Research Article

Some New Explicit Values of Quotients of Ramanujan's Theta Functions and Continued Fractions

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We evaluate some new explicit values of quotients of Ramanujan's theta functions and use them to find explicit values of Ramanujan's continued fractions.

1. Introduction

Ramanujan's general theta function $f(a, b)$ is defined by

$$f(a, b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1. \quad (1)$$

Two important special cases of $f(a, b)$ are the theta functions $\phi(q)$ and $\psi(q)$ [1, page 36, Entry 22] defined by, for $|q| < 1$,

$$\phi(q) := f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q; -q)_{\infty}}{(q; -q)_{\infty}}, \quad (2)$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}},$$

where $(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n)$.

In his notebooks [2], Ramanujan recorded many explicit values of theta functions $\phi(q)$ and $\psi(q)$. All these values were proved by Berndt [3, page 325] and Berndt and Chan [4]. Yi [5] introduced the parameter $h_{k,n}$ for positive real numbers k and n defined by

$$h_{k,n} = \frac{\phi(q)}{k^{1/4} \phi(q^k)}; \quad q = e^{-\pi \sqrt{n/k}} \quad (3)$$

and used the particular case $h_{n,n}$ to find explicit values $\phi(q)$. Baruah and Saikia [6] defined the parameter $g_{k,n}$ for positive real numbers k and n as

$$g_{k,n} = \frac{\psi(-q)}{k^{1/4} q^{(k-1)/8} \psi(-q^k)}; \quad q = e^{-\pi \sqrt{n/k}} \quad (4)$$

and used the particular case $g_{n,n}$ to find explicit values $\psi(q)$. Saikia [7] also established some explicit values $\psi(q)$.

In this paper, we consider the particular cases $h_{3,n}$ and $g_{3,n}$ of the parameters $h_{k,n}$ and $g_{k,n}$, respectively. By using theta function identities, we find some new explicit values of the parameters $h_{3,n}$ and $g_{3,n}$. Particularly, we evaluate $h_{3,n}$ and $g_{3,n}$ for $n = 3/2, 2/3, 6, 1/6, 5/2, 2/5, 10$, and $1/10$. Previously, Yi [5] evaluated $h_{3,n}$ for $n = 1, 3, 1/3, 9, 1/9, 5, 1/5, 25$, and $1/25$. Saikia [8] evaluated $h_{3,n}$ for $n = 2, 1/2, 4, 1/4, 7, 1/7, 49$, and $1/49$. Baruah and Saikia [6] evaluated $g_{3,n}$ for $n = 1, 3, 1/3, 9, 1/9, 5, 1/5, 25, 1/25, 7, 1/7, 13, 1/13, 49$, and $1/49$. As an application to our new values, we evaluate some old and new explicit values of Ramanujan's cubic continued fraction $G(q)$ and a continued fraction of order twelve $H(q)$ which are, respectively, defined by, for $|q| < 1$,

$$G(q) := \frac{q^{1/3}}{1 + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} + \dots},$$

$$H(q) := \frac{q f(-q, -q^{11})}{f(-q^5, -q^7)} \quad (5)$$

$$= \frac{q(1-q)}{1 - q^3} + \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)} + \frac{q^3(1-q^8)(1-q^{10})}{(1-q^3)(1+q^{12})} + \dots$$

The continued fraction $G(q)$ was recorded by Ramanujan on page 366 of lost notebook [9]. We refer to [10–14] for explicit evaluations of $G(q)$. The continued fraction $H(q)$ was

introduced by Naika et al. [15]. We refer to [8, 15] for explicit evaluations of $H(q)$.

The presentation of the paper is as follows. In Section 2 we record some preliminary results for ready references in this paper. Section 3 is devoted to explicit evaluations of the parameters $h_{3,n}$ and $g_{3,n}$. In Section 4 we use new explicit values of $h_{3,n}$ and $g_{3,n}$ to evaluate some explicit values of the continued fractions $G(q)$ and $H(q)$.

We end this introduction by noting the following remarks regarding $g_{3,n}$ and $h_{3,n}$ from [6, page 1764, Theorem 4.1] and [5, page 385, Remark 2.3], respectively.

Remark 1. The parameter $g_{3,n}$ has positive real value with $g_{3,1} = 1$ and $g_{3,n}$ increases as n increases.

Remark 2. The parameter $h_{3,n}$ has positive real value with $h_{3,1} = 1$ and the values of $h_{3,n}$ decrease as n increases.

