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*Research Article*

# Approximation of Signals (Functions) by Trigonometric Polynomials in $L_p$-Norm

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Mittal and Rhoades (1999, 2000) and Mittal et al. (2011) have initiated a study of error estimates $E_n(f)$ through trigonometric-Fourier approximation (tfa) for the situations in which the summability matrix $T$ does not have monotone rows. In this paper, the first author continues the work in the direction for $T$ to be a $N_p$-matrix. We extend two theorems on summability matrix $N_p$ of Deger et al. (2012) where they have extended two theorems of Chandra (2002) using $C_\lambda$-method obtained by deleting a set of rows from Cesàro matrix $C_1$. Our theorems also generalize two theorems of Leindler (2005) to $N_p$-matrix which in turn generalize the result of Chandra (2002) and Quade (1937).

*"In memory of Professor K. V. Mital, 1918 – 2010."*

## 1. Introduction

Let $f$ be a $2\pi$ periodic signal (function) and let $f \in L_p := L_p[0, 2\pi]$, $p \geq 1$. Let

$$s_n(f) := s_n(f; x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{n} u_k(f; x) \tag{1}$$

denote the partial sums, called trigonometric polynomials of degree (or order) $n$, of the first $(n+1)$ terms of the Fourier series of $f$ at a point $x$.

The integral modulus of continuity of $f$ is defined by

$$\omega_p(\delta; f) := \sup_{0<|h|\leq\delta} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x+h) - f(x)|^p dx \right\}^{1/p}. \tag{2}$$

If, for $\alpha > 0$,

$$\omega_p(\delta; f) = O(\delta^\alpha), \tag{3}$$

then $f \in \mathrm{Lip}(\alpha, p)$ $(p \geq 1)$. Throughout $\|\cdot\|_p$ will denote the $L_p$-norm, defined by

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right\}^{1/p}, \quad f \in L_p (p \geq 1). \tag{4}$$

A positive sequence $\mathbf{c} := \{c_n\}$ is called almost monotone decreasing (increasing) if there exists a constant $K := K(\mathbf{c})$, depending on the sequence $\mathbf{c}$ only, such that, for all $n \geq m$,

$$c_n \leq K c_m \, (K c_n \geq c_m). \tag{5}$$

Such sequences will be denoted by $\mathbf{c} \in$ AMDS and $\mathbf{c} \in$ AMIS, respectively. A sequence which is either AMDS or AMIS is called almost monotone sequence and will be denoted by $\mathbf{c} \in$ AMS. Let $\mathbb{F}$ be an infinite subset of $\mathbb{N}$ and $\mathbb{F}$ as the range of strictly increasing sequence of positive integers; say $\mathbb{F} = \{\lambda(n)\}_{n=1}^{\infty}$. The Cesàro submethod $C_\lambda$ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)} \sum_{k=1}^{\lambda(n)} x_k, \quad (n = 1, 2, 3, \ldots), \tag{6}$$

where $\{x_k\}$ is a sequence of real or complex numbers. Therefore, the $C_\lambda$-method yields a subsequence of the Cesàro method $C_1$, and hence it is regular for any $\lambda$. $C_\lambda$ is obtained by deleting a set of rows from Cesàro matrix. The basic properties of $C_\lambda$-method can be found in [1, 2]. In the present paper, we will consider approximation of $f \in L_p$ by trigonometric polynomials $N_n^\lambda(f;x)$ and $R_n^\lambda(f;x)$ of degree (or order) $n$, where

$$\begin{aligned} N_n^\lambda(f;x) &= \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}s_m(f;x), \\ R_n^\lambda(f;x) &= \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)} p_m s_m(f;x), \\ s_n(f;x) &= \frac{1}{\pi}\int_0^{2\pi} f(x+t)D_n(t)\,dt, \\ D_n(t) &= \frac{(\sin(n+1/2)t)}{2\sin(t/2)}, \\ P_{\lambda(n)} &= p_0+p_1+\cdots+p_{\lambda(n)} \neq 0 \quad (n\geq 0), \end{aligned} \tag{7}$$

and by convention $p_{-1} = 0 = P_{-1}$.

