Research Article

Hypersurfaces with Two Distinct Para-Blaschke Eigenvalues in $S^{n+1}(1)$

Junfeng Chen and Shichang Shu

School of Mathematics and Information Science, Xianyang Normal University, Xianyang, Shaanxi 712000, China

Correspondence should be addressed to Shichang Shu; shushichang@126.com

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Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n ($n \geq 3$)-dimensional immersed hypersurface without umbilical points and with vanishing Möbius form in a unit sphere $S^{n+1}(1)$, and let $\mathbf{A}$ and $\mathbf{B}$ be the Blaschke tensor and the Möbius second fundamental form of $\mathbf{x}$, respectively. We define a symmetric $(0, 2)$ tensor $\mathbf{D} = \mathbf{A} + \lambda \mathbf{B}$ which is called the para-Blaschke tensor of $\mathbf{x}$, where λ is a constant. An eigenvalue of the para-Blaschke tensor is called a *para-Blaschke eigenvalue* of $\mathbf{x}$. The aim of this paper is to classify the oriented hypersurfaces in $S^{n+1}(1)$ with two distinct para-Blaschke eigenvalues under some rigidity conditions.

1. Introduction

In the Möbius geometry of hypersurfaces, Wang [1] studied invariants of hypersurfaces in a unit sphere $S^{n+1}(1)$ under the Möbius transformation group. Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n -dimensional immersed hypersurface without umbilical points in $S^{n+1}(1)$. We choose a local orthonormal basis $\{e_i\}$ for the induced metric $I = d\mathbf{x} \cdot d\mathbf{x}$ with dual basis $\{\theta_i\}$. Let $II = \sum_{i,j} h_{ij} \theta_i \otimes \theta_j$ be the second fundamental form of the immersion $\mathbf{x}$ and $H = (1/n) \sum_i h_{ii}$ the mean curvature of the immersion $\mathbf{x}$. By putting $\rho^2 = (n/(n-1))\{\sum_{i,j} h_{ij}^2 - nH^2\}$, the *Möbius metric* of the immersion $\mathbf{x}$ is defined by $g = \rho^2 d\mathbf{x} \cdot d\mathbf{x}$ which is a Möbius invariant. $\Phi = \rho \sum_i C_i \theta_i$, $\mathbf{A} = \rho^2 \sum_{i,j} A_{ij} \theta_i \otimes \theta_j$, and $\mathbf{B} = \rho^2 \sum_{i,j} B_{ij} \theta_i \otimes \theta_j$ are called the Möbius form, the Blaschke tensor, and the Möbius second fundamental form of the immersion $\mathbf{x}$, respectively (see [1]), where

$$A_{ij} = -\rho^{-2} \{ \text{Hess}_{ij}(\log \rho) - e_i(\log \rho) e_j(\log \rho) - H h_{ij} \} \\ - \frac{1}{2} \rho^{-2} (|\nabla(\log \rho)|^2 - 1 + |\vec{H}|^2) \delta_{ij}, \quad (1)$$

$$B_{ij} = \rho^{-1} (h_{ij} - H \delta_{ij}), \quad (2)$$

$$C_i = -\rho^{-2} \left\{ H_{,i} + \sum_j (h_{ij} - H \delta_{ij}) e_j(\log \rho) \right\}, \quad (3)$$

and Hess_{ij} and ∇ are the Hessian matrix and the gradient with respect to the induced metric $d\mathbf{x} \cdot d\mathbf{x}$. It was proved by [1] that Φ , $\mathbf{A}$, and $\mathbf{B}$ are Möbius invariants.

In the study of the Möbius geometry of hypersurfaces, one of the important aims is to characterize hypersurfaces in terms of Möbius invariants. Concerning this topic, there are many important results; one can see [2–9]. We should notice that [5] classified all umbilic-free hypersurfaces with parallel Möbius second fundamental form. Recently, by making use of the two important Möbius invariants, the Blaschke tensor $\mathbf{A}$ and the Möbius second fundamental form $\mathbf{B}$ of $\mathbf{x}$, Zhong and Sun [10] defined a symmetric $(0, 2)$ tensor $\mathbf{D} = \mathbf{A} + \lambda \mathbf{B}$ which is called the para-Blaschke tensor of $\mathbf{x}$, where λ is a constant. An eigenvalue of the para-Blaschke tensor is called a para-Blaschke eigenvalue of $\mathbf{x}$. In [7], Li and Wang investigated and completely classified hypersurfaces $\mathbf{x} : M \mapsto S^{n+1}(1)$ without umbilical points and with vanishing Möbius form Φ in $S^{n+1}(1)$, which satisfy $\mathbf{A} + \lambda \mathbf{B} + \mu g = 0$. It should be noted that the condition $\mathbf{A} + \lambda \mathbf{B} + \mu g = 0$ implies that the para-Blaschke eigenvalues of $\mathbf{x}$ are all equal. If $\mathbf{x}$ has two distinct constant para-Blaschke eigenvalues, Zhong and Sun [10] obtained the following.

Theorem 1 (see [10]). Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n -dimensional immersed hypersurface without umbilical points. If $\mathbf{x}$ is of two distinct constant para-Blaschke eigenvalues and of vanishing Möbius form Φ , then $\mathbf{x}$ is locally Möbius equivalent to

- (1) the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or
- (2) the image of σ of the standard cylinder $S^k(r) \times \mathbf{R}^{n-k}$ in $\mathbf{R}^{n+1}$, or
- (3) the image of τ of the Riemannian product $S^k(r) \times \mathbf{H}^{n-k}(\sqrt{1+r^2})$ in $\mathbf{H}^{n+1}$, or
- (4) the hypersurface as indicated in Example 11, or
- (5) the hypersurface as indicated in Example 12.

Remark 2. In Theorem 1 and the following Theorems, the conformal diffeomorphisms $\sigma : \mathbf{R}^{n+1} \mapsto S^{n+1}(1) \setminus \{(-1, 0)\}$ and $\tau : \mathbf{H}^{n+1} \mapsto S^{n+1}_+(1)$ are defined by

$$\begin{aligned} \sigma(u) &= \left(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2} \right), \quad u \in \mathbf{R}^{n+1}, \\ \tau(y_0, y_1) &= \left(\frac{1}{y_0}, \frac{y_1}{y_0} \right), \quad (y_0, y_1) \in \mathbf{H}^{n+1}, \end{aligned} \quad (4)$$

respectively, where $S^{n+1}_+(1)$ is the open hemisphere in $S^{n+1}(1)$ whose first coordinate is positive (see [1]).

In this paper, we study hypersurfaces $\mathbf{x} : M \mapsto S^{n+1}(1)$ without umbilical points and with vanishing Möbius form. If $\mathbf{x}$ is of two distinct para-Blaschke eigenvalues, we obtain some classification Theorems in terms of some rigidity conditions.

