Research Article

Approximation of Signals (Functions) by Trigonometric Polynomials in $L_p$-Norm

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Mittal and Rhoades (1999, 2000) and Mittal et al. (2011) have initiated a study of error estimates $E_n(f)$ through trigonometric-Fourier approximation (tfa) for the situations in which the summability matrix $T$ does not have monotone rows. In this paper, the first author continues the work in the direction for $T$ to be a $N_p$-matrix. We extend two theorems on summability matrix $N_p$ of Deger et al. (2012) where they have extended two theorems of Chandra (2002) using $C_{\lambda}$-method obtained by deleting a set of rows from Cesàro matrix $C_1$. Our theorems also generalize two theorems of Leindler (2005) to $N_p$-matrix which in turn generalize the result of Chandra (2002) and Quade (1937).

"In memory of Professor K. V. Mital, 1918 – 2010."

1. Introduction

Let $f$ be a $2\pi$ periodic signal (function) and let $f \in L_p := L_p[0, 2\pi], p \geq 1$. Let

$$s_n(f) := s_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

$$= \sum_{k=0}^{n} u_k(f; x)$$

(1)

denote the partial sums, called trigonometric polynomials of degree (or order) $n$, of the first $(n + 1)$ terms of the Fourier series of $f$ at a point $x$.

The integral modulus of continuity of $f$ is defined by

$$\omega_p(\delta; f) := \sup_{0 < |h| \leq \delta} \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |f(x + h) - f(x)|^p dx \right\}^{1/p}.$$ 

(2)

If, for $\alpha > 0$,

$$\omega_p(\delta; f) = O(\delta^{\alpha}),$$

(3)

then $f \in \text{Lip}(\alpha, p) (p \geq 1)$. Throughout $\| \cdot \|_p$ will denote the $L_p$-norm, defined by

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_{0}^{2\pi} |f(x)|^p dx \right\}^{1/p}, \quad f \in L_p (p \geq 1).$$

(4)

A positive sequence $c := \{c_n\}$ is called almost monotone decreasing (increasing) if there exists a constant $K := K(c)$, depending on the sequence $c$ only, such that, for all $n \geq m$,

$$c_n \leq Kc_m (Kc_n \geq c_m).$$

(5)

Such sequences will be denoted by $c \in \text{AMDS}$ and $c \in \text{AMIS}$, respectively. A sequence which is either AMDS or AMIS is called almost monotone sequence and will be denoted by $c \in \text{AMS}$. Let $F$ be an infinite subset of $N$ and $F$ as the range of strictly increasing sequence of positive integers; say $F = \{\lambda(n)\}_{n=1}^{\infty}$. The Cesàro submethod $C_{\lambda}$ is defined as

$$(C_{\lambda}x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad (n = 1, 2, 3, \ldots),$$

(6)
where \( \{x_k\} \) is a sequence of real or complex numbers. Therefore, the \( C_\lambda \)-method yields a subsequence of the Cesàro method \( C_1 \), and hence it is regular for any \( \lambda \). \( C_\lambda \) is obtained by deleting a set of rows from Cesàro matrix. The basic properties of \( C_\lambda \)-method can be found in [1, 2]. In the present paper, we will consider approximation of \( f \in L_p \) by trigonometric polynomials \( N_n^\lambda(f; x) \) and \( R_n^\lambda(f; x) \) of degree (or order) \( n \), where

\[
N_n^\lambda(f; x) = \frac{1}{\lambda(n)} \sum_{m=0}^{\lambda(n)} P_{\lambda(n)-m} s_m(f; x),
\]

\[
R_n^\lambda(f; x) = \frac{1}{\lambda(n)} \sum_{m=0}^{\lambda(n)} p_m s_m(f; x),
\]

\[
s_n(f; x) = \frac{1}{n} \sum_{t=0}^{\pi n} (f + t) D_n(t) dt,
\]

\[
D_n(t) = \frac{(\sin(n+1/2)t)}{2 (n/2)},
\]

\[
p_{\lambda(n)} = p_0 + p_1 + \cdots + p_{\lambda(n)} \neq 0 \quad (n \geq 0),
\]

and by convention \( p_{-1} = 0 = p_{-2} \).

