Research Article

Linear Relative n -Widths for Linear Operators Preserving an Intersection of Cones

S. P. Sidorov

Department of Mechanics and Mathematics, Saratov State University, 410012 Saratov, Russia

Correspondence should be addressed to S. P. Sidorov; sidorovsp@yahoo.com

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We introduce the definition of linear relative n -width and find estimates of linear relative n -widths for linear operators preserving the intersection of cones of p -monotonicity functions.

1. Introduction

In various applications of CAGD (computer-aided geometric design) it is necessary to approximate functions preserving its properties such as monotonicity, convexity, and concavity. The survey of the theory of shape-preserving approximation can be found in [1].

Let X be a normed linear space and let V be a cone in X (a convex set, closed under nonnegative scalar multiplication). It is said that $f \in X$ has the shape in the sense of V whenever $f \in V$. Let X_n be a n -dimensional subspace of X . Classical problems of approximation theory are of interest in the theory of shape-preserving approximation as well:

- (1) problems of existence, uniqueness, and characterization of the best shape-preserving approximation $g^* \in X_n \cap V$ of $f \in V$ defined by

$$\|f - g^*\|_X = \inf_{g \in X_n \cap V} \|f - g\|_X; \quad (1)$$

- (2) estimation of the deviation of $A \cap V$ from $X_n \cap V$, that is,

$$E(A \cap V; X_n \cap V)_X := \sup_{f \in A \cap V} \inf_{g \in X_n \cap V} \|f - g\|_X; \quad (2)$$

- (3) estimation of relative n -width of A in X with the constraint V

$$d_n(A \cap V, V)_X := \inf_{X_n} E(A \cap V; X_n \cap V)_X, \quad (3)$$

the leftmost infimum taken over all affine subsets X_n of dimension $\leq n$, such that $X_n \cap V \neq \emptyset$;

- (4) estimation of linear relative n -widths with the constraint V in X .

The notion of relative n -width (3) was first introduced in 1984 by Konovalov [2]. Though he considered a problem not connected with preserving shapes, the concept of relative n -width arises in the theory of shape-preserving approximation naturally. Of course, it is impossible to obtain $d_n(A \cap V, V)_X$ and determine optimal subspaces X_n (if they exist) for all A, V, X . Nevertheless, some estimates of relative shape-preserving n -widths have been obtained in papers [3–5]. Estimates of relative (not necessary shape-preserving) widths have been obtained in works [6–11].

Let A be a subset of X and let $L : X \rightarrow X$ be a linear operator. The value

$$e(A, L) := \sup_{f \in A} \|f - Lf\|_X = \sup_{f \in A} \|(I - L)f\|_X \quad (4)$$

is the error of approximation of the identity operator I by the operator L on the set A .

Let V be a cone in X , $V \neq \emptyset$. We will say that the operator L preserves the shape in the sense of V , if $L(V) \subset V$. One might consider the problem of finding (if exists) a linear operator of finite rank n , which gives the minimal error of approximation of identity operator on some set over all finite rank n linear operators L preserving the shape in the sense V . It leads us naturally to the notion of *linear relative n -width*.

In this paper we introduce the definition of linear relative n -width and find estimates of linear relative n -widths for linear operators preserving an intersection of cones of p -monotonicity functions.

2. Notations and Definitions

Let $B[0, 1]$ denote the space of all real-valued bounded function, defined on $[0, 1]$, with the uniform norm on $[0, 1]$, and $\|f\|_{B[0,1]} = \sup_{x \in [0,1]} |f(x)|$. Denote by $B^k[0, 1]$, $k \geq 0$, the space of all real-valued functions, whose k th derivative is bounded on $[0, 1]$, endowed with the sup-norm

$$\|f\|_{B^k[0,1]} = \sum_{0 \leq i \leq k} \frac{1}{i!} \sup_{x \in [0,1]} |D^i f(x)|, \quad (5)$$

where D^i denotes the i th differential operator, $D^i f(x) = d^i f(x)/dx^i$, and $D^0 = I$ is the identity operator, and the derivatives are taken from the right at 0 and from the left at 1.

Denote by $C^k[0, 1]$, $k \geq 0$, the space of all real-valued and k -times continuously differentiable functions defined on $[0, 1]$ equipped with sup-norm (5).

