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Research Article

Some Properties of Certain Class of Analytic Functions

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We obtain some properties related to the coefficient bounds for certain subclass of analytic functions. We also work on the differential subordination for a certain class of functions.

1. Introductions

Let H denote the class of functions

$$f(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa, \tag{1}$$

which is analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let

$$H[a,n] = \left\{p \in H : p(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, n \in \mathbb{N}, a \neq 0\right\}. \tag{2}$$

Now let $H(\beta)$ be the class of functions defined by

$$F(z) = z^\beta + \sum_{\kappa=2}^{\infty} \lambda_\kappa a_\kappa z^{\beta+\kappa-1}, \quad \beta \in \mathbb{N},$$
$$\lambda_\kappa = \beta^{\kappa-1}, \quad z \neq \frac{1}{\beta}, \quad \beta|z| < 1. \tag{3}$$

The Hadamard product $f * g$ of two functions f and g is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \tag{4}$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ are analytic in U.

Let $\psi(z) = z^\beta + \sum_{\kappa=2}^{\infty} a_\kappa z^{\beta+\kappa-1}$, $\beta \in \mathbb{N}$, and then $\psi(z)$ is analytic in the open unit disc U. The function $F(z)$ defined in (3) is equivalent to

$$\psi(z) * \frac{z^\beta}{1-\beta z}, \quad z \neq \frac{1}{\beta}, \ \beta \in \mathbb{N}, \ \beta|z| < 1, \tag{5}$$

where $*$ is the Hadamard product and $F(z)$ is analytic in the open unit disc U.

We introduce a class of functions

$$Q^{(j)}(\alpha, \tau, \gamma; \beta) = \left\{F(z) \in H(\beta) : \mathfrak{Re} \times \left\{\frac{\alpha[F(z)]^{(j)}}{[z^\beta]^{(j)}} + \frac{\tau[F(z)]^{(j+1)}}{[z^\beta]^{(j+1)}}\right\} > \gamma, z \in U\right\}, \tag{6}$$

where

$$z \neq 0 \ \ \beta > j, \quad j \in \mathbb{N} \cup \{0\} \quad \text{also } \alpha + \tau \neq \gamma. \tag{7}$$

Authors like Saitoh [1] and Owa [2, 3] had previously studied the properties of the class of functions $Q^{(0)}(\alpha, \tau, \gamma; 1)$. They obtained many interesting results and Wang et al. [4]

studied the extreme points, coefficient bounds, and radius of univalency of the same class of functions. They obtained the following theorem among other results.

Theorem 1 (see [4]). *Let $f(z) \in H$. A function $f(z) \in Q^{(0)}(\alpha,\tau,\gamma;1)$ if and only if $f(z)$ can be expressed as*

$$f(z) = \frac{1}{\alpha+\tau}\int_{|x|}\left[(2\gamma-\alpha-\tau)z + 2(\alpha+\tau-\gamma) \times \sum_{\kappa=0}^{\infty}\frac{(\alpha+\tau)x^{\kappa}z^{\kappa+1}}{(\kappa+1)\tau+\alpha}\right]d\mu(x), \tag{8}$$

where $\mu(x)$ is the probability measure defined on

$$\chi = \{x : |x| = 1\}. \tag{9}$$

For fixed α, τ, and γ, the class $Q^{(0)}(\alpha,\tau,\gamma;1)$ and the probability measure $\{\mu\}$ defined on χ are one-to-one by expression (8).

Recently, Hayami et al. [5] studied the coefficient estimates of the class of function $f(z) \in H$ in the open unit disc U. They derived results based on properties of the class of functions $f(z) \in H[a,n]$, $a \neq 0$. Xu et al. [6] used the principle of differential subordination and the Dziok-Srivastava convolution operator to investigate some analytic properties of certain subclass of analytic functions. We also note that Stanciu et al. [7] used the properties of the class of functions $f(z) \in H[a,n]$, $a \neq 0$, to investigate the analytic and univalent properties of the following integral operator:

$$\Theta_{\alpha,\beta}(z) = \left(\beta\int_0^z t^{\beta-\alpha-1}[f(t)]^{\alpha}g(t)\,dt\right)^{1/\beta}, \tag{10}$$

where ($\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}\setminus\{0\}$, $f \in H$, $g \in H[1,n]$).