The case \( p_n = 1 \) for all \( n \geq 0 \) of either \( N_n^\lambda(f; x) \) or \( R_n^\lambda(f; x) \) yields

\[
\sigma_n^\lambda(f; x) = \frac{1}{\lambda(n) + 1} \sum_{m=0}^{\lambda(n)} s_m(f; x).
\]

We also use

\[
\Delta_n a_n = a_n - a_{n-1}, \quad \Delta_m g(n, m) = g(n, m) - g(n, m + 1).
\]

Mittal and Rhoades [3, 4] have initiated the study of error estimates \( E_n(f) \) through trigonometric-Fourier approximation (tfa) for the situations in which the summability matrix \( T \) does not have monotone rows. In this paper, the first author continues the work in the direction for \( T \) to be a \( N_p \)-matrix. Recently, Chandra [5] has proved three theorems on the trigonometric approximation using \( N_p \)-matrix. Some of them give sharper estimates than the results proved by Quade [6], Mohapatra and Russell [7], and himself earlier [8]. These results of Chandra [5] are improved in different directions by different investigators such as Leindler [9] who dropped the monotonicity on generating sequence \( \{p_n\} \) and Mittal et al. [10, 11] who used more general matrix while very recently Deger et al. [12] used more general \( C_\lambda \)-method in view of Armitage and Maddox [1].

### 2. Known Results

Leindler [9] proved the following.

**Theorem 1** (see [9]). If \( f \in \text{Lip}(\alpha, p) \) and \( \{p_n\} \) be positive. If one of the conditions

(i) \( p > 1, \alpha > 1, \) and \( \{p_n\} \in \text{AMIS}, \)

(ii) \( p > 1, \alpha > 1, \) and \( \{p_n\} \in \text{AMDS}, \)

\[(n + 1) p_n = O(P_n) \quad \text{holds,} \tag{10}\]

(iii) \( p > 1, \alpha = 1, \) and \( \sum_{k=1}^{n-1} k |\Delta p_k| = O(P_n), \)

(iv) \( p > 1, \alpha = 1, \) and \( \sum_{k=0}^{n-1} |\Delta p_k| = O(P_{n/\alpha}(n)), \) and (10) holds,

(v) \( p = 1, \alpha < 1, \) and \( \sum_{k=0}^{n-1} |\Delta p_k| = O(P_{n/\alpha}(n)) \)

maintains, then

\[
\|f - N_n(f)\|_p = O\left(\left(\lambda(n)\right)^{-\alpha}\right). \tag{11}\]

**Theorem 2** (see [9]). Let \( f \in \text{Lip}(\alpha, 1) \), \( 0 < \alpha < 1 \). If the positive \( \{p_n\} \) satisfies conditions (10) and \( \sum_{k=0}^{n-1} |\Delta p_k| = O(P_{n/\alpha}) \) hold, then

\[
\|f - R_n(f)\|_1 = O\left(\left(\lambda(n)\right)^{-\alpha}\right). \tag{12}\]

Deger et al. [12] proved.

**Theorem 3** (see [12]). Let \( f \in \text{Lip}(\alpha, p) \) and let \( \{p_n\} \) be positive such that

\[(\lambda(n) + 1) p_{\lambda(n)} = O(P_{\lambda(n)}). \tag{13}\]

If either (i) \( p > 1, \alpha > 1, \) and \( \{p_n\} \) is monotonic or

(ii) \( p = 1, \alpha < 1, \) and \( \{p_n\} \) is nondecreasing, then

\[
\|f - N_n^\lambda(f)\|_p = O\left(\left(\lambda(n)\right)^{-\alpha}\right). \tag{14}\]

**Theorem 4** (see [12]). Let \( f \in \text{Lip}(\alpha, 1) \), \( 0 < \alpha < 1 \). If the positive \( \{p_n\} \) satisfies condition (13) and is nondecreasing, then

\[
\|f - R_n^\lambda(f)\|_1 = O\left(\left(\lambda(n)\right)^{-\alpha}\right). \tag{15}\]

### 3. Main Results

In this paper we generalize Theorems 3 and 4 of Deger et al. [12], by dropping monotonicity on the elements of the matrix rows which in turn generalize Theorems 1 and 2, respectively, of Leindler [9] to a more general \( C_\lambda \)-method. We prove the following.

