Research Article

On a Subclass of Analytic Functions Related to a Hyperbola

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The object of the present investigation is to solve Fekete-Szegő problem and determine the sharp upper bound to the second Hankel determinant for a new class $\widetilde{\mathcal{R}}(a, c, \rho)$ of analytic functions in the unit disk. We also obtain a sufficient condition for an analytic function to be in this class.

1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of functions f of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \mathcal{A}$ is said to be starlike function of order ρ and convex function of order ρ , respectively, if and only if $\operatorname{Re}\{zf'(z)/f(z)\} > \rho$ and $\operatorname{Re}\{1 + (zf''(z)/f'(z))\} > \rho$, for $0 \leq \rho < 1$ and for all $z \in \mathcal{U}$. By usual notations, we denote these classes of functions by $\mathcal{S}^*(\rho)$ and $\mathcal{K}(\rho)$ ($0 \leq \rho < 1$), respectively. We write $\mathcal{S}^*(0) = \mathcal{S}^*$ and $\mathcal{K}(0) = \mathcal{K}$, the familiar subclasses of starlike functions and convex functions in $\mathcal{U}$.

Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}(\rho)$, if it satisfies the inequality:

$$\operatorname{Re}\{f'(z)\} > \rho \quad (0 \leq \rho < 1; z \in \mathcal{U}). \quad (2)$$

Note that $\mathcal{R}(\rho)$ is a subclass of close-to-convex functions of order ρ ($0 \leq \rho < 1$) in $\mathcal{U}$.

Let $\mathcal{P}$ denote the class of analytic functions of the form:

$$\phi(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathcal{U}) \quad (3)$$

satisfying the condition $\operatorname{Re}\{\phi(z)\} > 0$ in $\mathcal{U}$.

Let the functions f and g be analytic in $\mathbb{U}$. We say that f is subordinate to g , written as $f < g$ or $f(z) < g(z)$ ($z \in \mathbb{U}$), if there exists a Schwarz function ω , which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$, $z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence relation (cf., e.g., [1]):

$$f(z) < g(z) \iff f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}). \quad (4)$$

For the functions f, g analytic in $\mathcal{U}$ and given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad (5)$$

their Hadamard product (or convolution), denoted by $f * g$ is defined as

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z) \quad (z \in \mathcal{U}). \quad (6)$$

Note that $f * g$ is analytic in $\mathcal{U}$.

The Gauss hypergeometric function ${}_2F_1$ is defined by the infinite series

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} \quad (7)$$

$$(a, b, c \in \mathbb{C}, c \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}; z \in \mathcal{U}),$$

where $(\kappa)_n$ denotes the *Pochhammer symbol* (or *shifted factorial*) given, in terms of the Gamma function Γ , by

$$(\kappa)_n = \frac{\Gamma(\kappa + n)}{\Gamma(\kappa)} = \begin{cases} \kappa(\kappa + 1) \cdots (\kappa + n - 1), & n \in \mathbb{N} \\ 1, & n = 0. \end{cases} \quad (8)$$

We note that the series, given by (7), converges absolutely for $z \in \mathcal{U}$ and hence the function ${}_2F_1$ represents an analytic function in the unit disc $\mathcal{U}$ [2].

We further observe that the *Gauss hypergeometric function* ${}_2F_1$ plays an important role in the study of various properties and characteristics of subclasses of univalent/multivalent functions in geometric function theory (cf., e.g. [3–5]). In our present investigation, we consider the incomplete beta function ψ , defined by

$$\psi(a, c; z) = z {}_2F_1(a, 1; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} \quad (9)$$

$$(a, c \in \mathbb{C}, c \notin \mathbb{Z}_0^-; z \in \mathcal{U}).$$

By making use of the Hadamard product and the function ψ , Carlson and Shaffer [6] defined the linear operator $\mathcal{L}(a, c) : \mathcal{A} \rightarrow \mathcal{A}$ by

$$\mathcal{L}(a, c) f(z) = \psi(a, c; z) * f(z) \quad (f \in \mathcal{A}; z \in \mathcal{U}). \quad (10)$$

If $f \in \mathcal{A}$ is given by (1), then it follows from (10) that

$$\mathcal{L}(a, c) f(z) = z + \sum_{n=1}^{\infty} \frac{(a)_n}{(c)_n} a_{n+1} z^{n+1} \quad (z \in \mathcal{U}), \quad (11)$$

$$\begin{aligned} z(\mathcal{L}(a, c) f)'(z) \\ = a\mathcal{L}(a+1, c) f(z) - (a-1)\mathcal{L}(a, c) f(z) \quad (z \in \mathcal{U}). \end{aligned} \quad (12)$$