The case $p_n = 1$ for all $n$ ($\geq 0$) of either $N_n^\lambda(f;x)$ or $R_n^\lambda(f;x)$ yields

$$\sigma_n^\lambda(f;x) = \frac{1}{\lambda(n)+1}\sum_{m=0}^{\lambda(n)} s_m(f;x). \tag{8}$$

We also use

$$\Delta a_n = a_n - a_{n+1}, \qquad \Delta_m g(n,m) = g(n,m) - g(n,m+1). \tag{9}$$

Mittal and Rhoades [3, 4] have initiated the study of error estimates $E_n(f)$ through trigonometric-Fourier approximation (tfa) for the situations in which the summability matrix $T$ does not have monotone rows. In this paper, the first author continues the work in the direction for $T$ to be a $N_p$-matrix. Recently, Chandra [5] has proved three theorems on the trigonometric approximation using $N_p$-matrix. Some of them give sharper estimates than the results proved by Quade [6], Mohapatra and Russell [7], and himself earlier [8]. These results of Chandra [5] are improved in different directions by different investigators such as Leindler [9] who dropped the monotonicity on generating sequence $\{p_n\}$ and Mittal et al. [10, 11] who used more general matrix while very recently Deger et al. [12] used more general $C_\lambda$-method in view of Armitage and Maddox [1].

## 2. Known Results

Leindler [9] proved the following.

**Theorem 1** (see [9]). *If $f \in \mathrm{Lip}(\alpha,p)$ and $\{p_n\}$ be positive. If one of the conditions*

(i) $p>1$, $0<\alpha<1$, *and* $\{p_n\} \in AMDS$,

(ii) $p>1$, $0<\alpha<1$, *and* $\{p_n\} \in AMIS$,

$$(n+1)p_n = O(P_n) \text{ holds,} \tag{10}$$

(iii) $p>1$, $\alpha=1$, *and* $\sum_{k=1}^{n-1} k|\Delta p_k| = O(P_n)$,

(iv) $p>1$, $\alpha=1$, $\sum_{k=0}^{n-1}|\Delta p_k| = O(P_n/n)$, *and* (10) *holds,*

(v) $p=1$, $0<\alpha<1$, *and* $\sum_{k=-1}^{n-1}|\Delta p_k| = O(P_n/n)$

*maintains, then*

$$\left\|f - N_n(f)\right\|_p = O\left(n^{-\alpha}\right). \tag{11}$$

**Theorem 2** (see [9]). *Let $f \in \mathrm{Lip}(\alpha,1)$, $0<\alpha<1$. If the positive $\{p_n\}$ satisfies conditions* (10) *and* $\sum_{k=0}^{n-1}|\Delta p_k| = O(P_n/n)$ *hold, then*

$$\left\|f - R_n(f)\right\|_1 = O\left(n^{-\alpha}\right). \tag{12}$$

Deger et al. [12] proved.

**Theorem 3** (see [12]). *Let $f \in \mathrm{Lip}(\alpha,p)$ and let $\{p_n\}$ be positive such that*

$$(\lambda(n)+1)p_{\lambda(n)} = O\left(P_{\lambda(n)}\right). \tag{13}$$

*If either* (i) $p>1$, $0<\alpha\leq 1$, *and* $\{p_n\}$ *is monotonic or* (ii) $p=1$, $0<\alpha<1$, *and* $\{p_n\}$ *is nondecreasing, then*

$$\left\|f - N_n^\lambda(f)\right\|_p = O\left(n^{-\alpha}\right). \tag{14}$$

**Theorem 4** (see [12]). *Let $f \in \mathrm{Lip}(\alpha,1)$, $0<\alpha<1$. If the positive $\{p_n\}$ satisfies condition* (13) *and is nondecreasing, then*

$$\left\|f - R_n^\lambda(f)\right\|_1 = O\left(n^{-\alpha}\right). \tag{15}$$

## 3. Main Results

In this paper we generalize Theorems 3 and 4 of Deger et al. [12], by dropping monotonicity on the elements of the matrix rows which in turn generalize Theorems 1 and 2, respectively, of Leindler [9] to a more general $C_\lambda$-method. We prove the following.

