \frac{-1}{2} (|f(a)| - |f(b)|) \\
&\quad + b |f'(b)| \frac{\ln b - \ln a}{2} \\
&\quad \times \int_{1/2}^1 \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t [t^\alpha - (1-t)^\alpha] dt \\
&\quad + b |f'(b)| \frac{\ln b - \ln a}{2} \\
&\quad \times \int_0^{1/2} \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t [(1-t)^\alpha - t^\alpha] dt \\
&\leq \frac{1}{2} (|f(b)| - |f(a)|) \\
&\quad + b |f'(b)| \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \int_0^{1/2} \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t t^\alpha dt \right. \\
&\quad - \int_0^1 \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t (1-t)^\alpha dt \\
&\quad - 2 \int_0^{1/2} \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t t^\alpha dt \\
&\quad \left. + 2 \int_0^{1/2} \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^t (1-t)^\alpha dt \right\}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{2} (|f(b)| - |f(a)|) \\
&\quad + b |f'(b)| \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \frac{a |f'(a)|}{b |f'(b)|} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \right. \\
&\quad - \sum_{i=1}^{\infty} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \\
&\quad - \left(\frac{1}{2} \right)^{\alpha-1} \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^{1/2} \\
&\quad \times \sum_{i=1}^{\infty} \frac{[-(1/2) \ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \\
&\quad + 2 \sum_{i=0}^{\infty} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \\
&\quad \times \left[1 - \left(\frac{1}{2} \right)^{\alpha+i} \left(\frac{a |f'(a)|}{b |f'(b)|} \right)^{1/2} \right] \Bigg\} \\
&\leq \frac{1}{2} (|f'(b)| - |f'(a)|) \\
&\quad + b |f'(b)| \frac{\ln b - \ln a}{2} \\
&\quad \times \left\{ \frac{a |f'(a)|}{b |f'(b)|} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \right. \\
&\quad + \sum_{i=1}^{\infty} \frac{[\ln(a |f'(a)| / b |f'(b)|)]^{i-1}}{(\alpha+1)_i} \\
&\quad - \frac{(1/2)^{\alpha-1} (a |f'(a)| / b |f'(b)|)^{1/2}}{\alpha+1} \\
&\quad \left. + 2 \frac{1 - (1/2)^\alpha (a |f'(a)| / b |f'(b)|)^{1/2}}{\ln(a |f'(a)| / b |f'(b)|)} \right\}.
\end{aligned} \tag{25}$$

The proof is done. $\square$

Theorem 17. Let $f : [0, b] \rightarrow \mathbb{R}^+$ be a differentiable mapping. If $|f'|^q$ is measurable and $|f'|^q$, ($q > 1$) is GG-convex on $[a, b]$,

for some fixed $\alpha \in (0, \infty)$ and $t \in [0, 1]$, $0 \leq a < b$, then the following integrals hold:

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{1}{2} (|f(b)| - |f(a)|) \\ & \quad + \frac{\ln b - \ln a}{2} \\ & \quad \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[(1/2) \ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\ & \quad \left. + b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} - \frac{(1/2)^{p\alpha-1} (ab)^{p/2}}{p\alpha+1} \right. \\ & \quad \left. - \left[\ln\left(\frac{a}{b}\right)^p \right]^{-1} \left[\left(\frac{1}{2}\right)^{p\alpha-1} (ab)^{p/2} + 2b^p \right] \right)^{1/p} \\ & \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q}, \end{aligned} \quad (26)$$

where $1/p + 1/q = 1$.

Proof. By using Definition 4 and Lemmas 2, 7, 8, and 9, we have

$$\begin{aligned} & \left| \frac{\Gamma(\alpha+1)}{2(\ln b - \ln a)^\alpha} [{}_H J_{a^+}^\alpha f(b) + {}_H J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{b-a}{2} \int_0^1 |kf'(ta + (1-t)b)| dt \\ & \quad + \frac{\ln b - \ln a}{2} \int_0^1 a^t b^{1-t} |(1-t)^\alpha - t^\alpha| |f'(a^t b^{1-t})| dt \\ & \leq \frac{b-a}{2} \int_0^1 |k| |f'(ta + (1-t)b)| dt \\ & \quad + \frac{\ln b - \ln a}{2} \int_0^1 a^t b^{1-t} |(1-t)^\alpha - t^\alpha| |f'(a^t b^{1-t})| dt \\ & \leq \frac{b-a}{2} \int_0^1 |k| |f'(ta + (1-t)b)| dt \\ & \quad + \frac{\ln b - \ln a}{2} \left(\int_0^1 (a^t b^{1-t} |(1-t)^\alpha - t^\alpha|)^p dt \right)^{1/p} \\ & \quad \times \left(\int_0^1 |f'(a^t b^{1-t})|^q dt \right)^{1/q} \end{aligned}$$