Theorem 3. Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n ($n \geq 3$)-dimensional immersed hypersurface without umbilical points and with vanishing Möbius form Φ and constant Möbius scalar curvature $n(n-1)R$ in $S^{n+1}(1)$. If $\mathbf{x}$ is of two distinct para-Blaschke eigenvalues and the multiplicities of these two distinct para-Blaschke eigenvalues are greater than 1, then $\mathbf{x}$ is locally Möbius equivalent to

- (1) the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or
- (2) the image of σ of the standard cylinder $S^k(r) \times \mathbf{R}^{n-k}$ in $\mathbf{R}^{n+1}$, or
- (3) the image of τ of the Riemannian product $S^k(r) \times \mathbf{H}^{n-k}(\sqrt{1+r^2})$ in $\mathbf{H}^{n+1}$, or
- (4) the hypersurface as indicated in Example 11, or
- (5) the hypersurface as indicated in Example 12, where $1 < k < n-1$.

Theorem 4. Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n ($n \geq 3$)-dimensional immersed complete hypersurface without umbilical points and with vanishing Möbius form Φ and constant Möbius scalar curvature $n(n-1)R$ in $S^{n+1}(1)$. If $\mathbf{x}$ is of two distinct para-Blaschke eigenvalues one of which is simple, then we have the following.

- (1) If the Möbius sectional curvature of $\mathbf{x}$ is nonnegative and $R > ((n-2)/n^2)(1+n|\lambda|)$, then $\mathbf{x}$ is locally Möbius equivalent to

- (i) the Riemannian product $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or
- (ii) the image of σ of the standard cylinder $S^{n-1}(r) \times \mathbf{R}$ in $\mathbf{R}^{n+1}$.

- (2) If

$$|\bar{D}| \leq \sqrt{\frac{n-1}{n}} \left\{ \frac{n}{n-2} R - \frac{1+n|\lambda|}{n} \right\}, \quad (5)$$

then $\mathbf{x}$ is locally Möbius equivalent to:

- (i) the Riemannian product $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or
- (ii) the image of σ of the standard cylinder $S^{n-1}(r) \times \mathbf{R}$ in $\mathbf{R}^{n+1}$, in this case $\lambda \geq 0$, or
- (iii) the image of τ of the Riemannian product $S^{n-1}(r) \times \mathbf{H}^1(\sqrt{1+r^2})$ in $\mathbf{H}^{n+1}$, in this case $\lambda \leq 0$, where $\bar{D}_{ij} = D_{ij} - (1/n) \sum_k D_{kk} \delta_{ij}$.

If $\lambda = 0$, then the para-Blaschke tensor is the Blaschke tensor of $\mathbf{x}$. From Theorems 3 and 4, we have the following.

Corollary 5. Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n ($n \geq 3$)-dimensional immersed hypersurface without umbilical points and with vanishing Möbius form Φ and constant Möbius scalar curvature $n(n-1)R$ in $S^{n+1}(1)$. If $\mathbf{x}$ is of two distinct Blaschke eigenvalues and the multiplicities of these two distinct Blaschke eigenvalues are greater than 1, then $\mathbf{x}$ is locally Möbius equivalent to

- (1) the Riemannian product $S^k(r) \times S^{n-k}(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or
- (2) the image of σ of the standard cylinder $S^k(r) \times \mathbf{R}^{n-k}$ in $\mathbf{R}^{n+1}$, or
- (3) the image of τ of the Riemannian product $S^k(r) \times \mathbf{H}^{n-k}(\sqrt{1+r^2})$ in $\mathbf{H}^{n+1}$, or
- (4) the minimal hypersurface as indicated in Example 13, or
- (5) the nonminimal hypersurface as indicated in Example 14, where $1 < k < n-1$.

Corollary 6. Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n ($n \geq 3$)-dimensional immersed complete hypersurface without umbilical points and with vanishing Möbius form Φ and constant Möbius scalar curvature $n(n-1)R$ in $S^{n+1}(1)$. If $\mathbf{x}$ is of two distinct Blaschke eigenvalues one of which is simple, then we have the following.

- (1) If the Möbius sectional curvature of $\mathbf{x}$ is nonnegative and $R > (n-2)/n^2$, then $\mathbf{x}$ is locally Möbius equivalent to:

- (i) the Riemannian product $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or
 (ii) the image of σ of the standard cylinder $S^{n-1}(r) \times \mathbf{R}$ in $\mathbf{R}^{n+1}$.

(2) If

$$|\widetilde{\mathbf{A}}| \leq \sqrt{\frac{n-1}{n}} \left\{ \frac{n}{n-2} R - \frac{1}{n} \right\}, \quad (6)$$

then $\mathbf{x}$ is locally Möbius equivalent to

- (i) the Riemannian product $S^{n-1}(r) \times S^1(\sqrt{1-r^2})$ in $S^{n+1}(1)$, or
 (ii) the image of σ of the standard cylinder $S^{n-1}(r) \times \mathbf{R}$ in $\mathbf{R}^{n+1}$, or
 (iii) the image of τ of the Riemannian product $S^{n-1}(r) \times \mathbf{H}^1(\sqrt{1+r^2})$ in $\mathbf{H}^{n+1}$, where $\widetilde{A}_{ij} = A_{ij} - (1/n) \sum_k A_{kk} \delta_{ij}$.

2. Preliminaries

In this section, we review the definitions of Möbius invariants and the fundamental formulas on Möbius geometry of hypersurfaces in $S^{n+1}(1)$; for more details see [1].

Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n -dimensional hypersurface of $S^{n+1}(1)$ without umbilical points. We use the following range of indices throughout this paper:

$$1 \leq i, j, k, l, m \leq n. \quad (7)$$

The structure equations on M with respect to the Möbius metric g can be written as follows:

$$\begin{aligned} dY &= \sum_i \omega_i Y_i, \\ dN &= \sum_i \psi_i Y_i + \phi E_{n+1}, \\ dY_i &= -\psi_i Y - \omega_i N + \sum_j \omega_{ij} Y_j + \omega_{in+1} E_{n+1}, \\ dE_{n+1} &= -\phi Y - \sum_i \omega_{in+1} Y_i, \end{aligned} \quad (8)$$

where $\{\psi_i, \omega_{ij}, \omega_{in+1}, \phi\}$ are 1-forms on M with

$$\omega_{ij} + \omega_{ji} = 0. \quad (9)$$

By exterior differentiation of these equations, we get

$$\sum_i \omega_i \wedge \psi_i = 0, \quad \sum_i \omega_{in+1} \wedge \omega_i = 0, \quad (10)$$

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad (11)$$

$$d\psi_i = \sum_j \omega_{ij} \wedge \psi_j + \omega_{in+1} \wedge \phi, \quad (12)$$

$$d\phi = -\sum_i \omega_{in+1} \wedge \psi_i, \quad (13)$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \omega_{in+1} \wedge \omega_{jn+1} - \omega_i \wedge \psi_j - \psi_i \wedge \omega_j, \quad (14)$$

$$d\omega_{in+1} = \sum_j \omega_{ij} \wedge \omega_{jn+1} - \omega_i \wedge \phi, \quad (15)$$

where

$$\begin{aligned} \psi_i &= \sum_j A_{ij} \omega_j, & A_{ij} &= A_{ji}, & \omega_{in+1} &= \sum_j B_{ij} \omega_j, \\ B_{ij} &= B_{ji}, & \phi &= \sum_i C_i \omega_i, \end{aligned} \quad (16)$$

and A_{ij} , B_{ij} , and C_i are locally defined functions and satisfy (1), (2), and (3). We have

