Some Properties of Certain Class of Analytic Functions

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We obtain some properties related to the coefficient bounds for certain subclass of analytic functions. We also work on the differential subordination for a certain class of functions.

1. Introductions

Let \(H\) denote the class of functions

\[
f(z) = z + \sum_{\kappa=2}^{\infty} a_{\kappa} z^{\kappa},
\]

which is analytic in the unit disc \(U = \{ z \in \mathbb{C} : |z| < 1 \}\). Let

\[
H[a,n] = \{ p \in H : p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, n \in \mathbb{N}, a \neq 0 \}.
\]

Now let \(H(\beta)\) be the class of functions defined by

\[
F(z) = z^\beta + \sum_{\kappa=2}^{\infty} \lambda_\kappa a_{\kappa} z^{\kappa+1}, \beta \in \mathbb{N},
\]

\[
\lambda_\kappa = \beta^\kappa, \quad z \neq \frac{1}{\beta}, \quad \beta |z| < 1.
\]

The Hadamard product \(f \ast g\) of two functions \(f\) and \(g\) is defined by

\[
(f \ast g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k,
\]

where \(f(z) = \sum_{k=0}^{\infty} a_k z^k\) and \(g(z) = \sum_{k=0}^{\infty} b_k z^k\) are analytic in \(U\).

Let \(\psi(z) = z^\beta + \sum_{\kappa=2}^{\infty} a_{\kappa} z^{\kappa+1}, \beta \in \mathbb{N},\) and then \(\psi(z)\) is analytic in the open unit disc \(U\). The function \(F(z)\) defined in (3) is equivalent to

\[
\psi(z) \ast z^\beta 1 - \beta z, \quad z \neq \frac{1}{\beta}, \beta \in \mathbb{N}, \beta |z| < 1,
\]

where \(\ast\) is the Hadamard product and \(F(z)\) is analytic in the open unit disc \(U\).

We introduce a class of functions

\[
Q^{(\delta)}(\alpha, \tau, \gamma; \beta)
\]

\[
= \left\{ F(z) \in H(\beta) : \Re
\frac{\alpha [F(z)]^j \beta^j}{[z^\beta]^j} + \tau [F(z)]^{(j+1)} \right\} > \gamma, z \in U,
\]

where

\[
z \neq 0, \beta > j, \quad j \in \mathbb{N} \cup \{0\} \quad \text{also} \quad \alpha + \tau \neq \gamma.
\]

Authors like Saitoh \([1]\) and Owa \([2, 3]\) had previously studied the properties of the class of functions \(Q^{(0)}(\alpha, \tau, \gamma; 1)\). They obtained many interesting results and Wang et al. \([4]\)
studied the extreme points, coefficient bounds, and radius of univalence of the same class of functions. They obtained the following theorem among other results.

**Theorem 1** (see [4]). Let $f(z) \in H$. A function $f(z) \in Q^{(0)}(\alpha, \tau; 1)$ if and only if $f(z)$ can be expressed as

\[
f(z) = \frac{1}{\alpha + \tau} \int_{[\chi]} \left[ (2\gamma - \alpha - \tau)z + 2(\alpha + \tau - \gamma) \right] \times \sum_{k=0}^{\infty} \left( \frac{(\alpha + \tau)x^k}{(k + 1)\tau + \alpha} \right) d\mu(x),
\]

where $\mu(x)$ is the probability measure defined on $\chi = \{x : |x| = 1\}$.

For fixed $\alpha, \tau, \gamma$, the class $Q^{(0)}(\alpha, \tau; 1)$ and the probability measure $\{\mu\}$ are one-to-one by expression (8).