$$\begin{aligned} & \leq \frac{-1}{2} (|f(a)| - |f(b)|) \\ & \quad + \frac{\ln b - \ln a}{2} \\ & \quad \times \left(b^p \int_{1/2}^1 \left(\frac{a}{b}\right)^{pt} t^{p\alpha} dt - b^p \int_{1/2}^1 \left(\frac{a}{b}\right)^{pt} (1-t)^{p\alpha} dt \right. \\ & \quad \left. + b^p \int_0^{1/2} \left(\frac{a}{b}\right)^{pt} (1-t)^{p\alpha} dt - b^p \int_0^{1/2} \left(\frac{a}{b}\right)^{pt} t^{p\alpha} dt \right)^{1/p} \\ & \quad \times \left(|f'(b)|^q \int_0^1 \left[\left| \frac{f'(a)}{f'(b)} \right|^q \right]^{qt} dt \right)^{1/q} \\ & \leq \frac{-1}{2} (|f(a)| - |f(b)|) \\ & \quad + \frac{\ln b - \ln a}{2} \\ & \quad \times \left(b^p \int_0^1 \left(\frac{a}{b}\right)^{pt} t^{p\alpha} dt - b^p \int_0^{1/2} \left(\frac{a}{b}\right)^{pt} t^{p\alpha} dt \right. \\ & \quad \left. - b^p \int_0^1 \left(\frac{a}{b}\right)^{pt} (1-t)^{p\alpha} dt + \int_0^{1/2} \left(\frac{a}{b}\right)^{pt} (1-t)^{p\alpha} dt \right. \\ & \quad \left. + b^p \int_0^{1/2} \left(\frac{a}{b}\right)^{pt} (1-t)^{p\alpha} dt - b^p \int_0^{1/2} \left(\frac{a}{b}\right)^{pt} t^{p\alpha} dt \right)^{1/p} \\ & \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\ & \leq \frac{-1}{2} (|f(a)| - |f(b)|) \\ & \quad + \frac{\ln b - \ln a}{2} \\ & \quad \times \left(b^p \left(\frac{a}{b}\right)^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \\ & \quad \left. - b^p \sum_{i=0}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} - 2b^p \int_0^{1/2} \left(\frac{a}{b}\right)^{pt} t^{p\alpha} dt \right. \\ & \quad \left. + 2b^p \int_0^{1/2} \left(\frac{a}{b}\right)^{pt} (1-t)^{p\alpha} dt \right)^{1/p} \\ & \quad \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left| \frac{f'(a)}{f'(b)} \right|^q - 1 \right] \right)^{1/q} \\ & \leq \frac{-1}{2} (|f(a)| - |f(b)|) \\ & \quad + \frac{\ln b - \ln a}{2} \\ & \quad \times \left(b^p \left(\frac{a}{b}\right)^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \right. \end{aligned}$$