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \quad R_{ijkl} = -R_{jikl}, \quad (17)$$

$$\sum_i B_{ii} = 0, \quad \sum_{i,j} B_{ij}^2 = \frac{n-1}{n}, \quad \text{tr } \mathbf{A} = \frac{1}{2n} (1 + n^2 R). \quad (18)$$

Let $C_{i,j}$, $A_{ij,k}$, $B_{ij,k}$ be the covariant derivative of C_i , A_{ij} , B_{ij} . From the structure equations (8), we infer

$$A_{ij,k} - A_{ik,j} = B_{ik} C_j - B_{ij} C_k, \quad (19)$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik} A_{kj} - B_{kj} A_{ki}), \quad (20)$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij} C_k - \delta_{ik} C_j, \quad (21)$$

$$R_{ijkl} = B_{ik} B_{jl} - B_{il} B_{jk} + \delta_{ik} A_{jl} + \delta_{jl} A_{ik} - \delta_{il} A_{jk} - \delta_{jk} A_{il}, \quad (22)$$

where R_{ijkl} denotes the curvature tensor with respect to the Möbius metric g on M and $n(n-1)R = \sum_{i,j} R_{ijij}$ is the Möbius scalar curvature of the immersion $\mathbf{x} : M \rightarrow S^{n+1}(1)$.

Since the Möbius form $\Phi = \sum_i C_i \omega_i E_{n+1} \equiv 0$, by (19)–(21), we have for all indices i, j, k that

$$A_{ij,k} = A_{ik,j}, \quad B_{ij,k} = B_{ik,j}, \quad \sum_k B_{ik} A_{kj} = \sum_k B_{kj} A_{ki}. \quad (23)$$

Denote by $\mathbf{D} = \sum_{i,j} D_{ij} \omega_i \otimes \omega_j$ the $(0, 2)$ para-Blaschke tensor; then,

$$D_{ij} = A_{ij} + \lambda B_{ij}, \quad 1 \leq i, j \leq n, \quad (24)$$

where λ is constant. The covariant derivative of D_{ij} is defined by

$$\sum_k D_{ij,k} \omega_k = dD_{ij} + \sum_k D_{ik} \omega_{kj} + \sum_k D_{kj} \omega_{ki}. \quad (25)$$

From (24), we have

$$\begin{aligned} D_{ij,k} &= A_{ij,k} + \lambda B_{ij,k}, \\ D_{ij,k} - D_{ik,j} &= A_{ij,k} - A_{ik,j} + \lambda (B_{ij,k} - B_{ik,j}). \end{aligned} \quad (26)$$

From (23), we have for all indices i, j, k that

$$D_{ij,k} = D_{ik,j}. \quad (27)$$

3. Lemma and Examples

In this section, we make the following convention on the ranges of indices:

$$1 \leq i, j \leq n, \quad 1 \leq a, b, c \leq k, \quad k+1 \leq s, t \leq n. \quad (28)$$

The following Lemma 7 due to Li et al. [6] will be used.

Lemma 7 (see [6]). *Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n -dimensional hypersurface with two distinct principal curvatures with multiplicities k and $n-k$, respectively. Then, the principal curvatures of the Möbius second fundamental form $\mathbf{B}$ of $\mathbf{x}$ are constant, which are given by*

$$\mu_1 = \frac{1}{n} \sqrt{\frac{(n-1)(n-k)}{k}}, \quad \mu_2 = -\frac{1}{n} \sqrt{\frac{(n-1)k}{n-k}}. \quad (29)$$

In [4], Cheng and Shu studied the Möbius invariants on some typical examples.

Example 8 (see [4]). Let $\mathbf{x}_1 : S^k(1) \mapsto \mathbf{R}^{k+1}$ and $\mathbf{x}_2 : S^{n-k}(1) \mapsto \mathbf{R}^{n-k+1}$ be the standard embedding of the unit spheres. Then, the Riemannian product $\mathbf{x} : S^k(r) \times S^{n-k}(\sqrt{1-r^2}) \mapsto S^{n+1}(1)$ defined by $\mathbf{x} = (r\mathbf{x}_1, \sqrt{1-r^2}\mathbf{x}_2)$, for any $1 \leq k \leq n-1$ and any $0 < r < 1$, is an immersed hypersurface without umbilical points and with vanishing Möbius form in $S^{n+1}(1)$. By (3.4) in [4], we know that the Möbius sectional curvature of $\mathbf{x}$ is nonnegative. From (3.9)

to (3.11) in [4], Lemma 7, and (24) and by a simple and direct calculation, we see that

$$\begin{aligned} D_{ab} &= \left\{ \frac{n-1}{2k(n-k)n^2} [k(2n-k) - n^2 r^2] \right. \\ &\quad \left. + \lambda \frac{1}{n} \sqrt{\frac{(n-1)(n-k)}{k}} \right\} \delta_{ab}, \\ D_{st} &= \left\{ \frac{n-1}{2k(n-k)n^2} (n^2 r^2 - k^2) - \lambda \frac{1}{n} \sqrt{\frac{(n-1)k}{n-k}} \right\} \delta_{st}, \\ D_{as} &= 0. \end{aligned} \quad (30)$$

Thus, $\mathbf{x}$ has exactly two distinct constant para-Blaschke eigenvalues. Defining the trace-free tensor $\tilde{D}_{ij} = D_{ij} - (1/n) \text{tr } D \delta_{ij}$, we see that

$$\begin{aligned} (n-2k)^2 |\tilde{\mathbf{D}}|^2 &= \left\{ \sqrt{\frac{k(n-k)}{n}} \left(nR - \frac{n-2}{n} \right) \right. \\ &\quad \left. - (n-2k) \lambda \sqrt{\frac{n-1}{n}} \right\}^2, \end{aligned} \quad (31)$$

or

$$\begin{aligned} (n-2k)^2 |\tilde{\mathbf{D}}|^2 &= \left\{ \sqrt{\frac{k(n-k)}{n}} \left(nR - \frac{n-2}{n} \right) \right. \\ &\quad \left. + (n-2k) \lambda \sqrt{\frac{n-1}{n}} \right\}^2. \end{aligned} \quad (32)$$

Thus, we know that $|\tilde{\mathbf{D}}| = \sqrt{(n-1)/n}((n/(n-2))R - ((1+n\lambda)/n))$, if and only if $k = 1$ or $k = n-1$, and $|\tilde{\mathbf{D}}| = \sqrt{(n-1)/n}((n/(n-2))R - ((1-n\lambda)/n))$, if and only if $k = 1$ or $k = n-1$.