A continuous function $f : [0, 1] \rightarrow \mathbb{R}$ is said to be k -monotone, $k \geq 1$, on $[0, 1]$ if and only if for all choices of $k+1$ distinct $t_0, \dots, t_k$ in $[0, 1]$ the inequality $[t_0, \dots, t_k]f \geq 0$ holds, where $[t_0, \dots, t_k]f$ denotes the k th divided difference of f at $0 \leq t_0 < t_1 < \dots < t_k \leq 1$. Note that 2-monotone functions are just convex functions.

Let Δ^p denote the set of all p -monotone functions defined on $[0, 1]$ and $\Delta^0 := \{f \in C[0, 1] : f(t) \geq 0, t \in [0, 1]\}$ denotes the cone of all nonnegative functions. If $f \in C^p[0, 1]$, then $f \in \Delta^p$ if and only if $f^{(p)}(t) \geq 0, t \in [0, 1]$. It is said that a linear operator L of $C[0, 1]$ into $C[0, 1]$ preserves p -monotonicity, if $L(\Delta^p) \subset \Delta^p$.

Let $\sigma = (\sigma_i)_{i \geq 0}$ be a sequence of real numbers, $\sigma_i \in \{-1, 0, 1\}$, and let h, k be two integers such that $0 \leq h \leq k$ and $\sigma_h \cdot \sigma_k \neq 0$. Denote

$$\Delta^{h,k}(\sigma) := \{f \in C[0, 1] : \sigma_p f \in \Delta^p, h \leq p \leq k\}. \quad (6)$$

The cone $\Delta^{h,k}(\sigma)$ is the intersection of cones of p -monotonicity functions with $h \leq p \leq k$ taken with signs σ_p ; that is, $\Delta^{h,k}(\sigma) = \cap_{p=h}^k \sigma_p \Delta^p$.

Denote $e_i(x) = x^i$, $i = 0, 1, \dots$. Denote $\Pi_m := \text{span}\{e_0, e_1, \dots, e_m\}$ and $P_m := \{f \in \Pi_m : \|f\|_{B^m[0,1]} \leq 1\}$.

Recall that a linear operator mapping $C[0, 1]$ into a linear space of finite dimension n is called an *operator of finite rank n* .

3. The Example of Linear Operator Preserving the Cone $\Delta^{h,k}(\sigma)$

This section gives an example of linear operator of finite rank n preserving the cone $\Delta^{h,k}(\sigma)$ in the case $\sigma_k = \sigma_{k-2} = 1$.

Denote $s(i) := \min\{s > i : \sigma_s \neq 0\}$ for $i \in \{h, \dots, k-1\}$. Denote $t_j = j/(n-1)$, $j = 0, 1, \dots, n-1$. Let $(\delta_p)_{p=0}^{k-2}$ be the binary sequence defined by

$$\delta_p = \begin{cases} 0, & \text{if } p = k-2 \text{ or if } \sigma_p \sigma_{s(p)} \neq -1, \quad p \leq k-3 \\ 1, & \text{if } \sigma_p \sigma_{s(p)} = -1, \quad p \leq k-3. \end{cases} \quad (7)$$

Let $k, n \in \mathbb{N}$, $n \geq k+2$, and $\Lambda_n^{h,k} : C^{k-2}[0, 1] \rightarrow C^{k-2}[0, 1]$ be the linear operator defined by

$$\Lambda_n^{h,k} f(x) = \begin{cases} \sum_{l=0}^{k-2} \frac{1}{l!} e_l(x) \times \left(D^l f(t_0) - \delta_l \int_{t_0}^{t_1} D^{l+1} (\Lambda_n^{h,k} f - f)(t) dt \right) + \frac{(n-1) e_{k-1}(x)}{(k-1)!} \times [D^{k-2} f(t_1) - D^{k-2} f(t_0)], \\ \quad \text{if } x \in [t_0, t_1], \\ \sum_{l=0}^{k-2} \frac{1}{l!} (x - t_j)^l \times \left(D^l \Lambda_n^{h,k} f(t_j) - \delta_l \int_{t_j}^{t_{j+1}} D^{l+1} (\Lambda_n^{h,k} f - f)(t) dt \right) + \frac{(n-1) e_{k-1}(x - t_j)}{(k-1)!} \times [D^{k-2} f(t_{j+1}) - D^{k-2} f(t_j)], \\ \quad \text{if } x \in (t_j, t_{j+1}], \quad j = 1, 2, \dots, n-2. \end{cases} \quad (8)$$