Recently, Hayami et al. [5] studied the coefficient estimates of the class of functions $f(z) \in H$ in the open unit disc $U$. They derived results based on properties of the class of functions $f(z) \in H[a, n], a \neq 0$. Xu et al. [6] used the principle of differential subordination and the Dziok-Srivastava convolution operator to investigate some properties of the class of functions

\[
f(z) \in H[a, n]
\]

offunctions $f(\zeta) \in H[a, n]$, to investigate the analytic and univalent properties of the following integral operator:

\[
\Theta_{\alpha, \beta}(z) = \left( \frac{\beta}{\alpha} \right) \int_{0}^{\infty} \left[ f(\tau) \right]^{\alpha} g(\tau) d\tau
\]

where $\alpha \in \mathbb{C}, \beta \in \mathbb{C} \setminus \{0\}, f \in H, g \in H[1, n])$.

Motivated by the work in [1–7] and [8], we used the properties of the class of functions $f(z) \in H[a, n], a \neq 0$, to investigate the analytic and univalent properties of the following class of functions $Q^{(0)}(\alpha, \tau; 1)$.

We state the following known results required to prove our work.

**Definition 2.** If $g$ and $h$ are analytic in $U$, then $g$ is said to be subordinate to $h$, written as $g \prec h$ or $g(z) \prec h(z)$. If $g$ is univalent in $U$, then $g(0) = h(0)$ and $g(U) \subset h(U)$.

**Theorem 3** (see [8]). Consider $\Theta \in H[1, n]$ if and only if there is probability measure $\{\mu\}$ on $\chi$ such that

\[
p(z) = \int_{[\chi]} \left[ 1 + \frac{xz}{1 - xz} \right] d\mu(x), \quad (|z| < 1)
\]

and $\chi = \{x : |x| = 1\}$. The correspondence between $H[1, n]$ and the set of probability measures $\{\mu\}$ on $\chi$ given by Hallenbeck [9] is one-to-one.

**Theorem 4** (see [10, 11]). Let $h(z)$ be convex in $U$, $h(0) = a$, $c \neq 0$, and $\Re|c| \geq 0$. If $g(z) \in H[a, n]$ and

\[
g(z) + \frac{z^g(z)}{c} < h(z),
\]

then

\[
g(z) < q(z) = \frac{cz^{-c/n}}{c} \int_{0}^{h} t^{(c/n)/2} h(t) dt < h(z).
\]

The function $q$ is convex and the best $(a, n)$-dominant.

**Lemma 5** (see [10]). Let $h$ be starlike in $U$, with $h(0) = 0$ and $a \neq 0$. If $p \in H[a, n]$ satisfies

\[
z^p(z) < h(z),
\]

then

\[
p(z) < q(z) = a \exp \left[ \frac{1}{n} \int_{0}^{h} \xi^{-1/2} d\xi \right]
\]

and $q$ is the best $(a, n)$-dominant.

**Lemma 6** (see [12]). Let $p \in [1, n]$, with $\Re|p(z)| > 0$ in $U$. Then, for $|z| = r < 1$,

\[
(i) \quad (1 - r)/(1 + r) \leq \Re|p(z)| \leq |p(z)| \leq (1 + r)/(1 - r),
\]

\[
(ii) \quad |p(z)| \leq 2\Re|p(z)|/(1 - r^2).
\]

**Remark 7.** The combination (i) and (ii) of Lemma 6 gives

\[
\left| \frac{zp'(z)}{p(z)} \right| < \frac{2r \Re|p(z)|}{(1 - r^2)}.
\]

**Remark 8.** For convenience, we limit our result to the principal branch and otherwise stated the constrains on $\beta, j, \alpha, \tau, \gamma$, and $\lambda_x$ which remain the same throughout this paper.

## 2. Coefficient Bounds of the Class of Functions $Q^{(1)}(\alpha, \tau; 1)$

We begin with the following result.

**Theorem 9.** Let $F(z)$ be as defined in (3). A function $F(z) \in Q^{(1)}(\alpha, \tau; 1)$, if and only if $zf[F(z)]^{(1)}$ can be expressed as

\[
z^f[F(z)]^{(1)} = \frac{\beta + j}{(\alpha + \tau)} \int_{[\chi]} \left[ (2\gamma - \alpha - \tau)z^\beta + 2(\alpha + \tau - \gamma) \right.
\]

\[
\times \sum_{k=0}^{\infty} \left( \frac{(\beta - j)(\alpha + \tau)x^k}{(\beta - j)(\alpha + \tau) + \kappa \tau} \right) d\mu(x),
\]

where $^{n}p_r = n!(n - r)!$ and $\{\mu\}$ is the probability measure defined on $\chi = \{x : |x| = 1\}$. 