Example 9 (see [4]). Let $\hat{\mathbf{x}} : S^k(1) \times \mathbf{R}^{n-k} \mapsto \mathbf{R}^{n+1}$ be the standard cylinder. Then, the hypersurface $\mathbf{x} = \sigma \circ \hat{\mathbf{x}} : S^k(1) \times \mathbf{R}^{n-k} \mapsto S^{n+1}(1)$ is an immersed hypersurface without umbilical points and with vanishing Möbius form in $S^{n+1}(1)$. By (3.25) in [4], we know that the Möbius sectional curvature of $\mathbf{x}$ is nonnegative. From (3.31)–(3.33) in [4], Lemma 7 and (24), we can calculate that

$$\begin{aligned} D_{ab} &= \left\{ -\frac{(n-1)(k-2n)}{2(n-k)n^2} + \lambda \frac{1}{n} \sqrt{\frac{(n-1)(n-k)}{k}} \right\} \delta_{ab}, \\ D_{st} &= \left\{ -\frac{(n-1)k}{2(n-k)n^2} - \lambda \frac{1}{n} \sqrt{\frac{(n-1)k}{n-k}} \right\} \delta_{st}, \quad D_{as} = 0. \end{aligned} \quad (33)$$

Thus $\mathbf{x}$ has exactly two distinct constant para-Blaschke eigenvalues. By the similar calculation in Example 8, we have

$$(n-2k)^2 |\tilde{\mathbf{D}}|^2 = \left\{ \sqrt{\frac{k(n-k)}{n}} \left(nR - \frac{n-2}{n} \right) + (n-2k) \lambda \sqrt{\frac{n-1}{n}} \right\}^2. \quad (34)$$

We know that $|\tilde{\mathbf{D}}| = \sqrt{(n-1)/n}((n/(n-2))R - ((1+n\lambda)/n))$ if and only if $k = n-1$. If $k = 1$, from (3.23) in [4], we have $R = 0$.

Example 10 (see [4]). Let $\bar{\mathbf{x}} : S^k(r) \times \mathbf{H}^{n-k}(\sqrt{1+r^2}) \mapsto \mathbf{H}^{n+1}$ be the standard embedding. Then the hypersurface $\mathbf{x} = \tau \circ \bar{\mathbf{x}} : S^k(r) \times \mathbf{H}^{n-k}(\sqrt{1+r^2}) \mapsto S^{n+1}(1)$ is an immersed hypersurface without umbilical points and with vanishing Möbius form in $S^{n+1}(1)$. By (3.41) in [4], we know that the Möbius sectional curvature of $\mathbf{x}$ is not nonnegative. From (3.45)–(3.47) in [4], Lemma 7 and (24), we can calculate that

$$D_{ab} = \left\{ \frac{n-1}{2k(n-k)n^2} [k(2n-k) + n^2 r^2] + \lambda \frac{1}{n} \sqrt{\frac{(n-1)(n-k)}{k}} \right\} \delta_{ab}, \quad (35)$$

$$D_{st} = - \left\{ \frac{n-1}{2k(n-k)n^2} [k^2 + n^2 r^2] + \lambda \frac{1}{n} \sqrt{\frac{(n-1)k}{n-k}} \right\} \delta_{st}, \quad D_{as} = 0. \quad (36)$$

Thus, $\mathbf{x}$ has exactly two distinct constant para-Blaschke eigenvalues. By the similar calculation in Example 8, we have

$$(2k-n)^2 |\tilde{\mathbf{D}}|^2 = \left\{ \sqrt{\frac{k(n-k)}{n}} \left(nR - \frac{n-2}{n} \right) + (2k-n) \lambda \sqrt{\frac{n-1}{n}} \right\}^2. \quad (37)$$

We know that $|\tilde{\mathbf{D}}| = \sqrt{(n-1)/n}((n/(n-2))R - ((1-n\lambda)/n))$ if and only if $k = n-1$. If $k = 1$, from (3.39) in [4], we have $R < 0$.

Recently, Zhong and Sun [10] studied some important examples, which can be stated as follows.

Example 11 (see [10]). Hypersurface $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) : \tilde{M}_1^k \times \mathbf{H}^{n-k}(r) \mapsto S^{n+1}$, where $\tilde{\mathbf{x}}_1 = \mathbf{y}_1/\mathbf{y}_0$, $\tilde{\mathbf{x}}_2 = \mathbf{y}_2/\mathbf{y}_0$, $\mathbf{y}_0 \in \mathbf{R}^+$, $\mathbf{y}_1 \in \mathbf{R}^{k+2}$, $\mathbf{y}_2 \in \mathbf{R}^{n-k}$, $2 \leq k \leq n-1$, $r > 0$. Here,

$\mathbf{y}_1 : \tilde{M}_1^k \mapsto S^{k+1}(r) \hookrightarrow \mathbf{R}^{k+2}$ is an immersed hypersurface in $(k+1)$ -dimensional sphere $S^{k+1}(r)$, which is at least of two nonzero principal curvatures and with constant mean curvature H_1 and constant scalar curvature R_1 :

$$H_1 = -\frac{n}{k}\lambda, \quad (38)$$

$$R_1 = \frac{\{nk(k-1) - (n-1)r^2\}}{nr^2} + n(n-1)\lambda^2,$$

and $(\mathbf{y}_0, \mathbf{y}_2) : \mathbf{H}^{n-k}(r) \mapsto \mathbf{R}_1^{n-k+1}$ is the canonical embedding of $(n-k)$ -dimensional hyperbolic space $\mathbf{H}^{n-k}(r)$ to $(n-k+1)$ -dimensional Lorentzian space $\mathbf{R}_1^{n-k+1}$ and $\mathbf{y}_0 > 0$, $-\mathbf{y}_0^2 + \mathbf{y}_2 \cdot \mathbf{y}_2 = -r^2$. By [10], we know that the Möbius sectional curvature of $\tilde{\mathbf{x}}$ is not nonnegative. From (3.12) to (3.21) in [10], by a simple and direct calculation, we see that the para-Blaschke tensor of $\tilde{\mathbf{x}}$ is

$$D_{ab} = \left(\frac{1}{2r^2} + \frac{\lambda^2}{2} \right) \delta_{ab}, \quad (39)$$

$$D_{as} = 0, \quad D_{st} = \left(-\frac{1}{2r^2} + \frac{\lambda^2}{2} \right) \delta_{st}.$$

Clearly, the Möbius form $\Phi \equiv 0$ and $\tilde{\mathbf{x}}$ has exactly two distinct constant para-Blaschke eigenvalues. We can easily check that $|\tilde{\mathbf{D}}| \neq \sqrt{(n-1)/n}((n/(n-2))R - ((1+n|\lambda|)/n))$ if $k = 1$ or $k = n-1$.