Theorem 1. Consider that $\Lambda_n^{h,k} : C^{k-2}[0, 1] \rightarrow C^{k-2}[0, 1]$ is a continuous linear operator of finite rank n , such that $\Lambda_n^{h,k}(\Delta^{h,k}(\sigma)) \subset \Delta^{h,k}(\sigma)$ and for all $0 \leq i \leq k-2$ there exists $c > 0$ not depending on n such that

$$\sup_{f \in P_k} \|D^i (\Lambda_n^{h,k} f) - D^i f\|_{B[0,1]} \leq cn^{-2}. \quad (9)$$

Proof. Since $D^{k-2} \Lambda_n^{h,k} f$ is a piecewise linear function on $[0, 1]$ with the set of breakpoints $\{(t_j, D^{k-2} f(t_j))\}_{j=0, \dots, n-1}$, then for every f such that $D^{k-2} f \geq 0$ the inequality $D^{k-2} \Lambda_n^{h,k} f \geq 0$ holds. Moreover, $\Lambda_n^{h,k}(\sigma_p \Delta^p) \subset \sigma_p \Delta^p$ for $p = k+1, k+2$.

Note that for every $f \in \Delta^k$ we have $D^{k-2}(\Lambda_n^{h,k} f - f) \geq 0$ (since $D^{k-2} f$ is convex and $D^{k-2} \Lambda_n^{h,k} f$ is a piecewise linear interpolation).

Let $f \in \Delta^{h,k}(\sigma)$ and suppose (by induction) that for $i \geq h+1$ the inequality

$$\sigma_l D^l (\Lambda_n^{h,k} f - f) \geq 0, \quad l = i+1, \dots, k-2, \quad (10)$$

holds on $[t_0, t_1]$. For any $x \in [t_0, t_1]$ we have

$$\begin{aligned} D^i (\Lambda_n^{h,k} f - f)(x) &= D^i (\Lambda_n^{h,k} f - f)(t_0) \\ &\quad + \int_{t_0}^x D^{i+1} (\Lambda_n^{h,k} f - f)(z) dz. \end{aligned} \quad (11)$$

Consider the following three cases.

(1) If $\sigma_i = 0$, then $\delta_i = 0$ and it follows from $D^i \Lambda_n^{h,k} f(0) = D^i f(0)$ that

$$\operatorname{sgn} D^i (\Lambda_n^{h,k} f - f) = \operatorname{sgn} D^{i+1} (\Lambda_n^{h,k} f - f). \quad (12)$$

(2) If $\sigma_i \neq 0$ and $\sigma_i = \sigma_{s(i)}$, then it follows from (11) that

$$\begin{aligned} \operatorname{sgn} D^i (\Lambda_n^{h,k} f - f) &= \operatorname{sgn} D^{i+1} (\Lambda_n^{h,k} f - f) = \dots \\ &= \operatorname{sgn} D^{s(i)} (\Lambda_n^{h,k} f - f) = \sigma_i. \end{aligned} \quad (13)$$

(3) If $\sigma_i \neq 0$ and $\sigma_i = -\sigma_{s(i)}$ then $\operatorname{sgn} D^{i+1} (\Lambda_n^{h,k} f - f) = \sigma_{s(i)}$ and

$$\begin{aligned} D^i (\Lambda_n^{h,k} f - f)(x) &= D^i (\Lambda_n^{h,k} f - f)(t_0) \\ &\quad + \int_{t_0}^x D^{i+1} (\Lambda_n^{h,k} f - f)(z) dz \\ &= - \int_{t_0}^{t_1} D^{i+1} (\Lambda_n^{h,k} f - f)(z) dz \\ &\quad + \int_{t_0}^x D^{i+1} (\Lambda_n^{h,k} f - f)(z) dz \\ &= - \int_x^{t_1} D^{i+1} (\Lambda_n^{h,k} f - f)(z) dz \\ &= \sigma_i \int_x^{t_1} \sigma_{s(i)} D^{i+1} (\Lambda_n^{h,k} f - f)(z) dz. \end{aligned} \quad (14)$$

Using (10) we get $\sigma_i D^i (\Lambda_n^{h,k} f - f) \geq 0$ on $[t_0, t_1]$, or $\sigma_i D^i \Lambda_n^{h,k} f \geq \sigma_i D^i f$ on $[t_0, t_1]$; that is, $\Lambda_n^{h,k} f \in \sigma_i \Delta^i[t_0, t_1]$ if $f \in \sigma_i \Delta^i[t_0, t_1]$.