Motivated by the work in [1–7], we used the properties of the class of function $f(z) \in H[a,n]$, $a \neq 0$, to investigate the coefficient estimates of the class of functions $F(z) \in H(\beta)$ in the open unit disc U. We also use the principle of differential subordination to investigate some properties of the class of functions $Q^{(\jmath)}(\alpha,\tau,\gamma;\beta)$.

We state the following known results required to prove our work.

Definition 2. If g and h are analytic in U, then g is said to be subordinate to h, written as $g \prec h$ or $g(z) \prec h(z)$. If g is univalent in U, then $g(0) = h(0)$ and $g(U) \subset h(U)$.

Theorem 3 (see [8]). *Consider $p \in H[1,n]$ if and only if there is probability measure $\{\mu\}$ on χ such that*

$$p(z) = \int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x), \quad (|z| < 1) \tag{11}$$

and $\chi = \{x : |x| = 1\}$. The correspondence between $H[1,n]$ and the set of probability measures $\{\mu\}$ on χ given by Hallenbeck [9] is one-to-one.

Theorem 4 (see [10, 11]). *Let $h(z)$ be convex in U, $h(0) = a$, $c \neq 0$, and $\Re e\{c\} \geq 0$. If $g(z) \in H[a,n]$ and*

$$g(z) + \frac{zg'(z)}{c} \prec h(z), \tag{12}$$

then

$$g(z) \prec q(z) = \frac{cz^{-c/\kappa}}{\kappa}\int_0^z t^{(c-\kappa)/\kappa}h(t)\,dt \prec h(z). \tag{13}$$

The function q is convex and the best (a,n)-dominant.

Lemma 5 (see [10]). *Let h be starlike in U, with $h(0) = 0$ and $a \neq 0$. If $p \in H[a,n]$ satisfies*

$$\frac{zp'(z)}{p(z)} \prec h(z), \tag{14}$$

then

$$p(z) \prec q(z) = a\exp\left[\frac{1}{n}\int_0^z h(\xi)\xi^{-1}d\xi\right] \tag{15}$$

and q is the best (a,n)-dominant.

Lemma 6 (see [12]). *Let $p \in [1,n]$, with $\Re e\{p(z)\} > 0$ in U. Then, for $|z| = r < 1$,*

(i) $(1-r)/(1+r) \leq \Re e\{p(z)\} \leq |p(z)| \leq (1+r)/(1-r)$,

(ii) $|p'(z)| \leq 2\Re e\{p(z)\}/(1-r^2)$.

Remark 7. The combination (i) and (ii) of Lemma 6 gives

$$\left|\frac{zp'(z)}{p(z)}\right| \leq \frac{2r\Re e\{p(z)\}}{(1-r)^2}. \tag{16}$$

Remark 8. For convenience, we limit our result to the principal branch and otherwise stated the constrains on β, $\jmath$, α, τ, γ, and λ_{κ} which remain the same throughout this paper.

2. Coefficient Bounds of the Class of Functions $Q^{(\jmath)}(\alpha,\tau,\gamma;\beta)$

We begin with the following result.

Theorem 9. *Let $F(z)$ be as defined in* (3). *A function $F(z) \in Q^{(\jmath)}(\alpha,\tau,\gamma;\beta)$, if and only if $z^{\jmath}[F(z)]^{(\jmath)}$ can be expressed as*

$$z^{\jmath}[F(z)]^{(\jmath)} = \frac{{}^{\beta}P_{\jmath}}{(\alpha+\tau)}\int_{|x|=1}\left[(2\gamma-\alpha-\tau)z^{\beta} + 2(\alpha+\tau-\gamma) \times \sum_{\kappa=0}^{\infty}\frac{(\beta-\jmath)(\alpha+\tau)x^{\kappa}z^{\beta+\kappa}}{(\alpha+\tau)(\beta-\jmath)+\kappa\tau}\right]d\mu(x), \tag{17}$$

where ${}^{n}P_r = n!/(n-r)!$ and $\{\mu\}$ is the probability measure defined on $\chi = \{x : |x| = 1\}$.