Example 12 (see [10]). Hypersurface $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) : \tilde{M}_1^k \times S^{n-k}(r) \mapsto S^{n+1}$, where $\tilde{\mathbf{x}}_1 = \mathbf{y}_1/\mathbf{y}_0$, $\tilde{\mathbf{x}}_2 = \mathbf{y}_2/\mathbf{y}_0$, $\mathbf{y}_0 \in \mathbf{R}^+$, $\mathbf{y}_1 \in \mathbf{R}^{k+1}$, $\mathbf{y}_2 \in \mathbf{R}^{n-k+1}$, $2 \leq k \leq n-1$, $r > 0$. Here, $(\mathbf{y}_0, \mathbf{y}_1) : \tilde{M}_1^k \mapsto \mathbf{H}^{k+1}(r) \hookrightarrow \mathbf{R}_1^{k+2}$ is an immersed hypersurface in $(k+1)$ -dimensional hyperbolic space $\mathbf{H}^{k+1}(r)$, which is at least of two nonzero principal curvatures and with constant mean curvature H_1 and constant scalar curvature R_1 :

$$H_1 = -\frac{n}{k}\lambda, \quad (40)$$

$$R_1 = \frac{\{-nk(k-1) - (n-1)r^2\}}{nr^2} + n(n-1)\lambda^2,$$

and $\mathbf{y}_0 > 0$, $-\mathbf{y}_0^2 + \mathbf{y}_1 \cdot \mathbf{y}_1 = -r^2$ and $\mathbf{y}_2 : S^{n-k}(r) \mapsto \mathbf{R}^{n-k+1}$ is the canonical embedding of $(n-k)$ -dimensional sphere $S^{n-k}(r)$ to $(n-k+1)$ -dimensional Euclidean space $\mathbf{R}^{n-k+1}$. By [10], we know that the Möbius sectional curvature of $\tilde{\mathbf{x}}$ is not nonnegative. From (3.39) to (3.48) in [10], by a simple and direct calculation, we see that the para-Blaschke tensor of $\tilde{\mathbf{x}}$ is

$$D_{ab} = \left(-\frac{1}{2r^2} + \frac{\lambda^2}{2} \right) \delta_{ab}, \quad (41)$$

$$D_{as} = 0, \quad D_{st} = \left(\frac{1}{2r^2} + \frac{\lambda^2}{2} \right) \delta_{st}.$$

Clearly, the Möbius form $\Phi \equiv 0$ and $\tilde{\mathbf{x}}$ has exactly two distinct constant para-Blaschke eigenvalues. We can easily check that $|\tilde{\mathbf{D}}| \neq \sqrt{(n-1)/n}((n/(n-2))R - ((1+n|\lambda|)/n))$ if $k = 1$ or $k = n-1$.

If $\lambda = 0$, Examples 11 and 12 decrease to the following Examples (see [8]).

Example 13. Minimal hypersurface $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) : \tilde{M}_1^k \times \mathbf{H}^{n-k}(r) \mapsto S^{n+1}$, where $\tilde{\mathbf{x}}_1 = \mathbf{y}_1/\mathbf{y}_0$, $\tilde{\mathbf{x}}_2 = \mathbf{y}_2/\mathbf{y}_0$, $\mathbf{y}_0 \in \mathbf{R}^+$, $\mathbf{y}_1 \in \mathbf{R}^{k+2}$, $\mathbf{y}_2 \in \mathbf{R}^{n-k}$, $2 \leq k \leq n-1$, $r > 0$. Here, $\mathbf{y}_1 : \tilde{M}_1^k \mapsto S^{k+1}(r) \hookrightarrow \mathbf{R}^{k+2}$ is an immersed minimal hypersurface without umbilical points in $(k+1)$ -dimensional sphere $S^{k+1}(r)$ and with constant scalar curvature R_1 :

$$R_1 = \frac{\{nk(k-1) - (n-1)r^2\}}{nr^2}, \quad (42)$$

and $(\mathbf{y}_0, \mathbf{y}_2) : \mathbf{H}^{n-k}(r) \mapsto \mathbf{R}_1^{n-k+1}$ is the canonical embedding of $(n-k)$ -dimensional hyperbolic space $\mathbf{H}^{n-k}(r)$ to $(n-k+1)$ -dimensional Lorentzian space $\mathbf{R}_1^{n-k+1}$.

Example 14. Nonminimal hypersurface $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2) : \tilde{M}_1^k \times S^{n-k}(r) \mapsto S^{n+1}$, where $\tilde{\mathbf{x}}_1 = \mathbf{y}_1/\mathbf{y}_0$, $\tilde{\mathbf{x}}_2 = \mathbf{y}_2/\mathbf{y}_0$, $\mathbf{y}_0 \in \mathbf{R}^+$, $\mathbf{y}_1 \in \mathbf{R}^{k+1}$, $\mathbf{y}_2 \in \mathbf{R}^{n-k+1}$, $2 \leq k \leq n-1$, $r > 0$. Here, $(\mathbf{y}_0, \mathbf{y}_1) : \tilde{M}_1^k \mapsto \mathbf{H}^{k+1}(r) \hookrightarrow \mathbf{R}_1^{k+2}$ is an immersed minimal hypersurface without umbilical points in $(k+1)$ -dimensional hyperbolic space $\mathbf{H}^{k+1}(r)$ and with constant scalar curvature R_1 :

$$R_1 = \frac{\{-nk(k-1) - (n-1)r^2\}}{nr^2}, \quad (43)$$

and $\mathbf{y}_2 : S^{n-k}(r) \mapsto \mathbf{R}^{n-k+1}$ is the canonical embedding of $(n-k)$ -dimensional sphere $S^{n-k}(r)$ to $(n-k+1)$ -dimensional Euclidean space $\mathbf{R}^{n-k+1}$.

4. Proof of Theorems

Let A , B , and D denote the $n \times n$ -symmetric matrices (A_{ij}) , (B_{ij}) , and (D_{ij}) , respectively, where A_{ij} , B_{ij} , and D_{ij} are defined by (1), (2), and (24). Since from (23), we know that $BA = AB$, $DA = AD$, and $BD = DB$ hold, we can choose a local orthonormal basis $\{E_1, E_2, \dots, E_n\}$ such that $A_{ij} = A_i \delta_{ij}$, $B_{ij} = B_i \delta_{ij}$, and $D_{ij} = D_i \delta_{ij}$, where A_i , B_i , and D_i are the Blaschke eigenvalues, the Möbius principal curvatures, and the para-Blaschke eigenvalues of the immersion $\mathbf{x}$.

Since the Möbius form Φ is zero, from (27), we know that the para-Blaschke tensor is Codazzi tensor. By the similar proof of Proposition 3.1 in [11], we have the following.

Proposition 15. Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an n -dimensional hypersurface with vanishing Möbius form in a unit sphere $S^{n+1}(1)$. If the multiplicity of a para-Blaschke eigenvalue is constant and greater than 1, then this para-Blaschke eigenvalue is constant along its leaf.