The operator $\mathcal{L}(a, c)$ extends several operators introduced and studied by earlier researchers in geometric function theory. For example, $\mathcal{L}(m+1, 1)f(z) = \mathcal{D}^m f(z)$ ($f \in \mathcal{A}, m \in \mathbb{Z}, m > -1; z \in \mathcal{U}$), the well-known Ruscheweyh derivative operator [7] of f and $\mathcal{L}(2, 2-\lambda)f(z) = \Omega_z^\lambda f(z)$ ($f \in \mathcal{A}, 0 \leq \lambda < 1; z \in \mathcal{U}$), the familiar Owa-Srivastava fractional differential operator [8] of f .

With the aid of the linear operator $\mathcal{L}(a, c)$, we introduce a subclass of $\mathcal{A}$ as follows.

Definition 1. A function $f \in \mathcal{A}$ is said to be in the class $\widetilde{\mathcal{R}}(a, c, \rho)$, if it satisfies the following subordination relation:

$$\frac{\mathcal{L}(a, c) f(z)}{z} < \left\{ \frac{1 + (1-2\rho)z}{1-z} \right\}^{1/2} \quad (0 \leq \rho < 1; z \in \mathcal{U}), \quad (13)$$

where the power in the right hand side of (13) indicates the principal branch. Note that if $f \in \widetilde{\mathcal{R}}(a, c, \rho)$, then by (13)

$$\frac{\mathcal{L}(a, c) f(z)}{z} = \{\rho + (1-\rho)\phi(z)\}^{1/2} \quad (\phi \in \mathcal{P}; z \in \mathcal{U}). \quad (14)$$

We denote by $\widetilde{\mathcal{R}}(2, 1, \rho) = \widetilde{\mathcal{R}}(\rho)$, the class of functions $f \in \mathcal{A}$ satisfying the subordination condition:

$$f'(z) < \left\{ \frac{1 + (1-2\rho)z}{1-z} \right\}^{1/2} \quad (0 \leq \rho < 1; z \in \mathcal{U}). \quad (15)$$

In fact, by suitably specializing the parameters a , c , and ρ in the class $\widetilde{\mathcal{R}}(a, c, \rho)$, we can obtain several subclasses of $\mathcal{A}$.

Remark 2. To bring out the geometrical significance of the class $\widetilde{\mathcal{R}}(a, c, \rho)$, we set

$$h_\rho(z) = \left\{ \frac{1 + (1-2\rho)z}{1-z} \right\}^{1/2} \quad (0 \leq \rho < 1; z \in \mathcal{U}) \quad (16)$$

and note that

$$\omega = h_\rho(e^{i\theta}) = \frac{1 + (1-2\rho)e^{i\theta}}{1 - e^{i\theta}} \quad (0 \leq \theta \leq 2\pi) \quad (17)$$

which gives $e^{i\theta}(\omega^2 - 2\rho + 1) = \omega^2 - 1$ or $|\omega^2 - 1| = |\omega^2 + 1 - 2\rho|$. Letting $\omega = u + iv$, we deduce that

$$1 - (u^2 - v^2)^2 + 4u^2v^2 = (u^2 - v^2 + 1 - 2\rho)^2 + 4u^2v^2, \quad (18)$$

which on simplification reduces to $u^2 - v^2 = \rho$. Thus, $h_\rho(\mathcal{U})$ is the interior of the right half branch of the hyperbola $u^2 - v^2 = \rho$. Hence, if $f \in \widetilde{\mathcal{R}}(a, c, \rho)$, then the set of values $\mathcal{L}(a, c)f(z)/z$ for $z \in \mathcal{U}$ lie in $h_\rho(\mathcal{U})$, where h_ρ is given by (16).