2. Preliminary Results

Lemma 3 (see [16, Theorem 3.2]). If $P := \sqrt{q}\psi(-q)\psi(-q^9)/\psi^2(-q^3)$ and $Q := q\psi(-q^2)\psi(-q^{18})/\psi^2(-q^6)$, then

$$\begin{aligned} & \frac{P^2}{Q^2} + \frac{Q^2}{P^2} + \left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2} \right)^2 \\ &= \left(3PQ - \frac{1}{PQ} \right) \left(3PQ - \frac{1}{PQ} + \frac{3P}{Q} + \frac{3Q}{P} + \frac{P^3}{Q^3} + \frac{Q^3}{P^3} \right) \\ &+ 14. \end{aligned} \quad (6)$$

Lemma 4 (see [16, Theorem 3.1]). If $L := \phi(q)\phi(q^9)/\phi^2(q)$ and $M := \sqrt{q}\psi(-q)\psi(-q^9)/\psi^2(-q^3)$, then

$$L^2 = \frac{M^2 + 1}{1 - 3M^2}. \quad (7)$$

Lemma 5 (see [16, Theorem 3.2]). If $P := \psi(-q^3)\psi(-q^5)/q\psi(-q)\psi(-q^{15})$ and $Q := \psi(-q^6)\psi(-q^{10})/q^2\psi(-q^2)\psi(-q^{30})$, then

$$\begin{aligned} & \left(\frac{P^2}{Q^2} + \frac{Q^2}{P^2} \right) + \left(\frac{P}{Q} + \frac{Q}{P} \right) - \left(PQ + \frac{1}{PQ} \right) \\ &+ \left(\sqrt{PQ} - \frac{1}{\sqrt{PQ}} \right) \left(\sqrt{\frac{P^3}{Q^3}} + \sqrt{\frac{Q^3}{P^3}} + \sqrt{\frac{P}{Q}} + \sqrt{\frac{Q}{P}} \right) = 0. \end{aligned} \quad (8)$$

Lemma 6 (see [16, Theorem 3.9]). If $L := \phi(q^3)\phi(q^5)/\phi(q)\phi(q^{15})$ and $M := \psi(-q^3)\psi(-q^5)/q\psi(-q)\psi(-q^{15})$, then

$$M = \frac{1 + L}{1 - L}. \quad (9)$$

Lemma 7 (see [5, page 385, Theorem 2.2]). For all positive real numbers k and n , one has

$$h_{k,1/n} = \frac{1}{h_{k,n}}. \quad (10)$$

Lemma 8 (see [6, page 1764, Theorem 4.1]). For all positive real numbers k and n , one has

$$g_{k,1/n} = \frac{1}{g_{k,n}}. \quad (11)$$

Lemma 9 (see [5, page 393, Theorem 4.9(i)]). For any positive real number n , one has

$$\sqrt{3} \left(h_{3,n}h_{3,9n} + \frac{1}{h_{3,n}h_{9,n}} \right) = \left(\frac{h_{3,9n}}{h_{3,n}} \right)^2 + 3. \quad (12)$$

Lemma 10 (see [5, page 394, Theorem 4.12(i)]). For any positive real number n , one has

$$\begin{aligned} & 3 \left\{ (h_{3,n}h_{3,25n})^2 + \frac{1}{(h_{3,n}h_{3,25n})^2} \right\} \\ &= \left\{ \left(\frac{h_{3,25n}}{h_{3,n}} \right)^3 - \left(\frac{h_{3,n}}{h_{3,25n}} \right)^3 \right\} \\ &+ 5 \left\{ \left(\frac{h_{3,25n}}{h_{3,n}} \right)^2 + \left(\frac{h_{3,n}}{h_{3,25n}} \right)^2 \right\} \\ &+ 5 \left\{ \left(\frac{h_{3,25n}}{h_{3,n}} \right) - \left(\frac{h_{3,n}}{h_{3,25n}} \right) \right\}. \end{aligned} \quad (13)$$

Lemma 11 (see [6, page 1769, Theorem 5.1(i) and (iii)]). For any positive real number n , one has

$$(i) \quad (1 + \sqrt{3}g_{3,n}g_{3,9n})^3 = 1 + 3g_{3,9n}^4,$$

$$(ii) \quad 3\{(g_{3,n}g_{3,25n})^2 + 1/(g_{3,n}g_{3,25n})^2\} + 5\{(g_{3,25n}/g_{3,n})^2 + (g_{3,n}/g_{3,25n})^2\} = \{(g_{3,25n}/g_{3,n})^3 - (g_{3,n}/g_{3,25n})^3\} + 5\{(g_{3,25n}/g_{3,n}) - (g_{3,n}/g_{3,25n})\}.$$

3. Explicit Evaluations of the Parameters $g_{k,n}$ and $h_{k,n}$

In this section, we prove some general theorems for the explicit evaluations of the parameters $g_{k,n}$ and $h_{k,n}$ and evaluate some new explicit values therefrom.