Proof of Theorem 3. Let $\mathbf{x} : M \mapsto S^{n+1}(1)$ be an immersed hypersurface without umbilical points and with vanishing Möbius form Φ and constant Möbius scalar curvature $n(n-1)R$ in $S^{n+1}(1)$. Denote by μ, ν the para-Blaschke eigenvalues

of multiplicities k and $n-k$, respectively, where $1 < k < n-1$. We have from (18) and (24) that

$$k\mu + (n-k)\nu = \text{tr } \mathbf{D} = \text{tr } \mathbf{A} = \frac{1}{2n} (1 + n^2 R). \quad (44)$$

Denote by $\mathcal{D}_\mu$ and $\mathcal{D}_\nu$ the leaf of the para-Blaschke eigenvalues μ and ν , respectively. From Proposition 15, we know that μ is constant on $\mathcal{D}_\mu$. Since the Möbius scalar curvature $n(n-1)R$ is constant, (44) implies that ν is constant on $\mathcal{D}_\mu$. By Proposition 15, again, ν is constant on $\mathcal{D}_\nu$. Therefore, we know that ν is constant on M . By the same assertion, we know that μ is constant on M . Therefore $\mathbf{x} : M \mapsto S^{n+1}(1)$ has two distinct constant para-Blaschke eigenvalues. By Theorem 1, we know that Theorem 3 is proved. $\square$

Next, we consider the case that $\mathbf{x}$ is of two distinct para-Blaschke eigenvalues, one of which is simple. We assume that $\mathbf{x}$ is of vanishing Möbius form and constant Möbius scalar curvature $n(n-1)R$. Without loss of generality, we may assume

$$D_1 = D_2 = \dots = D_{n-1} = \mu, \quad D_n = \nu, \quad (45)$$

where D_i for $i = 1, 2, \dots, n$ are the para-Blaschke eigenvalues of $\mathbf{x}$. Therefore, we know that

$$(n-1)\mu + \nu = \text{tr } \mathbf{D} = \text{tr } \mathbf{A} = \frac{1}{2n} (1 + n^2 R), \quad (46)$$

$$|\mathbf{D}|^2 = (n-1)\mu^2 + \nu^2. \quad (47)$$

Since $\mu - \nu = ((2n\mu - ((1/n) + nR))/2) \neq 0$, we have $2n\mu - ((1/n) + nR) \neq 0$.

We choose a local orthonormal frame $\{E_1, \dots, E_n\}$ such that E_i is a unit principal vector with respect to D_i . From (25), we have

$$D_{ij,k} = E_k(D_i) \delta_{ij} + \Gamma_{ik}^j (D_i - D_j), \quad (48)$$

where Γ_{ik}^j is the Levi-Civita connection for the Möbius metric g given by

$$\omega_{ij} = \sum_k \Gamma_{ik}^j \omega_k, \quad \Gamma_{ik}^j = -\Gamma_{jk}^i. \quad (49)$$

From (27) and (48), we easily see that

$$\begin{aligned} D_{ab,k} &= 0, \quad \forall a \neq b, k, & D_{aa,a} &= D_{nn,a} = 0, \\ E_a(\mu) &= E_a(\nu) = 0, \\ D_{aa,n} &= E_n(\mu), & D_{nn,n} &= E_n(\nu), \end{aligned} \quad (50)$$

where $1 \leq a, b \leq n-1$. From (50) and the formula

$$\begin{aligned} \sum_k D_{an,k} \omega_k &= dD_{an} + \sum_k D_{kn} \omega_{ka} \\ &+ \sum_k D_{ak} \omega_{kn} = (\mu - \nu) \omega_{an}, \end{aligned} \quad (51)$$

we have

$$\omega_{an} = \frac{E_n(\mu)}{\mu - \nu} \omega_a = \frac{2E_n(\mu)}{2n\mu - ((1/n) + nR)} \omega_a, \quad 1 \leq a \leq n-1. \quad (52)$$

Therefore, from (11) we have

$$d\omega_n = \sum_{k=1}^{n-1} \omega_k \wedge \omega_{kn} + \omega_{nn} \wedge \omega_n = 0. \quad (53)$$

We may consider locally that μ is a function of the arc length s of the integral curve of the principal vector field E_n corresponding to the principal curvature ν . Therefore, we may put $\omega_n = ds$. Thus, for $\mu = \mu(s)$, we get

$$d\mu = E_n(\mu) ds, \quad E_n(\mu) = \frac{d\mu}{ds}, \quad (54)$$

and for $1 \leq a \leq n-1$

$$\begin{aligned} \omega_{an} &= \frac{2(d\mu/ds)}{2n\mu - ((1/n) + nR)} \omega_a \\ &= \frac{d \left\{ \log |2n\mu - ((1/n) + nR)|^{1/n} \right\}}{ds} \omega_a. \end{aligned} \quad (55)$$

From (17), (22), and (55), we have for $1 \leq a \leq n-1$ that

$$\begin{aligned} d\omega_{an} &= \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_{bn} - (B_a B_n + A_a + A_n) \omega_a \wedge ds \\ &= \frac{d \left\{ \log |2n\mu - ((1/n) + nR)|^{1/n} \right\}}{ds} \\ &\quad \times \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_b - (B_a B_n + \mu + \nu - \lambda B_a - \lambda B_n) \omega_a \wedge ds. \end{aligned} \quad (56)$$

From (55), we have for $1 \leq a \leq n-1$ that

$$\begin{aligned} d\omega_{an} &= \frac{d^2 \left\{ \log |2n\mu - ((1/n) + nR)|^{1/n} \right\}}{ds^2} ds \wedge \omega_a \\ &\quad + \frac{d \left\{ \log |2n\mu - ((1/n) + nR)|^{1/n} \right\}}{ds} d\omega_a \\ &= \left\{ - \frac{d^2 \left\{ \log |2n\mu - ((1/n) + nR)|^{1/n} \right\}}{ds^2} \right. \\ &\quad \left. + \left[\frac{d \left\{ \log |2n\mu - ((1/n) + nR)|^{1/n} \right\}}{ds} \right]^2 \right\} \omega_a \wedge ds \\ &\quad + \frac{d \left\{ \log |2n\mu - ((1/n) + nR)|^{1/n} \right\}}{ds} \sum_{b=1}^{n-1} \omega_{ab} \wedge \omega_b. \end{aligned} \quad (57)$$

From the above two equalities, we have for $1 \leq a \leq n-1$

$$\begin{aligned} &\frac{d^2 \left\{ \log |2n\mu - ((1/n) + nR)|^{1/n} \right\}}{ds^2} \\ &\quad - \left\{ \frac{d \left\{ \log |2n\mu - ((1/n) + nR)|^{1/n} \right\}}{ds} \right\}^2 \\ &\quad - (B_a B_n + \mu + \nu - \lambda B_a - \lambda B_n) = 0. \end{aligned} \quad (58)$$

We define $\omega = |2n\mu - ((1/n) + nR)|^{-1/n}$ and obtain from the above equation that

$$\frac{d^2 \omega}{ds^2} + \omega (B_a B_n + \mu + \nu - \lambda B_a - \lambda B_n) = 0, \quad 1 \leq a \leq n-1. \quad (59)$$

On the other hand, from (55), we have $\nabla_{E_n} E_n = \sum_{i=1}^n \omega_{ni}(E_n) E_i = 0$. By the definition of geodesic, we know that any integral curve of the principal vector field corresponding to the principal curvature μ is a geodesic. Thus, we see that $\omega(s)$ is a function defined in $(-\infty, +\infty)$ since x is complete and any integral curve of the principal vector field corresponding to μ is a geodesic.

We can prove the following.