$$\begin{aligned}
& -b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& - \left(\frac{1}{2}\right)^{p\alpha} b^p \left(\frac{a}{b}\right)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2)\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& + 2b^p \sum_{i=0}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \left[1 - \left(\frac{a}{b}\right)^{p/2} \left(1 - \frac{1}{2}\right)^{p\alpha+i}\right]^{1/p} \\
& \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q} \\
& \leq \frac{-1}{2} (|f(a)| - |f(b)|) \\
& + \frac{\ln b - \ln a}{2} \\
& \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} - b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i}\right. \\
& \quad - \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2)\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad + 2b^p \sum_{i=0}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad \left. - 2b^p \left(\frac{a}{b}\right)^{p/2} \sum_{i=0}^{\infty} \left(1 - \frac{1}{2}\right)^{p\alpha+i} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i}\right)^{1/p} \\
& \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q} \\
& \leq \frac{-1}{2} (|f(a)| - |f(b)|) \\
& + \frac{\ln b - \ln a}{2} \\
& \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} - b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i}\right. \\
& \quad - \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2)\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad + 2b^p \sum_{i=0}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad \left. - 2b^p \left(\frac{a}{b}\right)^{p/2} \sum_{i=0}^{\infty} \left(1 - \frac{1}{2}\right)^{p\alpha+i} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i}\right)^{1/p} \\
& \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{1}{2} (|f(b)| - |f(a)|) \\
& + \frac{\ln b - \ln a}{2} \\
& \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} + b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i}\right. \\
& \quad - \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[-(1/2)\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad - \left(\frac{1}{2}\right)^{p\alpha} (ab)^{p/2} \sum_{i=1}^{\infty} \frac{[(1/2)\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} \\
& \quad - \left(\frac{1}{2}\right)^{p\alpha-1} (ab)^{p/2} \left[\ln\left(\frac{a}{b}\right)^p\right]^{-1} \\
& \quad \left. + 2b^p \left[\ln\left(\frac{a}{b}\right)^p\right]^{-1}\right)^{1/p} \\
& \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q} \\
& \leq \frac{1}{2} (|f(b)| - |f(a)|) \\
& + \frac{\ln b - \ln a}{2} \\
& \times \left(a^p \sum_{i=1}^{\infty} (-1)^{i-1} \frac{[(1/2)\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i}\right. \\
& \quad + b^p \sum_{i=1}^{\infty} \frac{[\ln(a/b)^p]^{i-1}}{(p\alpha+1)_i} - \frac{(1/2)^{p\alpha-1} (ab^{p/2})}{p\alpha+1} \\
& \quad - \left[\ln\left(\frac{a}{b}\right)^p\right]^{-1} \left[\left(\frac{1}{2}\right)^{p\alpha-1} (ab)^{p/2} + 2b^p\right]^{-1} \\
& \quad \left. \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q}\right. \\
& \quad \left. \times \left(\frac{|f'(b)|^q}{\ln(|f'(a)|/|f'(b)|)^q} \left[\left|\frac{f'(a)}{f'(b)}\right|^q - 1\right]\right)^{1/q}\right.
\end{aligned}$$

(27)

The proof is done. $\square$

4. Applications to Special Means

Consider the following special means (see [23]) for arbitrary real numbers $x, y, x \neq y$ as follows:

$$(M_1) \quad A(x, y) = (x + y)/2, \quad x, y \in \mathbb{R};$$

$$(M_2) \quad H(x, y) = 2/(1/x + 1/y), \quad x, y \in \mathbb{R} \setminus \{0\};$$

$$(M_3) \quad G(x, y) = \sqrt{xy};$$

$$(M_4) \quad L(x, y) = (y - x)/(\ln |y| - \ln |x|), \quad |x| \neq |y|, \quad xy \neq 0;$$

$$(M_5) \quad L_n(x, y) = [(y^{n+1} - x^{n+1})/(n+1)(y-x)]^{1/n}, n \in \mathbb{Z} \setminus \{-1, 0\}, x, y \in \mathbb{R}, x \neq y.$$

We give some applications to special means of real numbers.

Proposition 18. Let $a, b \in \mathbb{R}^+ \setminus \{0\}$, $0 \leq a < b$, $x \in [0, b]$. Then,

$$\begin{aligned} |A(a, b) - L(a, b)| &\leq \frac{\ln b - \ln a}{2} \frac{63a - b}{192}; \\ |A(a, b) - L(a, b)| &\leq \frac{\ln b - \ln a}{2} \left(\frac{1}{2}\right)^{1/q} \\ &\quad \times \left(\frac{a^p - 2b^p}{p+2} + \frac{b^p}{p+1} - \frac{2b^p + a^p}{2^{p+1}(p+1)} \right. \\ &\quad \left. - \frac{b^p - a^p}{2^{p+2}(p+2)} + \frac{a^p}{(p+1)(p+2)} \right. \\ &\quad \left. - \frac{a^p}{2^{p+1}(p+1)(p+2)} \right)^{1/p}; \\ |L(a, b) - A(a, b)| &\leq \frac{\ln b - \ln a}{2} \frac{63a - b}{192}; \\ |L(a, b) - A(a, b)| &\leq \frac{\ln b - \ln a}{2} \left(\frac{1}{2}\right)^{1/q} \\ &\quad \times \left(\frac{a^p - 2b^p}{p+2} + \frac{b^p}{p+1} - \frac{2b^p + a^p}{2^{p+1}(p+1)} \right. \\ &\quad \left. - \frac{b^p - a^p}{2^{p+2}(p+2)} + \frac{a^p}{(p+1)(p+2)} \right. \\ &\quad \left. - \frac{a^p}{2^{p+1}(p+1)(p+2)} \right)^{1/p}; \\ |A(a, b) - L(a, b)| &\leq b \frac{\ln b - \ln a}{2} \\ &\quad \times \left[\frac{a/b - 1}{\ln(a/b)} \right. \\ &\quad \left. + 2 \frac{(a/b)^{1/2} - a/b + (a/b)^2 - 1}{(\ln(a/b))^2} \right]. \end{aligned} \quad (28)$$