**Theorem 5.** If \( f \in \text{Lip}(\alpha, p) \) and \( \{p_n\} \) is positive and if one of the following conditions

(i) \( p > 1, \alpha < 1, \) and \( \{p_n\} \in \text{AMDS}, \)

(ii) \( p > 1, \alpha < 1, \) and \( \{p_n\} \in \text{AMIS}, \) and (13) holds,

(iii) \( p > 1, \alpha = 1, \) and \( \sum_{k=1}^{\lambda(n)-1} k |\Delta p_k| = O(P_{\lambda(n)}), \)

(iv) \( p > 1, \alpha = 1, \) and \( \sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}), \) and (13) holds,

(v) \( p = 1, \alpha < 1, \) and \( \sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}), \)

maintains, then

\[
\|f - N_n^\lambda(f)\|_p = O\left(\left(\lambda(n)\right)^{-\alpha}\right). \tag{16}\]
**Theorem 6.** Let $f \in \text{Lip}(\alpha, 1)$, $0 < \alpha < 1$. If the positive \{p_n\} satisfies (13) and the condition $\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)} / \lambda(n))$ holds, then

$$\|f - R^p_n(f)\|_p = O((\lambda(n))^{-\alpha}). \quad (19)$$

Remarks. (1) If $\lambda(n) = n$, then our Theorems 5 and 6 reduce to Theorems 1 and 2, respectively.

(2) Deger et al. [12] have used monotone sequences \{p_n\} in Theorems 3 and 4, while our Theorems 5 and 6 claim less if the sequence \{p_n\} is nonincreasing; that is,

$$\sum_{k=1}^{\lambda(n)-1} k |\Delta p_k| = \sum_{k=0}^{\lambda(n)-1} k (p_{k+1} - p_k) = P_{\lambda(n)} - P_0 = O(P_{\lambda(n)}), \quad (18)$$

while if sequence \{p_n\} is nondecreasing and condition (13) holds, then the condition in (iv) of Theorem 5 is also satisfied; that is,

$$\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = \sum_{k=0}^{\lambda(n)-1} (p_{k+1} - p_k) = P_{\lambda(n)} - P_0 \leq P_{\lambda(n)} \quad (19)$$

Thus our theorems generalize the two theorems of Deger et al. [12] under weaker assumptions and give sharper estimates because all the estimates of Deger et al. [12] are in terms of $n$, while our estimates are in terms of $\lambda(n)$ and $(\lambda(n))^{-\alpha} \leq n^{-\alpha}$ for $0 < \alpha \leq 1$.

### 4. Lemmas

We will use the following lemmas in the proof of our theorems.

**Lemma 1** (see [6]). If $f \in \text{Lip}(\alpha, p)$, for $0 < \alpha \leq 1$ and $p > 1$, then

$$\|f - s_n(f)\|_p = O(n^{-\alpha}). \quad (20)$$

**Lemma 2** (see [6]). If $f \in \text{Lip}(1, p)$, for $p > 1$, then

$$\|\sigma_n(f) - s_n(f)\|_p = O(n^{-1}). \quad (21)$$

**Lemma 3** (see [6]). If $f \in \text{Lip}(\alpha, 1)$, $0 < \alpha < 1$, then

$$\|f - \sigma_n(f)\|_1 = O(n^{-\alpha}). \quad (22)$$

**Lemma 4.** Let \{p_n\} $\in$ AMDS or let \{p_n\} $\in$ AMIS and satisfy (13). Then, for $0 < \alpha < 1$,

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha} P_{\lambda(n) - m} = O((\lambda(n))^{-\alpha} P_{\lambda(n)}). \quad (23)$$

holds.

**Proof.** Let $r$ denote the integral part of $(\lambda(n)/2)$. Then, if \{p_n\} $\in$ AMDS,

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha} P_{\lambda(n) - m} = \sum_{m=1}^{r} m^{-\alpha} P_{\lambda(n) - m} + \sum_{m=r+1}^{\lambda(n)} m^{-\alpha} P_{\lambda(n) - m} \leq K P_{\lambda(n) - r} \sum_{m=1}^{r} m^{-\alpha} + (r + 1)^{-\alpha} \sum_{m=r+1}^{\lambda(n)} P_{\lambda(n) - m} \leq K P_{\lambda(n) - r} \sum_{m=1}^{r} m^{-\alpha} + (r + 1)^{-\alpha} \sum_{m=0}^{\lambda(n)} P_{\lambda(n) - m} = K P_{\lambda(n) - r} (\lambda(n))^{-\alpha} P_{\lambda(n)} + O((\lambda(n))^{-\alpha} P_{\lambda(n)}) = O((\lambda(n))^{-\alpha} P_{\lambda(n)}). \quad (24)$$

Thus our theorems generalize the two theorems of Deger et al. [12] under weaker assumptions and give sharper estimates because all the estimates of Deger et al. [12] are in terms of $n$, while our estimates are in terms of $\lambda(n)$ and $(\lambda(n))^{-\alpha} \leq n^{-\alpha}$ for $0 < \alpha \leq 1$.