It can be shown analogously (by induction with j as induction variable) that $\Lambda_n^{h,k} f \in \sigma_i \Delta^i[t_j, t_{j+1}]$ if $f \in \sigma_i \Delta^i[t_j, t_{j+1}]$ for any $j = 1, \dots, n-2$. Therefore $\Lambda_n^{h,k}(\sigma_i \Delta^i[0, 1]) \subset \sigma_i \Delta^i[0, 1]$.

After completing induction steps with $i = k-3, k-4, \dots, h$ we can conclude that $\Lambda_n^{h,k}(\Delta^{h,k}(\sigma)) \subset \Delta^{h,k}(\sigma)$.

It can be easily verified that

(1) $\Lambda_n^{h,k} e_p = e_p$ for all $p = 0, 1, \dots, h-1$ (since $\pm e_p \in \Delta^{h,k}(\sigma)$ for all $p = 0, 1, \dots, h-1$);

(2) if $x \in [t_j, t_{j+1})$ for some $0 \leq j \leq n-2$, then

$$D^{k-2} (\Lambda_n^{h,k} e_k - e_k)(x) = \frac{k! (t_{j+1} - x)(x - t_j)}{2}. \quad (15)$$

It follows from (15) that the inequality

$$0 \leq D^{k-2} (\Lambda_n^{h,k} e_k - e_k) \leq \frac{k!}{8} (n-1)^{-2}, \quad (16)$$

holds on $[0, 1]$.

Suppose (by induction) that for a fixed $i \geq h+1$ there exist $c_l > 0$, $l = i+1, \dots, k-2$, such that for every $x \in [0, 1]$

$$\begin{aligned} 0 \leq \sigma_l D^l (\Lambda_n^{h,k} f - f)(x) &\leq c_l (n-1)^{-2}, \\ l &= i+1, \dots, k-2. \end{aligned} \quad (17)$$

It follows from

$$\begin{aligned} D^i (\Lambda_n^{h,k} e_k - e_k)(0) &= D^i e_k(0) \\ &\quad - \delta_i \int_0^1 D^{i+1} (\Lambda_n^{h,k} e_k - e_k)(t) dt \end{aligned} \quad (18)$$

that for $x \in [0, 1]$

$$\begin{aligned} |D^i (\Lambda_n^{h,k} e_k - e_k)(x)| &= \left| D^i (\Lambda_n^{h,k} e_k - e_k)(0) + \int_0^x D^{i+1} (\Lambda_n^{h,k} e_k - e_k)(t) dt \right| \\ &= 2 \left| \int_0^1 D^{i+1} (\Lambda_n^{h,k} e_{k+2} - e_{k+2})(t) dt \right| \\ &\leq 2 \frac{c_{i+1}}{(n-1)^2} \frac{1}{(k-i)!}. \end{aligned} \quad (19)$$

We have used the fact that if $f \in C[0, 1]$, $f \geq 0$, and there exists a constant $a \in \mathbb{R}$ such that $f \leq a$ on $[0, 1]$, then for every $p \in \mathbb{N}$

$$0 \leq \int_0^x \int_0^{t_{p-1}} \dots \int_0^{t_1} f(t_1) dt_1 \dots dt_p \leq a \frac{x^p}{p!}. \quad (20)$$

Then (19) implies with $c_i = 2(c_{i+1}/(k-i)!)$

$$\begin{aligned} \|D^i (\Lambda_n^{h,k} e_{k+2}) - D^i e_{k+2}\|_{B[0,1]} &\leq c_i (n-1)^{-2}, \\ \|\Lambda_n^{h,k} e_k - e_k\|_{B^{k-2}[0,1]} &:= \sum_{i=0}^{k-2} \frac{1}{i!} \|D^i (\Lambda_n^{h,k} e_k - e_k)\|_{B[0,1]} \\ &\leq (n-1)^{-2} \sum_{i=0}^{k-2} \frac{c_i}{i!}. \end{aligned} \quad (21)$$

Direct verification shows that $\Lambda_n^{h,k} e_p = e_p$ for all $p = h, \dots, k-1$.

Thus, (9) is verified and theorem is proved. $\square$

4. The Main Result

Let X be a linear normed space. Recall that linear n -width of a set $A \subset X$ in X is defined by [12]

$$\delta_n(A)_X := \inf_{L_n} e(A, L_n), \quad (22)$$

where infimum is taken over all linear continuous operators $L_n : X \rightarrow X$ of finite rank n .