Proof. If \( F(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta) \), then

\[ p(z) = \frac{\left( \tau[F(z)]^{(j+1)}/[z^\beta]^{(j+1)} \right) + \left( \alpha[F(z)]^{(j)}/[z^\beta]^{(j)} \right)}{\alpha + \tau - \gamma} \]

\[ \in H[1, n] \, . \]

By Theorem 3,

\[ \frac{\left( \tau[F(z)]^{(j+1)}/[z^\beta]^{(j+1)} \right) + \left( \alpha[F(z)]^{(j)}/[z^\beta]^{(j)} \right)}{\alpha + \tau - \gamma} = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x) \, , \]

and (19) can be written as

\[ \frac{[F(z)]^{(j+1)}/[z^\beta]^{(j+1)}}{[\alpha + \tau - \gamma]} = \int_{|x|=1} \frac{(\alpha + \tau) + (\alpha + \tau - 2\gamma) xz}{1 - xz} d\mu(x) \, , \]

which yields

\[ z^{[(\alpha + \tau) - (\beta - j) + \tau]} \int_0^z \left[ \frac{[F(\xi)]^{(j+1)}}{[\xi^\beta]^{(j+1)}} + \frac{\alpha[F(\xi)]^{(j)}}{[\xi^\beta]^{(j)}} \right] \xi^{[(\alpha + \tau) - (\beta - j) + \tau]} d\xi \]

\[ = \frac{1}{\tau} \int_{|x|=1} \frac{(\alpha + \tau) + (\alpha + \tau - 2\gamma) xz}{1 - xz} d\mu(x) \, , \]

and so the expression (17).

If \( z^{[\partial^j]F(z)} \) can be expressed as (17), reverse calculation shows that \( F(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta) \).

Corollary 10. Let \( F \) be defined as in (3). A function \( F(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta) \) if and only if \( F(z) \) can be expressed as

\[ F(z) = \frac{1}{\alpha + \tau} \int_{|x|=1} \left( 2\gamma - \alpha - \tau \right) z^\beta + 2(\alpha + \tau - \gamma) \]

\[ \times \sum_{k=2}^{\infty} \frac{[x^k z^\beta]}{[\alpha^k + \beta + \tau \beta + \tau + \beta]} d\mu(x) \, , \]

where \([\mu]\) is the probability measure defined on \( \chi = \{|x| = 1\} \).

Proof. It is as is Theorem 9.

Corollary 11. Let \( F(z) \) be as defined in (3). If \( F(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta) \), then, for \( \kappa \geq 2 \) and \( ^nP_r = n!/(n - r)! \), we have

\[ |a_\kappa| \leq H(\alpha, \tau, \gamma; \beta) \, , \]

where

\[ H(\alpha, \tau, \gamma; \beta) = \frac{2(\alpha + \tau - \gamma)(\beta - j)}{\lambda x} \cdot \frac{[\beta P_j^{n-1}]}{[(\alpha + \tau)(\beta - j) + \tau(\kappa - 1)]} \]

(24)

Proof. Let \( F(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta) \) from (17) and

\[ z^\beta \sum_{k=2}^{\infty} \frac{2(\alpha + \tau - \gamma)(\beta - j)x^{k-1}z^{\beta k-1}}{(\alpha + \tau)(\beta - j) + \tau(\kappa - 1)} \]

(|x| = 1).

Comparing the coefficient yields the result.