**Theorem 5.** *If $f \in \mathrm{Lip}(\alpha,p)$ and $\{p_n\}$is positive and if one of the following conditions*

(i) $p>1$, $0<\alpha<1$, *and* $\{p_n\} \in AMDS$,

(ii) $p>1$, $0<\alpha<1$, $\{p_n\} \in AMIS$, *and* (13) *holds,*

(iii) $p>1$, $\alpha=1$, *and* $\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)})$,

(iv) $p>1$, $\alpha=1$, $\sum_{k=0}^{\lambda(n)-1}|\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$, *and* (13) *holds,*

(v) $p=1$, $0<\alpha<1$, *and* $\sum_{k=-1}^{\lambda(n)-1}|\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$

*maintains, then*

$$\left\|f - N_n^\lambda(f)\right\|_p = O\left((\lambda(n))^{-\alpha}\right). \tag{16}$$

**Theorem 6.** *Let $f \in \mathrm{Lip}(\alpha, 1)$, $0 < \alpha < 1$. If the positive $\{p_n\}$ satisfies* (13) *and the condition $\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$ holds, then*

$$\left\| f - R_n^{\lambda}(f) \right\|_1 = O\left((\lambda(n))^{-\alpha}\right). \tag{17}$$

*Remarks.* (1) If $\lambda(n) = n$, then our Theorems 5 and 6 reduce to Theorems 1 and 2, respectively.

(2) Deger et al. [12] have used monotone sequences $\{p_n\}$ in Theorems 3 and 4, while our Theorems 5 and 6 claim less than the requirements of their theorems. For example, the condition of the sum in (iii) of Theorem 5 is always satisfied if the sequence $\{p_n\}$ is nonincreasing; that is,

$$\begin{aligned} \sum_{k=1}^{\lambda(n)-1} k\left|\Delta p_k\right| &= \sum_{k=1}^{\lambda(n)-1} k\left(p_k - p_{k+1}\right) \\ &= P_{\lambda(n)-1} - (\lambda(n) - 1)\, p_{\lambda(n)} = O\left(P_{\lambda(n)}\right), \end{aligned} \tag{18}$$

while if sequence $\{p_n\}$ is nondecreasing and condition (13) holds, then the condition in (iv) of Theorem 5 is also satisfied; that is,

$$\begin{aligned} \sum_{k=0}^{\lambda(n)-1} \left|\Delta p_k\right| &= \sum_{k=0}^{\lambda(n)-1} \left(p_{k+1} - p_k\right) = p_{\lambda(n)} - p_0 \le p_{\lambda(n)} \\ &= O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right). \end{aligned} \tag{19}$$

Thus our theorems generalize the two theorems of Deger et al. [12] under weaker assumptions and give sharper estimates because all the estimates of Deger et al. [12] are in terms of $n$, while our estimates are in terms of $\lambda(n)$ and $(\lambda(n))^{-\alpha} \le n^{-\alpha}$ for $0 < \alpha \le 1$.

## 4. Lemmas

We will use the following lemmas in the proof of our theorems.

**Lemma 1** (see [6]). *If $f \in \mathrm{Lip}(\alpha, p)$, for $0 < \alpha \le 1$ and $p > 1$, then*

$$\left\| f - s_n(f) \right\|_p = O\left(n^{-\alpha}\right). \tag{20}$$

**Lemma 2** (see [6]). *If $f \in \mathrm{Lip}(1, p)$, for $p > 1$, then*

$$\left\| \sigma_n(f) - s_n(f) \right\|_p = O\left(n^{-1}\right). \tag{21}$$

**Lemma 3** (see [6]). *If $f \in \mathrm{Lip}(\alpha, 1)$, $0 < \alpha < 1$, then*

$$\left\| f - \sigma_n(f) \right\|_1 = O\left(n^{-\alpha}\right). \tag{22}$$

**Lemma 4.** *Let $\{p_n\} \in$ AMDS or let $\{p_n\} \in$ AMIS and satisfy* (13). *Then, for $0 < \alpha < 1$,*

$$\sum_{m=1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} = O\left((\lambda(n))^{-\alpha} P_{\lambda(n)}\right) \tag{23}$$