By using Lemmas 7, 8, and 9, we have

$$\begin{aligned} |A(a, b) - L(a, b)| &\leq b \frac{\ln b - \ln a}{2} \\ &\quad \times \left(I[p+1] - H\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) \right. \\ &\quad \left. - 2J\left(p+1, \left(\frac{a}{b}\right)^p\right) \right. \\ &\quad \left. + 2bR\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) \right)^{1/p}; \end{aligned}$$

$$\begin{aligned} |L(a, b) - A(a, b)| &\leq b \frac{\ln b - \ln a}{2} \\ &\quad \times \left[\frac{a/b - 1}{\ln(a/b)} \right. \\ &\quad \left. + 2 \frac{(a/b)^{1/2} - a/b + (a/b)^2 - 1}{(\ln(a/b))^2} \right]; \\ |L(a, b) - A(a, b)| &\leq b \frac{\ln b - \ln a}{2} \\ &\quad \times \left(I[p+1] - H\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) \right. \\ &\quad \left. - 2J\left(p+1, \left(\frac{a}{b}\right)^p\right) \right. \\ &\quad \left. + 2bR\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) \right)^{1/p}. \end{aligned} \quad (29)$$

Proof. Applying Theorems 10, 11, 14, 15, 12, 13, 16, and 17, for $f(x) = x$ and $\alpha = 1$, one can obtain the results immediately. $\square$

Proposition 19. Let $a, b \in \mathbb{R}^+ \setminus \{0\}$, $0 \leq a < b$, $x \in [0, b]$, $n \geq 2$. Then,

$$\begin{aligned} |A(a^n, b^n) - L_{n-1}^{n-1}(a, b) L(a, b)| \\ &\leq \frac{\ln b - \ln a}{2} \frac{22na^n + 12nba^{n-1} + 9nab^{n-1} - 12nb^n}{96}; \\ |A(a^n, b^n) - L_{n-1}^{n-1}(a, b) L(a, b)| \\ &\leq \frac{\ln b - \ln a}{2} \left(\frac{n^q a^{q(n-1)} + n^q b^{q(n-1)}}{2} \right)^{1/q} \\ &\quad \times \left(\frac{a^p - 2b^p}{p+2} + \frac{b^p}{p+1} \right. \\ &\quad \left. - \frac{1}{2^{p+1}} \frac{b^p}{p+1} + \frac{1}{2^{p+2}} \frac{b^p - a^p}{p+2} - \frac{1}{2^{p+3}} \frac{1}{p+3} \right. \\ &\quad \left. - \frac{a^p}{p(p+1)} + \frac{2a^p}{(p+1)(p+2)} \right)^{1/p}; \\ |L_{n-1}^{n-1}(a, b) L(a, b) - A^n(x, y)| \\ &\leq \frac{nb^{n-1} - na^{n-1}}{2} \\ &\quad + \frac{\ln b - \ln a}{2} \frac{22na^n + 12nba^{n-1} + 9nab^{n-1} - 12nb^n}{96}; \end{aligned}$$