Theorem 12. One has

$$\begin{aligned} & \left(\frac{g_{3,n}g_{3,36n}}{g_{3,4n}g_{3,9n}} \right)^2 + \left(\frac{g_{3,4n}g_{3,9n}}{g_{3,n}g_{3,36n}} \right)^2 \\ & + \left(\left(\frac{g_{3,n}g_{3,36n}}{g_{3,4n}g_{3,9n}} \right)^2 + \left(\frac{g_{3,4n}g_{3,9n}}{g_{3,n}g_{3,36n}} \right)^2 \right)^2 - 14 \\ & = \left\{ 3 \left(\frac{g_{3,n}g_{3,4n}}{g_{3,9n}g_{3,36n}} \right) - \left(\frac{g_{3,9n}g_{3,36n}}{g_{3,n}g_{3,4n}} \right) \right\} \\ & \times \left\{ 3 \left(\frac{g_{3,n}g_{3,4n}}{g_{3,9n}g_{3,36n}} \right) - \left(\frac{g_{3,9n}g_{3,36n}}{g_{3,n}g_{3,4n}} \right) + 3 \left(\frac{g_{3,n}g_{3,36n}}{g_{3,4n}g_{3,9n}} \right) \right\} \\ & + \left\{ 3 \left(\frac{g_{3,n}g_{3,4n}}{g_{3,9n}g_{3,36n}} \right) - \left(\frac{g_{3,9n}g_{3,36n}}{g_{3,n}g_{3,4n}} \right) \right\} \\ & \times \left\{ 3 \left(\frac{g_{3,4n}g_{3,9n}}{g_{3,n}g_{3,36n}} \right) + \left(\frac{g_{3,n}g_{3,36n}}{g_{3,4n}g_{3,9n}} \right)^3 + \left(\frac{g_{3,4n}g_{3,9n}}{g_{3,n}g_{3,36n}} \right)^3 \right\}. \end{aligned} \quad (14)$$

Proof. We set $q := e^{-\pi\sqrt{n/3}}$ in Lemma 3 and use the definition of $g_{3,n}$. $\square$

Corollary 13. One has

$$\begin{aligned} g_{3,3/2} &= \frac{\left(\sqrt{3\sqrt{2+2\sqrt{2}} - \sqrt{6+6\sqrt{2}}} \right)}{\sqrt{2}}, \\ g_{3,6} &= \frac{\sqrt{6(1+\sqrt{2})}}{\sqrt{3\sqrt{2+2\sqrt{2}} - \sqrt{6+6\sqrt{2}}}}, \\ g_{3,2/3} &= \frac{\sqrt{2}}{\left(\sqrt{3\sqrt{2+2\sqrt{2}} - \sqrt{6+6\sqrt{2}}} \right)}, \\ g_{3,6} &= \frac{\sqrt{3\sqrt{2+2\sqrt{2}} - \sqrt{6+6\sqrt{2}}}}{\sqrt{6(1+\sqrt{2})}}. \end{aligned} \quad (15)$$

Proof. Setting $n = 1/6$ in Theorem 12 and simplifying using Lemma 8, we obtain

$$\begin{aligned} & \left(\frac{3}{(g_{3,3/2}g_{3,6})^2} - (g_{3,3/2}g_{3,6})^2 \right) \\ & \times \left(\frac{3}{(g_{3,3/2}g_{3,6})^2} - (g_{3,3/2}g_{3,6})^2 + 8 \right) + 8 = 0. \end{aligned} \quad (16)$$

Solving (16) for $g_{3,3/2}g_{3,6}$ and noting the fact in Remark 1, we find that

$$g_{3,3/2}g_{3,6} = \sqrt{3(1+\sqrt{2})}. \quad (17)$$

Again, setting $n = 1/6$ in Lemma 11(i) and simplifying using Lemma 8, we deduce that

$$\left(1 + \sqrt{3} \left(\frac{g_{3,3/2}}{g_{3,6}} \right) \right)^3 = 1 + 3g_{3,3/2}^4. \quad (18)$$