*holds.*

*Proof.* Let $r$ denote the integral part of $(\lambda(n)/2)$. Then, if $\{p_n\} \in$ AMDS,

$$\begin{aligned} \sum_{m=1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} &= \sum_{m=1}^{r} m^{-\alpha} p_{\lambda(n)-m} + \sum_{m=r+1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} \\ &\le K p_{\lambda(n)-r} \sum_{m=1}^{r} m^{-\alpha} + (r+1)^{-\alpha} \sum_{m=r+1}^{\lambda(n)} p_{\lambda(n)-m} \\ &\le K p_{\lambda(n)-r} \sum_{m=1}^{\lambda(n)} m^{-\alpha} + (r+1)^{-\alpha} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \\ &= K p_{\lambda(n)-r} (\lambda(n))^{1-\alpha} + O\left((\lambda(n))^{-\alpha} P_{\lambda(n)}\right) \\ &= O\left((\lambda(n))^{-\alpha} P_{\lambda(n)}\right). \end{aligned} \tag{24}$$

If $\{p_n\} \in$ AMIS and (13) holds, then

$$\begin{aligned} \sum_{m=1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} &= \sum_{m=1}^{r} m^{-\alpha} p_{\lambda(n)-m} + \sum_{m=r+1}^{\lambda(n)} m^{-\alpha} p_{\lambda(n)-m} \\ &\le K p_{\lambda(n)} \sum_{m=1}^{r} m^{-\alpha} + (r+1)^{-\alpha} \sum_{m=r+1}^{\lambda(n)} p_{\lambda(n)-m} \\ &= O\left(\frac{P_{\lambda(n)}}{\lambda(n)+1}\right) \sum_{m=1}^{\lambda(n)} m^{-\alpha} + O\left((\lambda(n))^{-\alpha} P_{\lambda(n)}\right) \\ &= O\left((\lambda(n))^{-\alpha} P_{\lambda(n)}\right). \end{aligned} \tag{25}$$

This completes the proof of Lemma 4. □

## 5. Proof of the Main Results

*Proof of Theorem 5.* We prove cases (i) and (ii) together. Since

$$N_n^{\lambda}(f; x) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \left\{ s_m(f; x) - f(x) \right\}, \tag{26}$$

thus in view of Lemmas 1 and 4 and condition (13), we have

$$\begin{aligned} \left\| f - N_n^{\lambda}(f) \right\|_p &\le \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \left\| f - s_m(f) \right\|_p \\ &= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} \left\| f - s_m(f) \right\|_p \\ &\quad + \frac{p_{\lambda(n)}}{P_{\lambda(n)}} \left\| s_0 - f \right\|_p \\ &= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} p_{\lambda(n)-m} O\left(m^{-\alpha}\right) + O\left(\frac{p_{\lambda(n)}}{P_{\lambda(n)}}\right) \\ &= O\left((\lambda(n))^{-\alpha}\right). \end{aligned} \tag{27}$$

Next we consider case (iv).

Let $p > 1$ and $\alpha = 1$. By Abel's transformation, we get

$$N_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} P_{\lambda(n)-m} u_m(f;x), \tag{28}$$

and thus

$$s_n^\lambda(f;x) - N_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} (P_{\lambda(n)} - P_{\lambda(n)-m}) u_m(f;x). \tag{29}$$

Hence again by Abel's transformation, we get

$$\begin{aligned} s_n^\lambda(f;x) - N_n^\lambda(f;x) &= \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} \Delta_m \left(m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m})\right) \\ &\quad \times \sum_{k=1}^{m} k u_k(f;x) + \frac{1}{(\lambda(n)+1)} \\ &\quad \times \sum_{k=1}^{\lambda(n)} k u_k(f;x). \end{aligned} \tag{30}$$