Proof. If $F(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta)$, then

$$p(z) = \frac{\left(\tau[F(z)]^{(j+1)}/\left[z^{\beta}\right]^{(j+1)}\right) + \left(\alpha[F(z)]^{(j)}/\left[z^{\beta}\right]^{(j)}\right) - \gamma}{\alpha + \tau - \gamma} \in H[1, n]. \tag{18}$$

By Theorem 3,

$$\frac{\left(\tau[F(z)]^{(j+1)}/\left[z^{\beta}\right]^{(j+1)}\right) + \left(\alpha[F(z)]^{(j)}/\left[z^{\beta}\right]^{(j)}\right) - \gamma}{\alpha + \tau - \gamma} = \int_{|x|=1} \frac{1 + xz}{1 - xz} d\mu(x), \tag{19}$$

and (19) can be written as

$$\frac{[F(z)]^{(j+1)}}{\left[z^{\beta}\right]^{(j+1)}} + \frac{\alpha[F(z)]^{(j)}}{\tau\left[z^{\beta}\right]^{(j)}} = \frac{1}{\tau}\int_{|x|=1} \frac{(\alpha + \tau) + (\alpha + \tau - 2\gamma)xz}{1 - xz} d\mu(x) \tag{20}$$

which yields

$$\begin{aligned} & z^{((\alpha+\tau)(j-\beta)+\tau)/\tau} \int_0^z \left[\frac{[F(\xi)]^{(j+1)}}{\left[\xi^{\beta}\right]^{(j+1)}} + \frac{\alpha[F(\xi)]^{(j)}}{\tau\left[\xi^{\beta}\right]^{(j)}}\right] \\ & \quad \times \xi^{((\alpha+\tau)(\beta-j)-\tau)/\tau} d\xi \\ & = \frac{1}{\tau}\int_{|x|=1} z^{((\alpha+\tau)(j-\beta)+\tau)/\tau} \\ & \quad \times \int_0^z \frac{(\alpha + \tau) + (\alpha + \tau - 2\gamma)x\xi}{1 - x\xi} \\ & \quad \times \xi^{((\alpha+\tau)(\beta-j)-\tau)/\tau} d\xi\, d\mu(x), \end{aligned} \tag{21}$$

and so the expression (17). □

If $z^{j}[F(z)]^{(j)}$ can be expressed as (17), reverse calculation shows that $F(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta)$.

Corollary 10. *Let F be defined as in* (3). *A function $F(z) \in Q^{(0)}(\alpha, \tau, \gamma; \beta)$ if and only if $F(z)$ can be expressed as*

$$F(z) = \frac{1}{\alpha + \tau}\int_{|x|=1} \left[(2\gamma - \alpha - \tau)z^{\beta} + 2(\alpha + \tau - \gamma) \times \sum_{\kappa=0}^{\infty} \frac{(\alpha + \tau)x^{\kappa}z^{\beta+\kappa}}{\alpha\beta + \tau(\kappa + \beta)}\right] d\mu(x), \tag{22}$$

where $\{\mu\}$ is the probability measure defined on $\chi = \{x : |x| = 1\}$.

Proof. It is as in Theorem 9. □

Corollary 11. *Let $F(z)$ be as defined in* (3). *If $F(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta)$, then, for $\kappa \geq 2$ and ${}^{n}P_r = n!/(n-r)!$, we have*

$$|a_{\kappa}| \leq H(\alpha, \tau, \gamma; \beta), \tag{23}$$

where

$$H(\alpha, \tau, \gamma; \beta) = \frac{2(\alpha + \tau - \gamma)(\beta - j)\left[{}^{\beta}P_j\right]}{\lambda_{\kappa}\left[{}^{[\beta+\kappa-1]}P_j\right]\left[(\alpha + \tau)(\beta - j) + \tau(\kappa - 1)\right]}. \tag{24}$$

Proof. Let $F(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta)$ from (17) and

$$\frac{z^{j}\left[F_x(z)\right]^{(j)}}{{}^{\beta}P_j} = z^{\beta} + \sum_{\kappa=2}^{\infty} \frac{2(\alpha + \tau - \gamma)(\beta - j)x^{\kappa-1}z^{\beta+\kappa-1}}{(\alpha + \tau)(\beta - j) + \tau(\kappa - 1)} \quad (|x| = 1). \tag{25}$$

Comparing the coefficient yields the result. □

Theorem 12. *Let $G(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta)$ and $\Psi(z) = G(z)/(\alpha + \tau - \gamma)$. Then for $|z| = r < 1$ we have*

$$\left|\frac{z\Psi'(z)}{\Psi(z)}\right| \leq \frac{\sum_{\kappa=2}^{\infty} 2(\beta - j)\left[[\kappa - 1]r^{\kappa-1}\right]}{1 + \sum_{\kappa=2}^{\infty} 2(\beta - j)r^{\kappa-1}} \leq \frac{2r\Re e\{\Psi(z)\}}{(1 - r)^2}. \tag{26}$$