Eliminating $g_{3,6}$ from (18) using (17) and simplifying, we obtain

$$g_{3,3/2}^4 + 3 \left(\sqrt{1+\sqrt{2}} - (1+\sqrt{2})^{3/2} \right) g_{3,3/2}^2 + (3+3\sqrt{2}) = 0. \quad (19)$$

Solving (19) and noting the fact in Remark 1, we obtain

$$g_{3,3/2} = \frac{\left(\sqrt{3\sqrt{2+2\sqrt{2}} - \sqrt{6+6\sqrt{2}}} \right)}{\sqrt{2}}. \quad (20)$$

Employing (20) in (17) and simplifying, we obtain

$$g_{3,6} = \frac{\sqrt{6(1+\sqrt{2})}}{\sqrt{3\sqrt{2+2\sqrt{2}} - \sqrt{6+6\sqrt{2}}}}. \quad (21)$$

Now the values of $g_{3,2/3}$ and $g_{3,1/6}$ follow from the values of $g_{3,3/2}$ and $g_{3,6}$, respectively, and Lemma 8. $\square$

Theorem 14. One has

$$\left(\frac{h_{3,n}}{h_{3,9n}} \right)^2 \left\{ 1 - 3 \left(\frac{g_{3,n}}{g_{3,9n}} \right)^2 \right\} - \left(\frac{g_{3,n}}{g_{3,9n}} \right)^2 - 1 = 0. \quad (22)$$

Proof. We set $q := e^{-\pi\sqrt{n/3}}$ in Lemma 4 and use the definitions of $h_{3,n}$ and $g_{3,n}$. $\square$

Corollary 15. One has

$$\begin{aligned} h_{3,3/2} &= 2^{-5/8} (3\sqrt{2}-4)^{3/4} (6+6\sqrt{2}+\sqrt{6})^{1/2}, \\ h_{3,6} &= 2^{1/8} 3^{1/2} (3\sqrt{2}+4)^{1/4} (6+6\sqrt{2}+\sqrt{6})^{-1/2}, \\ h_{3,2/3} &= 2^{5/8} (3\sqrt{2}-4)^{-3/4} (6+6\sqrt{2}+\sqrt{6})^{-1/2}, \\ h_{3,1/6} &= 2^{-1/8} 3^{-1/2} (3\sqrt{2}+4)^{-1/4} (6+6\sqrt{2}+\sqrt{6})^{1/2}. \end{aligned} \quad (23)$$

Proof. Setting $n = 1/6$ in Theorem 14 and simplifying using Lemma 7, we obtain

$$\frac{1}{(h_{3,6}h_{3,3/2})^2} \left(1 - \frac{3}{(g_{3,6}g_{3,3/2})^2} \right) - \frac{1}{(g_{3,6}g_{3,3/2})^2} - 1 = 0. \quad (24)$$

Employing (17) in (24), solving the resulting equation, and noting the fact in Remark 2, we deduce that

$$h_{3,3/2}h_{3,6} = 2^{1/4} \sqrt{\frac{3}{4+3\sqrt{2}}}. \quad (25)$$

Again setting $n = 1/6$ in Lemma 9 and simplifying using Lemma 7, we obtain

$$\sqrt{3} \left(\frac{h_{3,3/2}}{h_{3,6}} + \frac{h_{3,6}}{h_{3,3/2}} \right) = (h_{3,3/2} h_{3,6})^2 + 3. \quad (26)$$

Employing (25) in (26), solving the resulting equation, and noting the fact in Remark 2, we deduce that

$$\frac{h_{3,3/2}}{h_{3,6}} = \frac{6 + 6\sqrt{2} + \sqrt{6}}{\sqrt{3}(4 + 3\sqrt{2})}. \quad (27)$$

Multiplying (25) and (27), we evaluate the value of $h_{3,3/2}$, and dividing (25) by (27) and simplifying, we evaluate the value of $h_{3,6}$. Now the values of $h_{3,2/3}$ and $h_{3,1/6}$ easily follow from the values of $h_{3,3/2}$ and $h_{3,6}$, respectively, and Lemma 7. $\square$