Dealing with the problem of approximation of smooth functions by some class of linear operators, we may find that operators of this class have some property which limits the degree of approximation of smooth functions by operators of this class. Let us cite the well-known instances. By definition, every positive linear operator is shape-preserving with respect to the cone of all nonnegative functions Δ^0 . It was shown by Korovkin [13] that if linear polynomial operator preserves positiveness, the degree of approximation of continuous functions by this operator is low. He proved that the order of approximation by positive linear polynomial operators of degree n cannot be better than n^{-2} in $C[0, 1]$ even for the functions $1, x$, and x^2 . Moreover, Videnskiĭ [14] has shown that the result of [13] does not depend on the properties of the polynomials but rather on the limitation of dimension.

To determine the negative impact of the property of shape-preserving on the order of linear approximation we introduce the following definition based on ideas of Korovkin.

Let X be a linear normed space. Let V be a cone in X and let $A \subset X$ be a set and $A \cap V \neq \emptyset$.

Definition 2. Let one define Korovkin linear relative n -width of set A in X with the constraint V by

$$\delta_n(A, V)_X := \inf_{L_n(V) \subset V} e(A, L_n), \quad (23)$$

where infimum is taken over all linear continuous operators $L_n : X \rightarrow X$ of finite rank n satisfying $L_n(V) \subset V$.

Note that if $V = X$ then $\delta_n(A, V)_X = \delta_n(A)_X$.

If we compare the value of Korovkin linear relative n -width $\delta_n(A, V)_X$ of set A in X with the constraint V to the value of linear n -width $\delta_n(A)_X$ of the set A in X we can evaluate the negative impact of the shape-preserving constraint $L_n(V) \subset V$ on the intrinsic error of approximation by means of the *shape-preserving* linear operators of finite rank n compared to the error of *unconstrained* linear finite-rank approximation on the same set.

This section examines approximation properties of linear finite dimensional operators L_n preserving the cone $\Delta^{h,k}(\sigma)$, that is, such that

$$L_n(\Delta^{h,k}(\sigma)) \subset \Delta^{h,k}(\sigma). \quad (24)$$

In this section we will find estimates of Korovkin linear relative n -widths for linear operators preserving the cone $\Delta^{h,k}(\sigma)$ in the space $C^{k-2}[0, 1]$, that is, estimates of $\delta_n(A, \Delta^{h,k}(\sigma))_{C^{k-2}[0, 1]}$.

First we will prove a preliminary result.

Lemma 3. Let $L_n : C^{k-2}[0, 1] \rightarrow B^{k-2}[0, 1]$ be a linear operator of finite rank n , $n > k+2$, satisfying (24). Then there exists $c > 0$ such that

$$\sup_{x \in [0, 1]} |D^{k-2} L_n e_p(x) - D^{k-2} e_p(x)| \geq cn^{-2} \quad \forall h \leq p \leq k. \quad (25)$$

Proof. (1) It is sufficient to show that (25) holds for any linear operator $L_n : C^{k-2}[0, 1] \rightarrow B^{k-2}[0, 1]$ of finite rank n , such that $L_n(\Delta^{h,k}(\sigma)) \subset \sigma_{k-2} \Delta^{k-2}$. Denote $Z = \{z_i\}$, where $z_i = i/n$, $i = 0, \dots, n$. Let $\{v_1, \dots, v_n\}$ be the system of functions generating the linear space $\{D^{k-2} L_n f : f \in C[0, 1]\}$; that is, $\text{span}\{v_1, \dots, v_n\} = \{D^{k-2} L_n f : f \in C[0, 1]\}$.

Consider the matrix $A = (v_j(z_i))_{j=1, \dots, n}^{i=0, \dots, n}$. Rank of matrix A is not equal to 0, $\text{rank } A \neq 0$. Indeed, if $\text{rank } A = 0$, then $D^{k-2} L_n f(z_i) = \sum_{j=1}^n a_j(f) v_j(z_i) = 0$, $i = 0, \dots, n$, for every $f \in C^{k-2}[0, 1]$, which is impossible.