Thus

$$\begin{aligned} \left\| s_n^\lambda(f) - N_n^\lambda(f) \right\|_p &\le \frac{1}{P_{\lambda(n)}} \sum_{m=1}^{\lambda(n)} \left| \Delta_m \left(m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m})\right) \right| \\ &\quad \cdot \left\| \sum_{k=1}^{m} k u_k(f) \right\|_p \\ &\quad + \frac{1}{(\lambda(n)+1)} \left\| \sum_{k=1}^{\lambda(n)} k u_k(f) \right\|_p. \end{aligned} \tag{31}$$

Since

$$s_n(f;x) - \sigma_n(f;x) = \frac{1}{n+1} \sum_{k=1}^{n} k u_k(f;x), \tag{32}$$

thus by Lemma 2

$$\left\| \sum_{k=1}^{n} k u_k \right\|_p = (n+1) \left\| \sigma_n(f) - s_n(f) \right\|_p = O(1). \tag{33}$$

In view of (31) and (33), we obtain

$$\begin{aligned} &\left\| s_n^\lambda(f) - N_n^\lambda(f) \right\|_p \\ &\quad = O\left(\frac{1}{P_{\lambda(n)}}\right) \sum_{m=1}^{\lambda(n)} \left| \Delta_m \left(m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m})\right) \right| \\ &\qquad + O\left((\lambda(n))^{-1}\right). \end{aligned} \tag{34}$$

Now

$$\begin{aligned} \Delta_m \left( \frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) &= \frac{1}{m} \Delta_m (P_{\lambda(n)} - P_{\lambda(n)-m}) \\ &\quad + \frac{P_{\lambda(n)} - P_{\lambda(n)-m-1}}{m(m+1)} \\ &= \frac{P_{\lambda(n)-m-1} - P_{\lambda(n)-m}}{m} \\ &\quad + \frac{P_{\lambda(n)} - P_{\lambda(n)-m-1}}{m(m+1)} \\ &= \frac{P_{\lambda(n)} - P_{\lambda(n)-m-1}}{m(m+1)} - \frac{p_{\lambda(n)-m}}{m} \\ &= \frac{1}{m(m+1)} \left[ P_{\lambda(n)} - P_{\lambda(n)-m-1} \right. \\ &\qquad \left. - (m+1) p_{\lambda(n)-m} \right], \end{aligned} \tag{35}$$

$$\begin{aligned} \Delta_m \left( \frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) &= \frac{1}{m(m+1)} \\ &\quad \times \left[ \sum_{k=\lambda(n)-m}^{\lambda(n)} p_k - (m+1) p_{\lambda(n)-m} \right]. \end{aligned} \tag{36}$$

Next we will verify by the induction that

$$\begin{aligned} &\left| \sum_{k=\lambda(n)-m}^{\lambda(n)} p_k - (m+1) p_{\lambda(n)-m} \right| \\ &\quad \le \sum_{k=1}^{m} k \left| p_{\lambda(n)-k+1} - p_{\lambda(n)-k} \right|. \end{aligned} \tag{37}$$

If $m = 1$,

$$\left| \sum_{k=\lambda(n)-1}^{\lambda(n)} p_k - 2 p_{\lambda(n)-1} \right| = \left| p_{\lambda(n)} - p_{\lambda(n)-1} \right|. \tag{38}$$

Thus (37) holds for $m = 1$. Now let us assume that (37) is true for $m = j$ and we verify $m = j + 1$ $(\le \lambda(n))$. Since

$$\begin{aligned} &\left| \sum_{k=\lambda(n)-(j+1)}^{\lambda(n)} p_k - (j+2) p_{\lambda(n)-(j+1)} \right| \\ &\quad = \left| \sum_{k=\lambda(n)-j}^{\lambda(n)} p_k - (j+1) p_{\lambda(n)-(j+1)} \right| \\ &\quad = \left| \sum_{k=\lambda(n)-j}^{\lambda(n)} p_k - (j+1) p_{\lambda(n)-j} \right. \\ &\qquad \left. + (j+1) p_{\lambda(n)-j} - (j+1) p_{\lambda(n)-(j+1)} \right| \end{aligned}$$