$$\begin{aligned}
& \left| L_{n-1}^{n-1}(a, b) L(a, b) - A^n(x, y) \right| \\
& \leq \frac{nb^{n-1} - na^{n-1}}{2} \\
& \quad + \frac{\ln b - \ln a}{2} \left(\frac{n^q a^{q(n-1)} + n^q b^{q(n-1)}}{2} \right)^{1/q} \\
& \quad \times \left(\frac{a^p - 2b^p}{p+2} + \frac{b^p}{p+1} - \frac{1}{2^{p+1}} \frac{b^p}{p+1} \right. \\
& \quad \left. + \frac{1}{2^{p+2}} \frac{b^p - a^p}{p+2} - \frac{1}{2^{p+3}} \frac{1}{p+3} \right. \\
& \quad \left. - \frac{a^p}{p(p+1)} + \frac{2a^p}{(p+1)(p+2)} \right)^{1/p};
\end{aligned}$$

$$\begin{aligned}
& \left| A(a^n, b^n) - L_{n-1}^{n-1}(a, b) L(a, b) \right| \\
& \leq nb^n \frac{\ln b - \ln a}{2} \\
& \quad \times \left(\frac{a^n/b^n - 1}{\ln(a^n/b^n)} + \frac{4(a^n/b^n) - 2(a^n/b^n) - 2}{(\ln(a^n/b^n))^2} \right).
\end{aligned} \tag{30}$$

By using Lemmas 7, 8, and 9, we have

$$\begin{aligned}
& \left| A(a^n, b^n) - L_{n-1}^{n-1}(a, b) L(a, b) \right| \\
& \leq b \frac{\ln b - \ln a}{2} \left(I[p+1] - H\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) \right. \\
& \quad \left. - 2J\left(p+1, \left(\frac{a}{b}\right)^p\right) \right. \\
& \quad \left. + 2bR\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) \right)^{1/p} \\
& \quad \times \left(\frac{|nb^{n-1}|^q}{\ln(a^{n-1}/b^{n-1})^q} \left[\left(\frac{a^{n-1}}{b^{n-1}}\right)^q - 1 \right] \right)^{1/q}; \\
& \left| L_{n-1}^{n-1}(a, b) L(a, b) - A^n(x, y) \right| \\
& \leq \frac{nb^{n-1} - na^{n-1}}{2} \\
& \quad + nb^n \frac{\ln b - \ln a}{2} \\
& \quad \times \left(\frac{a^n/b^n - 1}{\ln(a^n/b^n)} + \frac{4(a^n/b^n) - 2(a^n/b^n) - 2}{(\ln(a^n/b^n))^2} \right);
\end{aligned}$$

$$\begin{aligned}
& \left| L_{n-1}^{n-1}(a, b) L(a, b) - A^n(x, y) \right| \\
& \leq \frac{nb^{n-1} - na^{n-1}}{2} \\
& \quad + b \frac{\ln b - \ln a}{2} \left(I[p+1] - H\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) \right. \\
& \quad \left. - 2J\left(p+1, \left(\frac{a}{b}\right)^p\right) \right. \\
& \quad \left. + 2bR\left(p+1, \left(\frac{a}{b}\right)^p, \frac{1}{2}\right) \right)^{1/p} \\
& \quad \times \left(\frac{|nb^{n-1}|^q}{\ln(a^{n-1}/b^{n-1})^q} \left[\left(\frac{a^{n-1}}{b^{n-1}}\right)^q - 1 \right] \right)^{1/q}.
\end{aligned} \tag{31}$$

Proof. Applying Theorems 10, 11, 14, 15, 12, 13, 16, and 17 for $f(x) = x^n$ and $\alpha = 1$, one can obtain the results immediately. $\square$

Proposition 20. Let $a, b \in \mathbb{R}^+ \setminus \{0\}$, $0 \leq a < b$, $x \in [0, b]$, $n \geq 2$. Then,