### 5. Proof of the Main Results

**Proof of Theorem 5.** We prove cases (i) and (ii) together. Since

$$N_n^f(f;x) = f(x) - f(x) = 1 \frac{\lambda(n)}{P_{\lambda(n)} m=0} [\lambda(n)]_n [s_m(f;x) - f(x)], \quad (26)$$

thus in view of Lemmas 1 and 4 and condition (13), we have

$$\|f - N_n^f(f)\|_p \leq \frac{\lambda(n)}{P_{\lambda(n)} m=0} \|s_m(f)\|_p$$

$$= \frac{\lambda(n)}{P_{\lambda(n)} m=0} \|s_m(f)\|_p + \frac{P_{\lambda(n)}}{P_{\lambda(n)}} \|f - s_m(f)\|_p$$

$$= \frac{\lambda(n)}{P_{\lambda(n)} m=0} \|s_m(f)\|_p + \frac{P_{\lambda(n)}}{P_{\lambda(n)}} \|s_m(f) - f\|_p$$

$$= \frac{\lambda(n)}{P_{\lambda(n)} m=0} O(m^{-\alpha}) + O\left(\frac{P_{\lambda(n)}}{P_{\lambda(n)}}\right)$$

$$= O((\lambda(n))^{-\alpha}). \quad (27)$$
Next we consider case (iv).

Let \( p > 1 \) and \( \alpha = 1 \). By Abel’s transformation, we get

\[
N_n^\lambda (f; x) = \frac{1}{P_{\lambda(n)} m} \sum_{m=0}^{\lambda(n)} (P_{\lambda(n)} - P_{\lambda(n)-m}) u_m (f; x),
\]

and thus

\[
\dot{s}_n^\lambda (f; x) - N_n^\lambda (f; x) = \frac{1}{P_{\lambda(n)} m} \sum_{m=1}^{\lambda(n)} \Delta_m \left( P_{\lambda(n)} - P_{\lambda(n)-m} \right) u_m (f; x).
\]

Hence again by Abel’s transformation, we get

\[
\dot{s}_n^\lambda (f; x) - N_n^\lambda (f; x) = \frac{1}{P_{\lambda(n)} m} \sum_{m=1}^{\lambda(n)} \Delta_m \left( P_{\lambda(n)} - P_{\lambda(n)-m} \right) u_m (f; x).
\]

Thus

\[
\| \dot{s}_n^\lambda (f) - N_n^\lambda (f) \|_p \leq \frac{1}{P_{\lambda(n)} m} \sum_{m=1}^{\lambda(n)} \| \Delta_m \left( P_{\lambda(n)} - P_{\lambda(n)-m} \right) u_m (f; x) \|_p.
\]

Since

\[
s_n (f; x) - \sigma_n (f; x) = \frac{1}{n + 1} \sum_{k=1}^{n} k u_k (f; x),
\]

thus by Lemma 2

\[
\| \sum_{k=1}^{n} k u_k \|_p = (n + 1) \| \sigma_n (f; x) - s_n (f; x) \|_p = O (1).
\]

In view of (31) and (33), we obtain

\[
\| \dot{s}_n^\lambda (f) - N_n^\lambda (f) \|_p = O \left( \frac{1}{P_{\lambda(n)} m} \sum_{m=1}^{\lambda(n)} \| \Delta_m \left( P_{\lambda(n)} - P_{\lambda(n)-m} \right) u_m (f; x) \|_p \right) + O \left( (\lambda (n))^1 \right).
\]
\[ \sum_{k=\lambda(n)-j}^{\lambda(n)} p_k - (j+1)P_{\lambda(n)-j} \leq \sum_{k=1}^{j} k\left| P_{\lambda(n)-k+1} - P_{\lambda(n)-k} \right| + (j+1)\left| P_{\lambda(n)-j} - P_{\lambda(n)-(j+1)} \right| \]

\[ = \sum_{k=1}^{j} k\left| P_{\lambda(n)-k+1} - P_{\lambda(n)-k} \right|. \]

(39)

thus (37) is proved for \( m = j + 1 \); that is, (37) is true for any \( 1 \leq m \leq \lambda(n) \). Using (36) and (37) and interchanging the order of summation, we get

\[ \sum_{m=1}^{\lambda(n)} \Delta_m \left( \frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) \leq \sum_{m=1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| P_{\lambda(n)-k+1} - P_{\lambda(n)-k} \right| \]

\[ \leq \sum_{m=1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{\lambda(n)-m} \left| \Delta_k \right| \]

\[ \leq \sum_{k=0}^{\lambda(n)-1} \left| \Delta_k \right|. \]