$$\begin{aligned}
&\leq \left| \sum_{k=\lambda(n)-j}^{\lambda(n)} p_k - (j+1)\, p_{\lambda(n)-j} \right| \\
&\quad + \left| (j+1)\, p_{\lambda(n)-j} - (j+1)\, p_{\lambda(n)-(j+1)} \right| \\
&\leq \sum_{k=1}^{j} k \left| p_{\lambda(n)-k+1} - p_{\lambda(n)-k} \right| \\
&\quad + (j+1) \left| p_{\lambda(n)-j} - p_{\lambda(n)-(j+1)} \right| \\
&= \sum_{k=1}^{j+1} k \left| p_{\lambda(n)-k+1} - p_{\lambda(n)-k} \right|,
\end{aligned} \tag{39}$$

thus (37) is proved for $m = j+1$; that is, (37) is true for any $1 \leq m \leq \lambda(n)$. Using (36) and (37) and interchanging the order of summation, we get

$$\begin{aligned}
\sum_{m=1}^{\lambda(n)} \left| \Delta_m \left( \frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) \right| &\leq \sum_{m=1}^{\lambda(n)} \frac{1}{m(m+1)} \\
&\quad \times \sum_{k=1}^{m} k \left| p_{\lambda(n)-k+1} - p_{\lambda(n)-k} \right| \\
&\leq \sum_{k=1}^{\lambda(n)} k \left| p_{\lambda(n)-k+1} - p_{\lambda(n)-k} \right| \\
&\quad \times \sum_{m=k}^{\infty} \frac{1}{m(m+1)} \\
&= \sum_{k=0}^{\lambda(n)-1} \left| \Delta p_k \right|.
\end{aligned} \tag{40}$$

Now combining this with the assumption $\sum_{k=0}^{\lambda(n)-1} |\Delta p_k| = O(P_{\lambda(n)}/\lambda(n))$, we get from (34)

$$\left\| s_n^{\lambda}(f) - N_n^{\lambda}(f) \right\|_p = O\left( (\lambda(n))^{-1} \right). \tag{41}$$

This and Lemma 1 with $\alpha = 1$ yield

$$\left\| f - N_n^{\lambda}(f) \right\|_p = O\left( (\lambda(n))^{-1} \right). \tag{42}$$

In the proof of case (iii), we first verify that the condition $\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)})$ implies that

$$\sum_{m=1}^{\lambda(n)} \Delta_m \left( \frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) = O\left( \frac{P_{\lambda(n)}}{\lambda(n)} \right). \tag{43}$$

In view of (36) and (37)

$$\begin{aligned}
&\sum_{m=1}^{\lambda(n)} \Delta_m \left( \frac{P_{\lambda(n)} - P_{\lambda(n)-m}}{m} \right) \\
&\quad \leq \sum_{m=1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\
&\quad = \sum_{1}^{r} + \sum_{r+1}^{\lambda(n)} := B_1 + B_2, \text{ say.}
\end{aligned} \tag{44}$$

Denoting again by $r$ the integral part of $(\lambda(n)/2)$, then, by Abel's transformation, we have

$$\begin{aligned}
B_1 &= \sum_{m=1}^{r} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| \Delta_k p_{\lambda(n)-k} \right| \\
&\leq \sum_{k=1}^{r} \left| \Delta_k p_{\lambda(n)-k} \right| \leq \sum_{i=r-2}^{\lambda(n)-1} \left| \Delta p_i \right| \\
&= O\left( \frac{P_{\lambda(n)}}{\lambda(n)} \right),
\end{aligned} \tag{45}$$

at the last step; we have used the condition $\sum_{k=1}^{\lambda(n)-1} k|\Delta p_k| = O(P_{\lambda(n)})$. Consider the following:

$$\begin{aligned}
B_2 &= \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \sum_{k=1}^{m} k \left| \Delta p_{\lambda(n)-k} \right| \\
&= \sum_{m=r+1}^{\lambda(n)} \frac{1}{m(m+1)} \\
&\quad \times \left[ \sum_{k=1}^{r} k \left| \Delta p_{\lambda(n)-k} \right| + \sum_{k=r}^{m} k \left| \Delta p_{\lambda(n)-k} \right| \right] \\
&:= B_{21} + B_{22}.
\end{aligned} \tag{46}$$