$$\begin{aligned}
& \left| H^{-1}(a, b) - G^{-2}(a, b) L(a, b) \right| \\
& \leq \frac{\ln b - \ln a}{2} \\
& \quad \times \left(450(1/a) + 60(a/b^2) - 30(b/a^2) \right. \\
& \quad \left. - 32(1/a) + 42(1/a) \right) (1920)^{-1}; \\
& \left| H^{-1}(a, b) - G^{-2}(a, b) L(a, b) \right| \\
& \leq \frac{\ln b - \ln a}{2} \left(\frac{a^{-2q} + b^{-2q}}{2} \right)^{1/q} \\
& \quad \times \left(\frac{a^p - 2b^p}{p+2} + \frac{b^p}{p+1} - \frac{1}{2^{p+2}} \frac{2a^p + 3b^p}{p+1} \right. \\
& \quad \left. + \frac{1}{2^{p+2}} \frac{2b^p - a^p}{p+2} - \frac{1}{p(p+1)} \right. \\
& \quad \left. - \frac{2a^p + b^p}{2^{p+2}(p+1)(p+2)} \right)^{1/p}; \\
& \left| G^{-2}(a, b) L(a, b) - A^{-1}(a, b) \right| \\
& \leq \frac{b^2 - a^2}{2a^2b^2} + \frac{\ln b - \ln a}{2} \\
& \quad \times \left(450(1/a) + 60(a/b^2) - 30(b/a^2) \right. \\
& \quad \left. - 32(1/a) + 42(1/a) \right) (1920)^{-1};
\end{aligned}$$

$$\begin{aligned}
& \left| G^{-2}(a, b) L(a, b) - A^{-1}(a, b) \right| \\
& \leq \frac{b^2 - a^2}{2a^2b^2} \\
& \quad + \frac{\ln b - \ln a}{2} \left(\frac{a^{-2q} + b^{-2q}}{2} \right)^{1/q} \\
& \quad \times \left(\frac{a^p - 2b^p}{p+2} + \frac{b^p}{p+1} - \frac{1}{2^{p+2}} \frac{2a^p + 3b^p}{p+1} \right. \\
& \quad \left. + \frac{1}{2^{p+2}} \frac{2b^p - a^p}{p+2} - \frac{1}{p(p+1)} \right. \\
& \quad \left. - \frac{2a^p + b^p}{2^{p+2}(p+1)(p+2)} \right)^{1/p}; \\
& \left| H^{-1}(a, b) - G^{-2}(a, b) L(a, b) \right| \\
& \leq \frac{\ln b - \ln a}{2b} \left(\frac{b/a - 1}{\ln(b/a)} + \frac{4(b/a)^{1/2} - 2(b/a) - 2}{(\ln(b/a))^2} \right). \quad (32)
\end{aligned}$$

By using Lemmas 7, 8, and 9, we have

$$\begin{aligned}
& \left| H^{-1}(a, b) - G^{-2}(a, b) L(a, b) \right| \\
& \leq b \frac{\ln b - \ln a}{2} \left(I[p+1] - H \left(p+1, \left(\frac{a}{b} \right)^p, \frac{1}{2} \right) \right. \\
& \quad \left. - 2J \left(p+1, \left(\frac{a}{b} \right)^p \right) \right. \\
& \quad \left. + 2bR \left(p+1, \left(\frac{a}{b} \right)^p, \frac{1}{2} \right) \right)^{1/p} \\
& \quad \times \left(\frac{|1/b^2|^q}{\ln|b^2/a^2|^q} \left[\left| \frac{b^2}{a^2} \right|^q - 1 \right] \right)^{1/q}; \\
& \left| G^{-2}(a, b) L(a, b) - A^{-1}(a, b) \right| \\
& \leq \frac{b^2 - a^2}{2a^2b^2} \\
& \quad + \frac{\ln b - \ln a}{2b} \left(\frac{b/a - 1}{\ln(b/a)} + \frac{4(b/a)^{1/2} - 2(b/a) - 2}{(\ln(b/a))^2} \right); \\
& \left| G^{-2}(a, b) L(a, b) - A^{-1}(a, b) \right| \\
& \leq \frac{b^2 - a^2}{2a^2b^2} \\
& \quad + b \frac{\ln b - \ln a}{2} \left(I[p+1] - H \left(p+1, \left(\frac{a}{b} \right)^p, \frac{1}{2} \right) \right. \\
& \quad \left. - 2J \left(p+1, \left(\frac{a}{b} \right)^p \right) \right.
\end{aligned}$$

$$+ 2bR \left(p+1, \left(\frac{a}{b} \right)^p, \frac{1}{2} \right) \right)^{1/p}$$

$$\times \left(\frac{|1/b^2|^q}{\ln|b^2/a^2|^q} \left[\left| \frac{b^2}{a^2} \right|^q - 1 \right] \right)^{1/q}.$$

(33)

Proof. Applying Theorems 10, 11, 14, 15, 12, 13, 16, and 17 for $f(x) = 1/x$ and $\alpha = 1$, one can obtain the results immediately. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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