Theorem 16. One has

$$\begin{aligned} & \left(\frac{g_{3,4n}g_{3,25n}}{g_{3,n}g_{3,100n}} \right)^2 + \left(\frac{g_{3,n}g_{3,100n}}{g_{3,4n}g_{3,25n}} \right)^2 + \left(\frac{g_{3,4n}g_{3,25n}}{g_{3,n}g_{3,100n}} \right) \\ & + \left(\frac{g_{3,n}g_{3,100n}}{g_{3,4n}g_{3,25n}} \right) - \left(\frac{g_{3,25n}g_{3,100n}}{g_{3,n}g_{3,4n}} \right) - \left(\frac{g_{3,n}g_{3,4n}}{g_{3,25n}g_{3,100n}} \right) \\ & + \left\{ \left(\frac{g_{3,25n}g_{3,100n}}{g_{3,n}g_{3,4n}} \right)^{1/2} - \left(\frac{g_{3,n}g_{3,4n}}{g_{3,25n}g_{3,100n}} \right)^{1/2} \right\} \\ & \times \left\{ \left(\frac{g_{3,4n}g_{3,25n}}{g_{3,n}g_{3,100n}} \right)^{3/2} + \left(\frac{g_{3,n}g_{3,100n}}{g_{3,4n}g_{3,25n}} \right)^{3/2} \right. \\ & \left. + \left(\frac{g_{3,4n}g_{3,25n}}{g_{3,n}g_{3,100n}} \right)^{1/2} + \left(\frac{g_{3,n}g_{3,100n}}{g_{3,4n}g_{3,25n}} \right)^{1/2} \right\} = 0. \end{aligned} \quad (28)$$

Proof. We set $q := e^{-\pi\sqrt{n/3}}$ in Lemma 5 and use the definition of $g_{3,n}$. $\square$

Corollary 17. One has

$$\begin{aligned} g_{3,5/2} &= \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{1/2} (a + \sqrt{5b})^{1/4} c^{-1/4}, \\ g_{3,10} &= \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{1/2} (a + \sqrt{5b})^{-1/4} c^{1/4}, \\ g_{3,2/5} &= \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{-1/2} (a + \sqrt{5b})^{-1/4} c^{1/4}, \\ g_{3,1/10} &= \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{-1/2} (a + \sqrt{5b})^{1/4} c^{-1/4}, \end{aligned} \quad (29)$$

where

$$\begin{aligned} a &= 63 + 35\sqrt{3} + 30\sqrt{2 + \sqrt{3}} + 20\sqrt{6 + 3\sqrt{3}}, \\ b &= 2574 + 1485\sqrt{3} + 1330\sqrt{2 + \sqrt{3}} + 770\sqrt{6 + 3\sqrt{3}}, \\ c &= 25 + 15\sqrt{3} + 14\sqrt{2 + \sqrt{3}} + 7\sqrt{6 + 3\sqrt{3}}. \end{aligned} \quad (30)$$

Proof. Setting $n = 1/10$ in Theorem 16 and simplifying using Lemma 8, we obtain

$$\begin{aligned} & (g_{3,5/2}g_{3,10})^2 + \frac{1}{(g_{3,5/2}g_{3,10})^2} \\ & - 4 \left(g_{3,5/2}g_{3,10} - \frac{1}{g_{3,5/2}g_{3,10}} \right) - 4 = 0. \end{aligned} \quad (31)$$

Solving (31) and noting the fact in Remark 2, we find that

$$g_{3,5/2}g_{3,10} = 1 + \sqrt{3} + \sqrt{2 + \sqrt{3}}. \quad (32)$$

Again, setting $n = 1/10$ in Lemma 11 (ii) and simplifying using Lemma 8, we obtain

$$\begin{aligned} & 3 \left\{ \left(\frac{g_{3,5/2}}{g_{3,10}} \right)^2 + \left(\frac{g_{3,5/2}}{g_{3,10}} \right) \right\} \\ & + 5 \left((g_{3,5/2}g_{3,10})^2 + \frac{1}{(g_{3,5/2}g_{3,10})^2} \right) \\ & - \left((g_{3,5/2}g_{3,10})^3 - \frac{1}{(g_{3,5/2}g_{3,10})^3} \right) \\ & - 5 \left(g_{3,5/2}g_{3,10} - \frac{1}{g_{3,5/2}g_{3,10}} \right) = 0. \end{aligned} \quad (33)$$