Take a nontrivial vector $\delta = (\delta_0, \dots, \delta_n) \in R^{n+1}$, such that

$$\sum_{i=0}^n |\delta_i| = 1, \quad \sum_{i=0}^n \delta_i v_j(z_i) = 0, \quad j = 1, \dots, n. \quad (26)$$

Let $F_{k-2}(Z, \delta)$ denote the set of functions $f \in C^{k-2}[0, 1]$, such that $D^{k-2} f(z_i) = \text{sgn } \delta_i$, $i = 0, \dots, n$. Let a function $g \in F(Z, \delta)$ be such that

$$\sup_{0 \leq j \leq n-1} \sup_{x \in (z_j, z_{j+1})} [z_j, x, z_{j+1}] (D^{k-2} g) = \lambda(Z, \delta), \quad (27)$$

where

$$\lambda(Z, \delta) := \inf_{f \in F_{k-2}(Z, \delta)} \max_{0 \leq j \leq n-1} \sup_{x \in [z_j, z_{j+1}]} [z_j, x, z_{j+1}] (D^{k-2} f), \quad (28)$$

where points z_j, x , and z_{j+1} are arranged in ascending order.

Denote by D_n the set of all vectors $\delta = (\delta_i)_{i=0}^n$ with $\delta_i \in \{-1, 0, 1\}$.

It follows from Theorem 2.1 in [15] that

$$\max_{0 \leq j \leq n-1} \sup_{x \in [z_j, z_{j+1}]} [z_j, x, z_{j+1}] (D^{k-2} g) \leq \lambda(Z), \quad (29)$$

where

$$\lambda(Z) := \sup_{\delta \in D_n} \lambda(\delta, Z) \leq \frac{(3n^2)}{2}. \quad (30)$$

It follows from $D^{k-2} L_n g \in \text{span}\{v_1, \dots, v_n\}$ that $\sum_{i=1}^n \delta_i D^{k-2} L_n g(z_i) = 0$. Then

$$\begin{aligned} 1 &= \sum_{i=0}^n |\delta_i| = \sum_{i=0}^n \delta_i D^{k-2} g(z_i) \\ &= \sum_{i=0}^n \delta_i (D^{k-2} g(z_i) - D^{k-2} L_n g(z_i)) \\ &\leq \sum_{i=0}^n |\delta_i| |D^{k-2} L_n g(z_i) - D^{k-2} g(z_i)| \\ &\leq \|D^{k-2} L_n g - D^{k-2} g\|_{B[0, 1]}. \end{aligned} \quad (31)$$

(2) Given $z \in [0, 1]$, we have

$$\begin{aligned}
 & \left| D^{k-2} L_n g(z) - D^{k-2} g(z) \right| \\
 &= \left| D^{k-2} L_n \left(g - \frac{e_{k-2}}{(k-2)!} D^{k-2} g(z) \right) (z) \right. \\
 &\quad \left. + \frac{1}{(k-2)!} D^{k-2} g(z) (D^{k-2} L_n e_{k-2} - D^{k-2} e_{k-2})(z) \right| \\
 &= \left| D^{k-2} L_n \left(g - \frac{e_{k-2}}{(k-2)!} D^{k-2} g(z) \right) (z) \right| \\
 &\quad + \frac{1}{(k-2)!} \left| D^{k-2} g(z) \right| \left| (D^{k-2} L_n e_{k-2} - D^{k-2} e_{k-2})(z) \right| \\
 &\leq \left| D^{k-2} L_n \left(g - \frac{e_{k-2}}{(k-2)!} D^{k-2} g(z) \right) (z) \right| \\
 &\quad + \frac{1}{(k-2)!} \left| (D^{k-2} L_n e_{k-2} - D^{k-2} e_{k-2})(z) \right|, \tag{32}
 \end{aligned}$$

since $\|D^{k-2} g\|_{B[0,1]} = 1$.

Denote $g_z = g - (e_{k-2}/(k-2)!)D^{k-2}g(z)$. It follows from [16] that there exist functions $\varphi_{z,j} \in \text{span}\{e_h, \dots, e_k\}$, $j = 1, 2$, such that

$$\varphi_{z,j} + (-1)^j g_z \in \Delta^{h,k}(\sigma), \quad j = 1, 2, \tag{33}$$

and such that

(1) $\varphi_{z,j} \in \Delta^{h,k}(\sigma^{(j)})$, where $\sigma^{(j)} := (\sigma_i^{(j)})_{i \geq 0}$ with $\sigma_i^{(j)} = \sigma_i$ for $i \neq j$ and $\sigma_j^{(j)} = 0$;

(2) $D^{k-2} \varphi_{z,j}(z) = 0 < D^{k-2} \varphi_{z,j}(x)$ for all $x \in [0, 1] \setminus \{z\}$;

(3) $D^k \varphi_{z,j} = 2\sigma_k \cdot \max_{0 \leq j \leq n-1} \sup_{x \in [z_j, z_{j+1}]} [z_j, x, z_{j+1}] (D^{k-2} g)$.