Proposition 16. *If $R > ((n-2)/n^2) + ((n-2)/n)|\lambda|$, then the positive function ω is bounded.*

Proof. By (59) and taking summation over a , $1 \leq a \leq n-1$, we have

$$\begin{aligned} (n-1) \frac{d^2 \omega}{ds^2} + \omega \left\{ \sum_{a=1}^{n-1} B_a B_n + (n-1)\mu + (n-1)\nu \right. \\ \left. - \lambda \sum_{a=1}^{n-1} B_a - (n-1)\lambda B_n \right\} = 0; \end{aligned} \quad (60)$$

that is, from (18)

$$\frac{d^2 \omega}{ds^2} + \omega \left\{ -\frac{1}{n-1} B_n^2 - \frac{n-2}{n-1} \lambda B_n + \mu + \nu \right\} = 0. \quad (61)$$

From the definition of ω and (46), we see that

$$\frac{d^2 \omega}{ds^2} + \omega \left\{ -\frac{1}{n-1} B_n^2 - \frac{n-2}{n-1} \lambda B_n + \frac{2-n}{2n} \omega^{-n} + \frac{1}{n^2} + R \right\} = 0, \quad (62)$$

for $2n\mu - ((1/n) + nR) > 0$, or

$$\frac{d^2 \omega}{ds^2} + \omega \left\{ -\frac{1}{n-1} B_n^2 - \frac{n-2}{n-1} \lambda B_n - \frac{2-n}{2n} \omega^{-n} + \frac{1}{n^2} + R \right\} = 0, \quad (63)$$

for $2n\mu - ((1/n) + nR) < 0$.

From (18), we have $B_n^2 = (\sum_{i=1}^{n-1} B_i)^2 \leq (n-1) \sum_{i=1}^{n-1} B_i^2 = (n-1)((n-1)/n - B_n^2)$. Thus, we have $B_n^2 \leq ((n-1)/n)^2$. We infer that

$$-\frac{1}{n-1} B_n^2 - \frac{n-2}{n-1} \lambda B_n \geq -\frac{n-1}{n^2} - \frac{n-2}{n} |\lambda|. \quad (64)$$

From (63) and (64), we have

$$\frac{d^2\bar{\omega}}{ds^2} + \bar{\omega} \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) + \frac{2-n}{2n} \bar{\omega}^{1-n} \leq 0, \quad (65)$$

for $2n\mu - ((1/n) + nR) > 0$, or

$$\frac{d^2\bar{\omega}}{ds^2} + \bar{\omega} \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) - \frac{2-n}{2n} \bar{\omega}^{1-n} \leq 0, \quad (66)$$

for $2n\mu - ((1/n) + nR) < 0$.

Since $d\bar{\omega}/ds$ is existing, we know that for some fixed $s_0 \in (-\infty, +\infty)$, $\bar{\omega}(s_0)$ and $d\bar{\omega}(s_0)/ds$ are two constants. Multiplying (65) and (66) by $2d\bar{\omega}/ds$, we have two cases.

(i) If $d\bar{\omega}/ds \geq 0$, for any $s \in (-\infty, +\infty)$, integrating from s_0 to s , then

$$\int_{s_0}^s \left\{ 2 \frac{d\bar{\omega}}{ds} \frac{d^2\bar{\omega}}{ds^2} + 2 \frac{d\bar{\omega}}{ds} \bar{\omega} \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) + 2 \frac{d\bar{\omega}}{ds} \frac{2-n}{2n} \bar{\omega}^{1-n} \right\} ds \leq 0, \quad (67)$$

for $2n\mu - ((1/n) + nR) > 0$, or

$$\int_{s_0}^s \left\{ 2 \frac{d\bar{\omega}}{ds} \frac{d^2\bar{\omega}}{ds^2} + 2 \frac{d\bar{\omega}}{ds} \bar{\omega} \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) - 2 \frac{d\bar{\omega}}{ds} \frac{2-n}{2n} \bar{\omega}^{1-n} \right\} ds \leq 0, \quad (68)$$

for $2n\mu - ((1/n) + nR) < 0$.

(ii) If $d\bar{\omega}/ds \leq 0$, for any $s \in (-\infty, +\infty)$, integrating from s to s_0 , then

$$\int_s^{s_0} \left\{ 2 \frac{d\bar{\omega}}{ds} \frac{d^2\bar{\omega}}{ds^2} + 2 \frac{d\bar{\omega}}{ds} \bar{\omega} \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) + 2 \frac{d\bar{\omega}}{ds} \frac{2-n}{2n} \bar{\omega}^{1-n} \right\} ds \geq 0, \quad (69)$$

for $2n\mu - ((1/n) + nR) > 0$, or

$$\int_s^{s_0} \left\{ 2 \frac{d\bar{\omega}}{ds} \frac{d^2\bar{\omega}}{ds^2} + 2 \frac{d\bar{\omega}}{ds} \bar{\omega} \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) - 2 \frac{d\bar{\omega}}{ds} \frac{2-n}{2n} \bar{\omega}^{1-n} \right\} ds \geq 0, \quad (70)$$

for $2n\mu - ((1/n) + nR) < 0$.

For the two cases above, we all have

$$\begin{aligned} & \left(\frac{d\bar{\omega}(s)}{ds} \right)^2 + \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) \bar{\omega}^2(s) + \frac{1}{n} \bar{\omega}^{2-n}(s) \\ & \leq \left(\frac{d\bar{\omega}(s_0)}{ds} \right)^2 + \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) \bar{\omega}^2(s_0) \\ & \quad + \frac{1}{n} \bar{\omega}^{2-n}(s_0), \end{aligned} \quad (71)$$

for $2n\mu - ((1/n) + nR) > 0$, or

$$\begin{aligned} & \left(\frac{d\bar{\omega}(s)}{ds} \right)^2 + \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) \bar{\omega}^2(s) - \frac{1}{n} \bar{\omega}^{2-n}(s) \\ & \leq \left(\frac{d\bar{\omega}(s_0)}{ds} \right)^2 + \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) \bar{\omega}^2(s_0) \\ & \quad - \frac{1}{n} \bar{\omega}^{2-n}(s_0), \end{aligned} \quad (72)$$

for $2n\mu - ((1/n) + nR) < 0$.

Therefore, we have

$$\begin{aligned} & \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) \bar{\omega}^2(s) + \frac{1}{n} \bar{\omega}^{2-n}(s) \\ & \leq \left(\frac{d\bar{\omega}(s_0)}{ds} \right)^2 + \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) \bar{\omega}^2(s_0) \\ & \quad + \frac{1}{n} \bar{\omega}^{2-n}(s_0), \end{aligned} \quad (73)$$

for $2n\mu - ((1/n) + nR) > 0$, or

$$\begin{aligned} & \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) \bar{\omega}^2(s) - \frac{1}{n} \bar{\omega}^{2-n}(s) \\ & \leq \left(\frac{d\bar{\omega}(s_0)}{ds} \right)^2 + \left(R - \frac{n-2}{n^2} - \frac{n-2}{n} |\lambda| \right) \bar{\omega}^2(s_0) \\ & \quad - \frac{1}{n} \bar{\omega}^{2-n}(s_0), \end{aligned} \quad (74)$$

for $2n\mu - ((1/n) + nR) < 0$.