Employing (32) in (33), solving the resulting equation, and noting the fact in Remark 1, we obtain

$$\frac{g_{3,10}}{g_{3,5/2}} = \left(\frac{a + \sqrt{5b}}{c} \right)^{1/2}, \quad (34)$$

where $a = 63 + 35\sqrt{3} + 30\sqrt{2 + \sqrt{3}} + 20\sqrt{6 + 3\sqrt{3}}$, $b = 2574 + 1485\sqrt{3} + 1330\sqrt{2 + \sqrt{3}} + 770\sqrt{6 + 3\sqrt{3}}$, and $c = 25 + 15\sqrt{3} + 14\sqrt{2 + \sqrt{3}} + 7\sqrt{6 + 3\sqrt{3}}$.

Combining (32) and (34), we calculate the values of $g_{3,5/2}$ and $g_{3,10}$. Then, the values of $g_{3,2/5}$ and $g_{3,1/10}$ follow from the values of $g_{3,5/2}$ and $g_{3,10}$, respectively, and Lemma 8. $\square$

Theorem 18. One has

$$\left(\frac{g_{3,25n}}{g_{3,n}} \right) \left\{ 1 - \left(\frac{h_{3,25n}}{h_{3,n}} \right) \right\} - 1 - \left(\frac{h_{3,25n}}{h_{3,n}} \right) = 0. \quad (35)$$

Proof. We set $q := e^{-\pi\sqrt{n/3}}$ in Lemma 6 and use the definitions of $h_{3,n}$ and $g_{3,n}$. $\square$

Corollary 19. One has

$$\begin{aligned}
 h_{3,5/2} &= \left(\sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{1/2} \left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{-1/2} \\
 &\quad \times (f - 4\sqrt{5b})^{1/4} d^{-1/4}, \\
 h_{3,10} &= \left(\sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{1/2} \left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{-1/2} \\
 &\quad \times (f - 4\sqrt{5b})^{-1/4} d^{1/4}, \\
 h_{3,2/5} &= \left(\sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{-1/2} \left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{1/2} \\
 &\quad \times (f - 4\sqrt{5b})^{-1/4} d^{1/4}, \\
 h_{3,1/10} &= \left(\sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{-1/2} \left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^{1/2} \\
 &\quad \times (f - 4\sqrt{5b})^{1/4} d^{-1/4},
 \end{aligned} \tag{36}$$

where b is given in Corollary 17, $f = 1143 + 655\sqrt{3} + 582\sqrt{2 + \sqrt{3}} + 346\sqrt{6 + 3\sqrt{3}}$, and $d = 1117 + 645\sqrt{3} + 578\sqrt{2 + \sqrt{3}} + 334\sqrt{6 + 3\sqrt{3}}$.

Proof. Setting $n = 1/10$ in Theorem 18 and simplifying using Lemma 7, we get

$$g_{3,5/2}g_{3,10}(1 - h_{3,5/2}h_{3,10}) - 1 - h_{3,5/2}h_{3,10} = 0. \tag{37}$$

Employing (32) and solving the resulting equation, we obtain

$$h_{3,5/2}h_{3,10} = \frac{\sqrt{3} + \sqrt{2 + \sqrt{3}}}{2 + \sqrt{3} + \sqrt{2 + \sqrt{3}}}. \tag{38}$$

Again setting $n = 1/10$ in Lemma 10 and simplifying using Lemma 7, we find that

$$\begin{aligned}
 &3 \left\{ \left(\frac{h_{3,5/2}}{h_{3,10}} \right)^2 + \left(\frac{h_{3,10}}{h_{3,5/2}} \right)^2 \right\} \\
 &\quad - \left((h_{3,5/2}h_{3,10})^3 - \frac{1}{(h_{3,5/2}h_{3,10})^3} \right) \\
 &\quad - 5 \left((h_{3,5/2}h_{3,10})^2 + \frac{1}{(h_{3,5/2}h_{3,10})^2} \right) \\
 &\quad - 5 \left(h_{3,5/2}h_{3,10} - \frac{1}{h_{3,5/2}h_{3,10}} \right) = 0.
 \end{aligned} \tag{39}$$

Employing (38) in (39), solving the resulting equation, and noting the fact in Remark 2, we obtain

$$\frac{h_{3,5/2}}{h_{3,10}} = \sqrt{\frac{f - 4\sqrt{5b}}{d}}, \tag{40}$$

where b is given in Corollary 17, $f = 1143 + 655\sqrt{3} + 582\sqrt{2 + \sqrt{3}} + 346\sqrt{6 + 3\sqrt{3}}$, and $d = 1117 + 645\sqrt{3} + 578\sqrt{2 + \sqrt{3}} + 334\sqrt{6 + 3\sqrt{3}}$.