Theorem 12. Let \( G(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta) \) and \( \Psi(z) = G(z)/(\alpha + \tau - \gamma) \). Then for \(|z| < 1\) we have

\[ \left| \frac{z^\Psi(z)}{\Psi(z)} \right| \leq \frac{\sum_{k=2}^{\infty} 2(\beta - j)(\kappa - 1)^{r-1}}{1 + \sum_{k=2}^{\infty} (\beta - j)^{r-1}} \leq \frac{2\Re \{\Psi(z)\}}{(1 - r)^2} \, . \]

Proof. Since \( G(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta) \), then

\[ G(z) \equiv \frac{[\tau[F(z)]^{(j+1)}]}{[z^\beta]^{(j+1)}} + \frac{\alpha[F(z)]^{(j)}}{[z^\beta]^{(j)}} - \gamma \, , \]

and then

\[ \frac{z^\Psi(z)}{\Psi(z)} = \sum_{k=2}^{\infty} \frac{2(\beta - j)(\kappa - 1)^{r-1}}{1 + \sum_{k=2}^{\infty} (\beta - j)^{r-1}} \leq \frac{2\Re \{\Psi(z)\}}{(1 - r)^2} \, . \]

From (31) and (28), we got

\[ \left| \frac{z^\Psi(z)}{\Psi(z)} \right| \leq \frac{2\Re \{\Psi(z)\}}{(1 - r)^2} \, . \]

The application of Remark 7 to (27) gives

\[ \left| \frac{z^\Psi(z)}{\Psi(z)} \right| \leq \frac{2\Re \{\Psi(z)\}}{(1 - r)^2} \, . \]

Since

\[ \sum_{k=2}^{\infty} 2(\beta - j)(\kappa - 1)^{r-1} < 2\Re \{\Psi(z)\} \, , \]

\[ 1 + \sum_{k=2}^{\infty} (\beta - j)^{r-1} > (1 - r)^2 \, , \]

then Theorem 12 is proved.
3. Application of Differential Subordination to the Function $Q^{(j)}(\alpha, \tau, \gamma; \beta)$

Here we calculate some subordinate properties of the class $Q^{(j)}(\alpha, \tau, \gamma; \beta)$.

**Theorem 13.** Let $G(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta)$ and let $h(z)$ be starlike in $U$ with $h(0) = 0$ and $\alpha + \tau \not= \gamma$. If

$$G(z) < h(z),$$

then

$$G(z) < q(z) = (\alpha + \tau - \gamma) \left\{ 1 + \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right\}^{1/n}, \quad \alpha + \tau \not= \gamma,$$

and $q$ is the best $((\alpha + \tau - \gamma), n)$-dominant.

**Proof.** Let $G(z) \in Q(\alpha, \tau, \gamma; \beta)$; then

$$G(z) = (\alpha + \tau - \gamma) + \sum_{\nu=2}^{\infty} \frac{\lambda_{\nu}}{\beta P_{\nu}^{(j)}} a_\nu z^{\nu-1},$$

where

$$\lambda_{\nu} = \frac{\nu!}{(\beta+\nu-1) P_{\nu+1}^{(j)}} [\beta(P_{\nu+1}^{(j)})^{(j)} - \beta(P_{\nu}^{(j)})^{(j)}].$$

Let $g(z) = \frac{F(z)}{z^\beta}$, then $g(z) = 1 + \sum_{\nu=2}^{\infty} \frac{\lambda_{\nu}}{\beta P_{\nu}} a_\nu z^{\nu-1}$, and from (39) we have

$$g(z) < (\alpha + \tau)(1+Az)^{-\beta/(1+Bz)} = h(z),$$

and $h(z)$ is convex and univalent in $U$. So, by Lemma 6,

$$g(z) < (\alpha + \tau)(1+Az)^{-\beta/(1+Bz)} \leq (1+Az)^{-\beta/(1+Bz)}.$$

This completes the proof of Theorem 15.

**Corollary 16.** Let

$$Q^{(j)}(\alpha, \tau, 0; \beta) = \frac{a_\nu F(z)}{z^\beta} + \frac{\tau [F(z)]^{(j)}}{[z^\beta]^{(j)}},$$

then

$$\frac{F(z)}{z^\beta} < \frac{\alpha \beta + \tau (\beta - \kappa)}{\kappa} z^{-\beta(\alpha + \tau)/\kappa} \leq \frac{1 + Az}{1+Bz}.$$
References