Since $R > ((n-2)/n^2) + ((n-2)/n)|\lambda|$, from (73) and (74), we see that the positive function $\bar{\omega}$ is bounded from above. Proposition 16 is proved. $\square$

Proof of Theorem 4. (1) Since the Möbius sectional curvature of $\mathbf{x}$ is nonnegative, from (22), we have $R_{ijij} = B_i B_j + A_i + A_j = B_i B_j + D_i - \lambda B_i + D_j - \lambda B_j \geq 0$ for $i \neq j$. Therefore, we know that $B_a B_n + \mu + \nu - \lambda B_a - \lambda B_n \geq 0$, $1 \leq a \leq n-1$. From (59), we have $d^2\bar{\omega}/ds^2 \leq 0$. Thus, $d\bar{\omega}/ds$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, as observed by Wei [12], $\bar{\omega}(s)$ must be monotonic when s tends to infinity. Since we assume that $R > ((n-2)/n^2) + ((n-2)/n)|\lambda|$, from Proposition 16, we know that the positive function $\bar{\omega}(s)$ is bounded. Since $\bar{\omega}(s)$ is bounded and monotonic when s tends to infinity, we know that both $\lim_{s \rightarrow -\infty} \bar{\omega}(s)$ and $\lim_{s \rightarrow +\infty} \bar{\omega}(s)$ exist and then we get

$$\lim_{s \rightarrow -\infty} \frac{d\bar{\omega}(s)}{ds} = \lim_{s \rightarrow +\infty} \frac{d\bar{\omega}(s)}{ds} = 0. \quad (75)$$

By the monotonicity of $d\bar{\omega}(s)/ds$, we have $d\bar{\omega}(s)/ds \equiv 0$ and $\bar{\omega}(s) = \text{constant}$. From $\bar{\omega} = |2n\mu - ((1/n) + nR)|^{-1/n}$, we know that μ is constant, and therefore ν is also constant

from (46). We obtain that $\mathbf{x} : M \mapsto S^{n+1}(1)$ has two distinct constant para-Blaschke eigenvalues. From Theorem 1 and Examples 8–12, we know that the result (1) of Theorem 4 is proved.

(2) From (46) and (47), we have

$$\begin{aligned} n(n-1)\mu^2 - \frac{n-1}{n}(1+n^2R)\mu \\ + \frac{1}{4n^2}(1+n^2R)^2 - |\mathbf{D}|^2 = 0. \end{aligned} \quad (76)$$

Putting $\widetilde{D}_{ij} = D_{ij} - (1/n)\text{tr } \mathbf{D}\delta_{ij}$, we have $|\mathbf{D}|^2 = |\widetilde{\mathbf{D}}|^2 + (1/n)(\text{tr } \mathbf{D})^2 = |\widetilde{\mathbf{D}}|^2 + (1/4n^3)(1+n^2R)^2$. Therefore, (76) can be rewritten as

$$\begin{aligned} n(n-1)\mu^2 - \frac{n-1}{n}(1+n^2R)\mu \\ + \frac{n-1}{4n^3}(1+n^2R)^2 - |\widetilde{\mathbf{D}}|^2 = 0. \end{aligned} \quad (77)$$

We have from (77)

$$\begin{aligned} \mu &= \frac{1}{2n^2}(1+n^2R) + \frac{|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}} \quad \text{or,} \\ \mu &= \frac{1}{2n^2}(1+n^2R) - \frac{|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}}. \end{aligned} \quad (78)$$

From (46), we obtain

$$\mu + \nu = \frac{1}{n^2}(1+n^2R) - \frac{(n-2)|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}}, \quad (79)$$

or

$$\mu + \nu = \frac{1}{n^2}(1+n^2R) + \frac{(n-2)|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}}. \quad (80)$$

If there exists a point p on M such that (79) and (80) hold at p ; that is, we have $|\widetilde{\mathbf{D}}| = 0$ at p . From $|\widetilde{\mathbf{D}}|^2 = \sum_i (\widetilde{D}_{ii})^2 = \sum_i (D_{ii} - (1/n)\text{tr } \mathbf{D})^2 = 0$, we have $D_{ii} = (1/n)\text{tr } \mathbf{D}$ at p ; this is in contradiction to the assumption that M has two distinct para-Blaschke eigenvalues. Therefore, we only consider two cases.

Case (i). If (79) holds on M , by (64) and (5), we have

$$\begin{aligned} -\frac{1}{n-1}B_n^2 - \frac{n-2}{n-1}\lambda B_n + \mu + \nu \\ \geq -\frac{n-1}{n^2} - \frac{n-2}{n}|\lambda| \\ + \frac{1}{n^2}(1+n^2R) - \frac{(n-2)|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}} \\ = \frac{1}{n}\left(nR - \frac{n-2}{n} - (n-2)|\lambda|\right) - \frac{(n-2)|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}} \\ = \frac{n-2}{\sqrt{n(n-1)}}\left\{\sqrt{\frac{n-1}{n}}\left(\frac{n}{n-2}R - \frac{1+n|\lambda|}{n}\right) - |\widetilde{\mathbf{D}}|\right\} \geq 0. \end{aligned} \quad (81)$$

Case (ii). If (80) holds on M , by (64) and (5), we have

$$\begin{aligned} -\frac{1}{n-1}B_n^2 - \frac{n-2}{n-1}\lambda B_n + \mu + \nu \\ \geq -\frac{n-1}{n^2} - \frac{n-2}{n}|\lambda| \\ + \frac{1}{n^2}(1+n^2R) + \frac{(n-2)|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}} \\ = \frac{1}{n}\left(nR - \frac{n-2}{n} - (n-2)|\lambda|\right) + \frac{(n-2)|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}} \\ \geq \frac{1}{n}\sqrt{\frac{n}{n-1}}(n-2)|\widetilde{\mathbf{D}}| + \frac{(n-2)|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}} = \frac{2(n-2)|\widetilde{\mathbf{D}}|}{\sqrt{n(n-1)}} \geq 0. \end{aligned} \quad (82)$$

From (61), (81), and (82), we all have $d^2\omega/ds^2 \leq 0$. Thus, $d\omega/ds$ is a monotonic function of $s \in (-\infty, +\infty)$. Therefore, we have the following: $\omega(s)$ must be monotonic when s tends to infinity. Since $|\widetilde{\mathbf{D}}| \neq 0$ and $|\widetilde{\mathbf{D}}| \leq \sqrt{(n-1)/n}\{(n/(n-2))R - ((1+n|\lambda|)/n)\}$, we have $R > ((n-2)/n^2) + ((n-2)/n)|\lambda|$. From Proposition 16, we know that the positive function $\omega(s)$ is bounded. Therefore, by the similar assertion in the proof of (1), we obtain that $\mathbf{x} : M \mapsto S^{n+1}(1)$ has two distinct constant para-Blaschke eigenvalues. From Theorem 1 and Examples 8–12, we know that Theorem 4 is proved. $\square$

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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