Combining (38) and (40), we evaluate the values of $h_{3,5/2}$ and $h_{3,10}$. Then the values of $h_{3,2/5}$ and $h_{3,1/10}$ follow from $h_{3,5/2}$ and $h_{3,10}$, respectively, and Lemma 7. $\square$

4. Explicit Evaluations of Continued Fractions

This section is devoted to finding some explicit values of the continued fractions $G(q)$ and $H(q)$ by using new values of $g_{3,n}$ and $h_{3,n}$ evaluated in Section 3.

Lemma 20 (see [6, page 1788, Theorem 9.1(i)]). One has

$$G^3 \left(e^{-\pi\sqrt{n/3}} \right) = \frac{-1}{1 + 3g_{3,n}^4}. \tag{41}$$

Theorem 21. One has

$$\begin{aligned}
 G^3 \left(-e^{-\pi/\sqrt{2}} \right) &= \frac{2}{9\sqrt{36 + 42\sqrt{2}} - (38 + 36\sqrt{2})}, \\
 G^3 \left(-e^{-\pi\sqrt{2}} \right) &= \frac{\left(3\sqrt{2 + 2\sqrt{2}} - \sqrt{6 + 6\sqrt{2}} \right)^2}{6 \left(\sqrt{(2 + 2\sqrt{2})(6 + 6\sqrt{2})} - (58 + 40\sqrt{2}) \right)}, \\
 G^3 \left(-e^{-\pi\sqrt{2}/3} \right) &= \frac{\left(3\sqrt{2 + 2\sqrt{2}} - \sqrt{6 + 6\sqrt{2}} \right)^2}{6 \left(\sqrt{(2 + 2\sqrt{2})(6 + 6\sqrt{2})} - (6 + 4\sqrt{2}) \right)}, \\
 G^3 \left(-e^{-\pi/3\sqrt{2}} \right) &= \frac{2(3 + 2\sqrt{2})}{\left(\sqrt{36 + 24\sqrt{2}} - (10 + 8\sqrt{2}) \right)}, \\
 G^3 \left(-e^{-\pi\sqrt{5/6}} \right) &= \frac{-c}{c + 3(a + \sqrt{5b}) \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^2}, \\
 G^3 \left(-e^{-\pi\sqrt{10/3}} \right) &= \frac{-(a + \sqrt{5b})}{(a + \sqrt{5b}) + 3c \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^2}, \\
 G^3 \left(-e^{-\pi\sqrt{2/15}} \right) &= \frac{-(a + \sqrt{5b}) \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^2}{(a + \sqrt{5b}) \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^2 + 3c}, \\
 G^3 \left(-e^{-\pi/\sqrt{30}} \right) &= \frac{-c \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^2}{c \left(1 + \sqrt{3} + \sqrt{2 + \sqrt{3}} \right)^2 + 3(a + \sqrt{5b})},
 \end{aligned} \tag{42}$$

where a , b , and c are given in Corollary 17.

Proof. We set $n = 3/2, 6, 2/3, 1/6, 5/2, 10, 2/5$, and $1/10$ in Lemma 20 and use the corresponding values of $g_{3,n}$ from Corollaries 13 and 17 to complete the proof. $\square$

The explicit values of $G^3(-e^{-\pi/3\sqrt{2}})$, $G^3(-e^{-\pi\sqrt{5/6}})$, $G^3(-e^{-\pi\sqrt{10/3}})$, $G^3(-e^{-\pi\sqrt{2/15}})$, and $G^3(-e^{-\pi/\sqrt{30}})$ are new.

Lemma 22 (see [8, page 144, Theorem 5.1]). *One has*

$$H\left(e^{-\pi\sqrt{n/3}}\right) = \frac{3^{1/4}h_{3,n} - 1}{3^{1/4}h_{3,n} + 1}. \quad (43)$$

By setting $n = 3/2, 6, 2/3, 1/6, 5/2, 10, 2/5$, and $1/10$ in Lemma 22 and employing the corresponding values of $h_{3,n}$ from Corollaries 15 and 19, we calculate the following new explicit values of the continued fraction $H(q)$.