Proof. Since $G(z) \in Q^{(j)}(\alpha, \tau, \gamma; \beta)$, then

$$G(z) \equiv \frac{\tau[F(z)]^{(j+1)}}{\left[z^{\beta}\right]^{(j+1)}} + \frac{\alpha[F(z)]^{(j)}}{\left[z^{\beta}\right]^{(j)}} - \gamma, \tag{27}$$

and then

$$\frac{z\Psi'(z)}{\Psi(z)} = \frac{\sum_{\kappa=2}^{\infty} \vartheta(\alpha, \tau, \gamma; \beta)[\kappa - 1]a_{\kappa}z^{\kappa-1}}{1 + \sum_{\kappa=2}^{\infty} \vartheta(\alpha, \tau, \gamma; \beta)a_{\kappa}z^{\kappa-1}}, \tag{28}$$

where

$$\vartheta(\alpha, \tau, \gamma; \beta) = \frac{\lambda_{\kappa}\left[{}^{[\beta+\kappa-1]}P_j\right]\left[(\alpha + \tau)(\beta - j) + \tau(\kappa - 1)\right]}{(\alpha + \tau - \gamma)\left[{}^{\beta}P_{(j)}\right]}. \tag{29}$$

From (23) and (28), we got

$$\left|\frac{z\Psi'(z)}{\Psi(z)}\right| \leq \frac{\sum_{\kappa=2}^{\infty} 2(\beta - j)[\kappa - 1]r^{\kappa-1}}{1 + \sum_{\kappa=2}^{\infty} 2(\beta - j)r^{\kappa-1}} \quad (\kappa \geq 2). \tag{30}$$

The application of Remark 7 to (27) gives

$$\left|\frac{z\Psi'(z)}{\Psi(z)}\right| \leq \frac{2r\Re e\{\Psi(z)\}}{(1 - r)^2}. \tag{31}$$

Since

$$\sum_{\kappa=2}^{\infty} 2(\beta - j)[\kappa - 1]r^{\kappa-1} < 2r\Re e\{\Psi(z)\}, \qquad 1 + \sum_{\kappa=2}^{\infty} 2(\beta - j)r^{\kappa-1} > (1 - r)^2, \tag{32}$$

then Theorem 12 is proved. □

3. Application of Differential Subordination to the Function $Q^{(\jmath)}(\alpha,\tau,\gamma;\beta)$

Here we calculate some subordinate properties of the class $Q^{(\jmath)}(\alpha,\tau,\gamma;\beta)$.

Theorem 13. *Let $G(z)\in Q^{(\jmath)}(\alpha,\tau,\gamma;\beta)$ and let $h(z)$ be starlike in U with $h(0)=0$ and $\alpha+\tau\neq\gamma$. If*

$$\left\{\frac{zG'(z)}{G(z)}\right\}\prec h(z), \tag{33}$$

then

$$G(z)\prec q(z)=(\alpha+\tau-\gamma)\exp\left[n^{-1}\int_0^z h(\xi)\xi^{-1}d\xi,\right] \tag{34}$$

and q is the best $([\alpha+\tau-\gamma],n)$-dominant.

Proof. Let $G(z)\in Q(\alpha,\tau,\gamma;\beta)$; then

$$G(z)=(\alpha+\tau-\gamma)+\sum_{\kappa=2}^{\infty}\frac{\lambda_\kappa\left[{}^{[\beta+\kappa-1]}P_{\jmath+1}\right]a_\kappa z^{\kappa-1}}{{}^{\beta}P_{\jmath+1}}. \tag{35}$$

Since $G(z)$ is analytic in U and $G(0)=\alpha+\tau-\gamma$, it suffices to show that

$$\left\{\frac{zG'(z)}{G(z)}\right\}\prec h(z). \tag{36}$$

Following the same argument in [10] (pages 76 and 77), (36) is true. Application of Lemma 5 proves Theorem 13 with $q(z)$ as the best $([\alpha+\tau-\gamma],n)$-dominant. □

Example 14. Let $h(z)=\sqrt{z}/(1-z)$; if

$$\left\{\frac{zG'(z)}{G(z)}\right\}\prec\frac{\sqrt{z}}{1-z} \tag{37}$$

then

$$G(z)\prec(\alpha+\tau-\gamma)\left\{\frac{1+\sqrt{z}}{1-\sqrt{z}}\right\}^{1/n},\quad \alpha+\tau\neq\gamma, \tag{38}$$

and q is the best $(\alpha+\tau-\gamma,n)$-dominant.