(40)

In view of (36) and (37)

\[ \sum_{m=1}^{\lambda(n)} \Delta_m \left( \frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) \leq \sum_{m=1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| \Delta_k P_{\lambda(n)-k} \right| \]

\[ \leq \sum_{r=1}^{\lambda(n)-1} \frac{1}{m(m+1)} \sum_{k=1}^{m} \left| \Delta_k P_{\lambda(n)-k} \right| \]

\[ = \frac{\lambda(n)}{\lambda(n)} \sum_{k=0}^{\lambda(n)-1} \left| \Delta_k \right|. \]

(44)

Denoting again by \( r \) the integral part of \( \lambda(n)/2 \), then, by Abel’s transformation, we have

\[ B_1 = \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} \left| \Delta_k P_{\lambda(n)-k} \right| \]

\[ \leq \sum_{k=1}^{\lambda(n)-1} \frac{1}{m(m+1)} \sum_{k=1}^{m} \left| \Delta_k P_{\lambda(n)-k} \right| \]

\[ = \frac{\lambda(n)}{\lambda(n)} \sum_{k=0}^{\lambda(n)-1} \left| \Delta_k \right| \]

\[ = \frac{\lambda(n)}{\lambda(n)} \sum_{k=0}^{\lambda(n)-1} \left| \Delta_k \right|. \]

(46)

Furthermore, using again our assumption, we get

\[ B_{21} \leq \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} \left| \Delta_k \right| = O \left( \frac{P_{\lambda(n)}}{\lambda(n)} \right), \]

\[ B_{22} \leq \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} \left| \Delta_k \right| = O \left( \frac{P_{\lambda(n)}}{\lambda(n)} \right). \]

(47)

Summing up our partial results, we verified (43). Thus (34) and Lemma 1 again yield

\[ \left\| f - N_{\alpha}^n(f) \right\|_p = O \left( (\lambda(n))^{-1} \right). \]

(48)
Now, we prove case (v), by using (26), $p_{-1} = 0$, and Abel’s transformation
\[
N^A_n(f; x) - f(x) = \frac{1}{p_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} [s_m(f; x) - f(x)]
= \frac{1}{p_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} \left( \Delta_m p_{\lambda(n)-m} \right) \sum_{k=0}^{m} [s_k(f; x) - f(x)]
= \frac{1}{p_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (m + 1) (\Delta_m p_{\lambda(n)-m})
\times [\sigma_m(f; x) - f(x)].
\]

Hence in view of Lemma 3
\[
\|N^A_n(f) - f\|_1 \leq \frac{1}{p_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (m + 1) |\Delta_m p_{\lambda(n)-m}| \|f - \sigma_m(f)\|_1
= O \left( \frac{1}{p_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (m + 1)^{1-\alpha} |\Delta_m p_{\lambda(n)-m}| \right)
= O \left( \frac{\lambda(n)^{1-\alpha}}{p_{\lambda(n)}} \sum_{m=0}^{\lambda(n)-1} |\Delta p_m| \right)
= O \left( \frac{\lambda(n)^{1-\alpha}}{p_{\lambda(n)}} \cdot O \left( \frac{P_{\lambda(n)}}{\lambda(n)} \right) \right)
= O \left( (\lambda(n))^{-\alpha} \right).
\]

Herewith case (v) is also verified and thus the proof of Theorem 5 is complete.

**Proof of Theorem 6.** Since $R^A_n(f; x) = (1/p_{\lambda(n)}) \sum_{m=0}^{\lambda(n)} P_m s_m(f; x)$, so, in view of the assumptions of Theorem 6, we get
\[
\|f - R^A_n(f)\|_1 = \frac{1}{p_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} \left[ \frac{1}{p_{\lambda(n)}} \sum_{m=0}^{\lambda(n)-1} |\Delta p_m| \cdot \|f - \sigma_m(f)\|_1 \right]
\leq \frac{1}{p_{\lambda(n)}} \sum_{m=0}^{\lambda(n)-1} (m + 1) |\Delta p_m| \cdot \|f - \sigma_m(f)\|_1
+ (\lambda(n) + 1) p_{\lambda(n)} \|f - \sigma^A_n(f)\|_1
= O \left( (\lambda(n))^{1-\alpha} P_{\lambda(n)}^{-1} \sum_{m=0}^{\lambda(n)-1} |\Delta p_m| \right)
+ O \left( (\lambda(n))^{1-\alpha} \right)
= O \left( (\lambda(n))^{-\alpha} \right).
\]

This proves Theorem 6.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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