Theorem 23. *One has*

$$H\left(e^{-\pi/\sqrt{2}}\right) = \frac{-2 + 2^{3/8}3^{1/4}(3\sqrt{2} - 4)^{3/4}\sqrt{1 + 6\sqrt{2} + \sqrt{6}}}{2 + 2^{3/8}3^{1/4}(3\sqrt{2} - 4)^{3/4}\sqrt{1 + 6\sqrt{2} + \sqrt{6}}},$$

$$H\left(e^{-\pi\sqrt{2}}\right) = \frac{2^{1/8}3^{3/4}(4 + 3\sqrt{2})^{1/4} - \sqrt{1 + 6\sqrt{2} + \sqrt{6}}}{2^{1/8}3^{3/4}(4 + 3\sqrt{2})^{1/4} + \sqrt{1 + 6\sqrt{2} + \sqrt{6}}},$$

$$H\left(e^{-\pi\sqrt{2/3}}\right) = \frac{2^{5/8}3^{1/4} - (3\sqrt{2} - 4)^{3/4}\sqrt{1 + 6\sqrt{2} + \sqrt{6}}}{2^{5/8}3^{1/4} + (3\sqrt{2} - 4)^{3/4}\sqrt{1 + 6\sqrt{2} + \sqrt{6}}},$$

$$H\left(e^{-\pi/\sqrt{18}}\right) = \frac{2^{7/8}\sqrt{1 + 6\sqrt{2} + \sqrt{6}} - 2(12 + 9\sqrt{2})^{1/4}}{2^{7/8}\sqrt{1 + 6\sqrt{2} + \sqrt{6}} + 2(12 + 9\sqrt{2})^{1/4}},$$

$$\begin{aligned} H\left(e^{-\pi\sqrt{5/6}}\right) &= \left(3^{1/4}\left(3 + \sqrt{2 + \sqrt{3}}\right)^{1/2} (f - 4\sqrt{5b})^{1/4} \right. \\ &\quad \left. - \left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}}\right)^{1/2} d^{1/4}\right) \\ &\quad \times \left(3^{1/4}\left(3 + \sqrt{2 + \sqrt{3}}\right)^{1/2} (f - 4\sqrt{5b})^{1/4} \right. \\ &\quad \left. + \left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}}\right)^{1/2} d^{1/4}\right)^{-1}, \end{aligned}$$

$$\begin{aligned} H\left(e^{-\pi\sqrt{10/3}}\right) &= \left(3^{1/4}\left(3 + \sqrt{2 + \sqrt{3}}\right)^{1/2} d^{1/4} \right. \\ &\quad \left. - \left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}}\right)^{1/2} (f - 4\sqrt{5b})^{1/4}\right) \\ &\quad \times \left(3^{1/4}\left(3 + \sqrt{2 + \sqrt{3}}\right)^{1/2} d^{1/4} \right. \\ &\quad \left. + \left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}}\right)^{1/2} (f - 4\sqrt{5b})^{1/4}\right)^{-1}, \\ H\left(e^{-\pi\sqrt{2/15}}\right) &= \left(3^{1/4}\left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}}\right)^{1/2} d^{1/4} \right. \\ &\quad \left. - \left(3 + \sqrt{2 + \sqrt{3}}\right)^{1/2} (f - 4\sqrt{5b})^{1/4}\right) \\ &\quad \times \left(3^{1/4}\left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}}\right)^{1/2} d^{1/4} \right. \\ &\quad \left. + \left(3 + \sqrt{2 + \sqrt{3}}\right)^{1/2} (f - 4\sqrt{5b})^{1/4}\right)^{-1}, \\ H\left(e^{-\pi/\sqrt{30}}\right) &= \left(3^{1/4}\left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}}\right)^{1/2} (f - 4\sqrt{5b})^{1/4} \right. \\ &\quad \left. - \left(3 + \sqrt{2 + \sqrt{3}}\right)^{1/2} d^{1/4}\right) \\ &\quad \times \left(3^{1/4}\left(2 + \sqrt{3} + \sqrt{2 + \sqrt{3}}\right)^{1/2} (f - 4\sqrt{5b})^{1/4} \right. \\ &\quad \left. + \left(3 + \sqrt{2 + \sqrt{3}}\right)^{1/2} d^{1/4}\right)^{-1}, \end{aligned} \quad (44)$$

where b is given in Corollary 17 and d and f are given in Corollary 19.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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