Let $b_{i,j}$, $i = h, \dots, k$, be such that $\varphi_{z,j} = \sum_{i=h}^k b_{i,j} e_i$, $j = 1, 2$. It follows from the proof of Theorem 2 in [16] that there exist positive real numbers a_i , $i = h, \dots, k$, such that

$$|b_{i,j}| \leq a_i |D^k \varphi_{z,j}|, \quad j = 1, 2. \tag{34}$$

Then $L_n(\varphi_{z,j} + (-1)^j g_z) \in \sigma_{k-2} \Delta^k$, $j = 1, 2$, and, consequently, we have

$$\begin{aligned}
 & \left| D^{k-2} L_n \left(g - \frac{e_{k-2}}{(k-2)!} D^{k-2} g(z) \right) (z) \right| \\
 &\leq \max \left\{ \left| D^{k-2} L_n \varphi_{z,1}(z) \right|, \left| D^{k-2} L_n \varphi_{z,2}(z) \right| \right\}. \tag{35}
 \end{aligned}$$

It follows from $D^{k-2} \varphi_{z,j}(z) = 0$ that

$$\begin{aligned}
 & \left| D^{k-2} L_n \varphi_{z,j}(z) \right| \\
 &= \left| D^{k-2} L_n \varphi_{z,j}(z) - D^{k-2} \varphi_{z,j}(z) \right| \\
 &\leq \|D^{k-2} L_n \varphi_{z,j} - D^{k-2} \varphi_{z,j}\|_{B[0,1]} \\
 &\leq \sum_{i=h}^k |b_{i,j}| \|D^{k-2} L_n e_i - D^{k-2} e_i\|_{B[0,1]} \\
 &\leq |D^k \varphi_{z,j}| \sum_{i=h}^k |a_i| \|D^{k-2} L_n e_i - D^{k-2} e_i\|_{B[0,1]} \\
 &\leq 2 \max_{0 \leq j \leq n-1} \sup_{x \in [z_j, z_{j+1}]} [z_j, x, z_{j+1}] (D^{k-2} g) \\
 &\quad \times \sum_{i=h}^k |a_i| \|D^{k-2} L_n e_i - D^{k-2} e_i\|_{B[0,1]},
 \end{aligned} \tag{36}$$

where a_i , $i = h, \dots, k$, do not depend on n .

It follows from (36) and (29) that

$$\begin{aligned}
 & \left| D^{k-2} L_n \varphi_{z,j}(z) \right| \\
 &\leq 2\lambda(Z) \sum_{i=h}^k |a_i| \|D^{k-2} L_n e_i - D^{k-2} e_i\|_{B[0,1]}. \tag{37}
 \end{aligned}$$

It follows from (31), (32), (35), (36), and (37) that

$$\frac{1}{2\lambda(Z)} \leq \sum_{i=h}^k |a_i| \|D^{k-2} L_n e_i - D^{k-2} e_i\|_{B[0,1]}. \tag{38}$$

Thus, as it follows from (30) there exists a constant $C > 0$ such that

$$Cn^{-2} \leq \sum_{i=h}^k \|D^{k-2} L_n e_i - D^{k-2} e_i\|_{B[0,1]}. \tag{39}$$

□

Theorem 4. Let $n \geq k + 2$. If $\sigma_k \sigma_{k-2} = 1$ then

(1) there exist $c_1, c_2 > 0$ not depending on n such that

$$c_1 n^{-2} < \delta(P_k, \Delta^{h,k}(\sigma))_{C^{k-2}[0,1]} < c_2 n^{-2}; \tag{40}$$