Solution. If $h(z)=\sqrt{z}/(1-z)$ and $z=re^{i\theta}$, $0<\theta<\pi$, then simple calculation shows that $h(0)=0$ and $h(z)$ is starlike and the argument in [10] shows (37). The proof also follows from Lemma 5.

Theorem 15. *Let $G(z)\in Q^{(\jmath)}(\alpha,\tau,0;\beta)$ and $-1\le B<A\le 1$, $\beta>\jmath+1$ with $\alpha+\tau\neq 0$. If*

$$G(z)\prec(\alpha+\tau)\frac{1+Az}{1+Bz}, \tag{39}$$

then

$$\begin{aligned}\frac{[F(z)]^{(\jmath)}}{[z^\beta]^{(\jmath)}}&\prec\frac{(\alpha+\tau)(\beta-\jmath)}{\tau\kappa}z^{(\alpha+\tau)(\jmath-\beta)/\tau\kappa}\\&\quad\times\int_0^z\xi^{(\alpha(\beta-\jmath)+\tau(\beta-\jmath-\kappa))/\tau\kappa}\left(\frac{1+A\xi}{1+B\xi}\right)d\xi.\end{aligned} \tag{40}$$

Proof. Let $G(z)\in Q^{(\jmath)}(\alpha,\tau,0;\beta)$, and, from (6),

$$\frac{1}{(\alpha+\tau)}G(z)=1+\sum_{\kappa=2}^{\infty}C_\kappa a_\kappa z^{\kappa-1}, \tag{41}$$

where

$$C_\kappa=\frac{\lambda_\kappa\left[{}^{(\beta+\kappa-1)}P_{\jmath+1}\right]\cdot[(\alpha+\tau)(\beta-\jmath)+\tau(\kappa-1)]}{(\alpha+\tau)\left[{}^{\beta}P_{\jmath+1}\right]}. \tag{42}$$

Let $g(z)=[F(z)]^{(\jmath)}/[z^\beta]^{(\jmath)}$; then $g(z)=1+\sum_{\kappa=2}^{\infty}\varpi_\kappa a_\kappa z^{\kappa-1}$, where $\varpi_\kappa=\lambda_\kappa[{}^{(\beta+\kappa-1)}P_{\jmath+1}]/[{}^{(\beta)}P_\jmath]$, and from (39) we have

$$g(z)+\frac{\tau}{(\alpha+\tau)(\beta-\jmath)}zg'(z)\prec\left(\frac{1+Az}{1+Bz}\right)=h(z), \tag{43}$$

and $h(z)$ is convex and univalent in U. So, by Lemma 6,

$$\begin{aligned}g(z)&\prec\frac{(\alpha+\tau)(\beta-\jmath)}{\tau\kappa}z^{-(\alpha+\tau)(\beta-\jmath)/\tau\kappa}\\&\quad\times\int_0^z\xi^{(\alpha(\beta-\jmath)+\tau(\beta-\jmath-\kappa))/\tau\kappa}\left(\frac{1+A\xi}{1+B\xi}\right)d\xi.\end{aligned} \tag{44}$$

This completes the proof of Theorem 15. □

Corollary 16. *Let*

$$Q^{(0)}(\alpha,\tau,0;\beta)=\frac{\alpha(F(z))}{z^\beta}+\frac{\tau[F(z)]'}{[z^\beta]'}. \tag{45}$$

If

$$Q^{(0)}(\alpha,\tau,0;\beta)\prec(\alpha+\tau)\left(\frac{1+Az}{1+Bz}\right), \tag{46}$$

then

$$\begin{aligned}\frac{F(z)}{z^\beta}&\prec\frac{\alpha\beta+\tau(\beta-\kappa)}{\tau\kappa}z^{-\beta(\alpha+\tau)/\tau\kappa}\\&\quad\times\int_0^z\xi^{(\beta(\alpha+\tau)-\tau\kappa)/\tau\kappa}\left(\frac{1+A\xi}{1+B\xi}\right)d\xi.\end{aligned} \tag{47}$$

Proof. The result follows from Theorem 15. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Authors' Contribution

Both authors read and approved the final paper.

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