Furthermore, using again our assumption, we get

$$\begin{aligned}
B_{21} &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{m+1} \sum_{i=r-2}^{\lambda(n)-1} \left| \Delta p_i \right| = O\left( \frac{P_{\lambda(n)}}{\lambda(n)} \right), \\
B_{22} &\leq \sum_{m=r}^{\lambda(n)} \frac{1}{m+1} \sum_{k=r}^{m} \left| \Delta p_{\lambda(n)-k} \right| \\
&= O\left( \frac{1}{\lambda(n)} \right) \left[ \left| \Delta p_0 \right| + 2 \left| \Delta p_1 \right| + \cdots + (r+1) \left| \Delta p_{r+1} \right| \right] \\
&= O\left( \frac{P_{\lambda(n)}}{\lambda(n)} \right).
\end{aligned} \tag{47}$$

Summing up our partial results, we verified (43). Thus (34) and Lemma 1 again yield

$$\left\| f - N_n^{\lambda}(f) \right\|_p = O\left( (\lambda(n))^{-1} \right). \tag{48}$$

Now, we prove case (v), by using (26), $p_{-1} = 0$, and Abel's transformation

$$\begin{aligned} N_n^\lambda(f;x) - f(x) &= \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} \{s_m(f;x) - f(x)\} \\ &= \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (\Delta_m p_{\lambda(n)-m}) \sum_{k=0}^{m} \{s_k(f;x) - f(x)\} \\ &= \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (m+1)(\Delta_m p_{\lambda(n)-m}) \\ &\qquad \times \{\sigma_m(f;x) - f(x)\}. \end{aligned} \tag{49}$$

Hence in view of Lemma 3

$$\begin{aligned} \left\| N_n^\lambda(f) - f \right\|_1 &\le \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} (m+1) \left|\Delta_m p_{\lambda(n)-m}\right| \left\| f - \sigma_m(f) \right\|_1 \\ &= O\left(\frac{1}{P_{\lambda(n)}}\right) \sum_{m=0}^{\lambda(n)} (m+1)^{1-\alpha} \left|\Delta_m p_{\lambda(n)-m}\right| \\ &= O\left(\frac{\lambda(n)^{1-\alpha}}{P_{\lambda(n)}}\right) \sum_{m=-1}^{\lambda(n)-1} \left|\Delta p_m\right| \\ &= O\left(\frac{\lambda(n)^{1-\alpha}}{P_{\lambda(n)}}\right) \cdot O\left(\frac{P_{\lambda(n)}}{\lambda(n)}\right) \\ &= O\left((\lambda(n))^{-\alpha}\right). \end{aligned} \tag{50}$$

Herewith case (v) is also verified and thus the proof of Theorem 5 is complete. □

*Proof of Theorem 6.* Since $R_n^\lambda(f;x) = (1/P_{\lambda(n)}) \sum_{m=0}^{\lambda(n)} p_m s_m(f;x)$, so, in view of the assumptions of Theorem 6, we get

$$\begin{aligned} \left\| f - R_n^\lambda(f) \right\|_1 &= \frac{1}{P_{\lambda(n)}} \left\| \sum_{m=0}^{\lambda(n)} p_m \{f - s_m(f)\} \right\|_1 \\ &\le \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)-1} (m+1) \left|\Delta p_m\right| \cdot \left\| f - \sigma_m(f) \right\|_1 \\ &\quad + (\lambda(n)+1) p_{\lambda(n)} P_{\lambda(n)}^{-1} \left\| f - \sigma_n^\lambda(f) \right\|_1 \\ &= O\left((\lambda(n))^{1-\alpha} P_{\lambda(n)}^{-1}\right) \sum_{m=0}^{\lambda(n)-1} \left|\Delta p_m\right| \\ &\quad + O\left((\lambda(n))^{-\alpha}\right) = O\left((\lambda(n))^{-\alpha}\right). \end{aligned} \tag{51}$$

This proves Theorem 6. □

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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