(2) for all $0 \leq m \leq k - 1$

$$\delta_n(P_m, \Delta^{h,k}(\sigma))_{C^k[0,1]} = 0. \tag{41}$$

If $\sigma_k \sigma_{k-2} \neq 1$ then

$$\delta(P_k, \Delta^{h,k}(\sigma))_{C^{k-2}[0,1]} = 0. \tag{42}$$

Proof. To prove (40) it is sufficient to show that there exist $c_1, c_2 > 0$ not depending on n such that

$$c_1 n^{-2} < \inf_{L_n(\Delta^{h,k}(\sigma)) \subset \Delta^{h,k}(\sigma)} \sup_{f \in P_k} \|f - L_n f\|_{B^{k-2}[0,1]} < c_2 n^{-2}, \quad (43)$$

where infimum is taken over all linear continuous operators L_n of $C^{k-2}[0, 1]$ into $B^{k-2}[0, 1]$ of finite rank n satisfying $L_n(\Delta^{h,k}(\sigma)) \subset \Delta^{h,k}(\sigma)$. Without loss of generality we will assume $\sigma_k = \sigma_{k-2} = 1$. Then the upper estimate in (40) follows from Theorem 1. The lower estimate follows from Lemma 3.

As it was shown in the proof of Theorem 1 there exists a continuous linear operator $\Lambda_n^{h,k} : C^{k-2}[0, 1] \rightarrow C^{k-2}[0, 1]$ of finite rank n , such that $\Lambda_n^{h,k}(\Delta^{h,k}(\sigma)) \subset \Delta^{h,k}(\sigma)$, and for all $0 \leq i \leq k-2$

$$\sup_{f \in P_m} \|D^i(\Lambda_n^{h,k} f) - D^i f\|_{B[0,1]} = 0, \quad 0 \leq m \leq k-1. \quad (44)$$

It follows from (44) that

$$\inf_{L_n(\Delta^{h,k}(\sigma)) \subset \Delta^{h,k}(\sigma)} \sup_{f \in P_m} \|f - L_n f\|_{B^{k-2}[0,1]} = 0, \quad (45)$$

$$0 \leq m \leq k-1,$$

where infimum is taken over all linear continuous operators L_n of $C^{k-2}[0, 1]$ into $B^{k-2}[0, 1]$ of finite rank n satisfying $L_n(\Delta^{h,k}(\sigma)) \subset \Delta^{h,k}(\sigma)$, and consequently (41) holds.

Finally, (42) follows from [16], where it is shown (remark after Proposition 1) that there exists a linear operator that maps the cone of positive and concave functions onto the same cone and holds the space P_3 . $\square$

Denote $B^{(k)} := \{f \in B^k[0, 1] : \|f\|_{B^k[0,1]} \leq 1\}$.

Theorem 5. Let $n \geq k+2$ and $\sigma_k \sigma_{k-2} = 1$. Then there exist $c_1, c_2 > 0$ not depending on n such that

$$c_1 n^{-2} < \delta_n(B^{(k)}, \Delta^{h,k}(\sigma))_{C^{k-2}[0,1]} < c_2 n^{-2}. \quad (46)$$

Proof. Note that $P_k \subset B^{(k)}$. Then the lower estimate in (46) follows from (40).

On the other hand, if $f \in B^{(k)}$ and $x, t \in [0, 1]$ then

$$f(t) = \sum_{r=0}^{k-1} \frac{D^r f(x)}{r!} x^r + \frac{1}{k!} \int_0^1 (x-t)_+^k D^k f(t) dt \quad (47)$$

and $\|D^k f\|_{B[0,1]} \leq 1$. The properties of linear operator $\Lambda_n^{h,k}$ defined in (8) imply that there exists $c_2 > 0$ such that

$$\|f - \Lambda_n^{h,k} f\|_{B^{k-2}[0,1]} \leq \frac{c_2}{(n-1)^2}. \quad (48)$$

$\square$

5. Conclusion

Estimation of linear relative n -widths is of interest in the theory of shape-preserving approximation as, knowing the value of relative linear n -width, we can judge how good or bad (in terms of optimality) this or that finite dimensional method preserving the shape in the sense V is.

The paper shows that if a linear operator with finite rank n preserves the shape in the sense of cone $\Delta^{h,k}(\sigma)$, the degree of simultaneous approximation of derivatives of order $0 \leq i \leq k-2$ of continuous functions by derivatives of this operator cannot be better than n^{-2} on both the set P_k and the ball $B^{(k)}$. Results show that the shape-preserving property of operators is negative in the sense that the error of approximation by means of such operators does not decrease with the increase of smoothness of approximated functions. In other words, there is saturation effect for linear finite-rank operators preserving the shape in the sense of cone $\Delta^{h,k}(\sigma)$. It is worth noting that nonlinear approximation preserving k -monotonicity does not have this shortcoming [17].

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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