Fekete and Szegő [9] defined the Hankel determinant of a function f , given by (1) as

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2 \quad (a_1 = 1). \quad (19)$$

In our present investigation, we also consider the second Hankel determinant of f , given by

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2. \quad (20)$$

It is known [10] that if f given by (1) is analytic and univalent in $\mathcal{U}$, then the sharp inequality $H_2(1) = |a_3 - a_2^2| \leq 1$ holds. For a family $\mathcal{F}$ of functions in $\mathcal{A}$ of the form (1), the more general problem of finding the sharp upper bounds for the functionals $(a_3 - \mu a_2^2)$ ($\mu \in \mathbb{R}/\mathbb{C}$) is popularly known as Fekete-Szegő problem for the class $\mathcal{F}$. The Fekete-Szegő problem for the known classes of univalent functions, starlike functions, convex functions, and close-to-convex functions has been completely settled [9, 11–18]. Recently, Janteng et al. [19, 20] have obtained the sharp upper bounds to the second Hankel determinant $H_2(2)$ for the family $\mathcal{R}$ of functions in $\mathcal{A}$ whose derivatives have positive real part in $\mathcal{U}$. For initial work on the class $\mathcal{R}$, one may refer to the paper by MacGregor [21].

Our objective in the present paper is to solve the Fekete-Szegő problem and also to determine the sharp upper bound

to the second Hankel determinant for the class $\widetilde{\mathcal{R}}(a, c, \rho)$ by following the techniques devised by Libera and Złotkiewicz [22, 23]. The criteria for functions in $\mathcal{A}$ to be in this class are also obtained.

To establish our main results, we will need the following results about the functions belonging to the class $\mathcal{P}$.

Lemma 3. Let the function ϕ , given by (3), be a member of the class $\mathcal{P}$. Then

$$|p_k| \leq 2 \quad (k \geq 1), \quad (21)$$

$$|p_2 - \nu p_1^2| \leq 2 \max \{1, |2\nu - 1|\} \quad (\nu \in \mathbb{C}), \quad (22)$$

$$p_2 = \frac{1}{2} \{p_1^2 + (4 - p_1^2)x\}, \quad (23)$$

$$p_3 = \frac{1}{4} \{p_1^3 + 2(4 - p_1^2)p_1x - (4 - p_1^2)p_1x^2 + 2(4 - p_1^2)(1 - |x|^2)z\} \quad (24)$$

for some complex numbers x, z satisfying $|x| \leq 1$ and $|z| \leq 1$. The estimates in (21) and (22) are sharp.

We note that the estimate (21) is contained in [10]; the estimate (22) is obtained by Ma and Minda [24]; the results in (23) and (24) are due to Libera and Złotkiewicz [23] (see also [22]).

2. Main Results

Unless otherwise mentioned, we assume throughout the sequel that

$$a > 0, \quad c > 0, \quad 0 \leq \rho < 1. \quad (25)$$

Now, we determine the sharp upper bound for the functional $|a_3 - \mu a_2^2|$ ($\mu \in \mathbb{C}$) for functions of the form (1) belonging to the class $\widetilde{\mathcal{R}}(a, c, \rho)$.

Theorem 4. Let $a > 0$ and $c > 0$. If the function f , given by (1), belongs to the class $\widetilde{\mathcal{R}}(a, c, \rho)$, then for any $\mu \in \mathbb{C}$

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ & \leq \frac{(c)_2(1-\rho)}{(a)_2} \\ & \quad \times \max \left\{ 1, \frac{|2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)|}{2a(c+1)} \right\}. \end{aligned} \quad (26)$$

The estimate in (26) is sharp.

Proof. Since $f \in \widetilde{\mathcal{R}}(a, c, \rho)$, by (14) we have

$$\frac{\mathcal{L}(a, c)f(z)}{z} = \{\rho + (1-\rho)\phi(z)\}^{1/2} \quad (z \in \mathcal{U}), \quad (27)$$

where $\phi \in \mathcal{P}$ is given by (3). It is easily seen that

$$\begin{aligned} & \{\rho + (1-\rho)\phi(z)\}^{1/2} \\ & = 1 + \frac{1}{2}(1-\rho)p_1z + \frac{1-\rho}{2} \left\{ p_2 - \frac{(1-\rho)}{4}p_1^2 \right\} z^2 \\ & \quad + \frac{1-\rho}{2} \left\{ p_3 - \frac{(1-\rho)}{2}p_1p_2 + \frac{(1-\rho)^2}{8}p_1^3 \right\} z^3 + \dots \\ & \quad (z \in \mathcal{U}). \end{aligned} \quad (28)$$

Writing the series expansion of $\mathcal{L}(a, c)f(z)$ given by (11), $\{\rho + (1-\rho)\phi(z)\}^{1/2}$, in (27) and equating the coefficients of z, z^2, z^3 in the resulting equation, we obtain

$$a_2 = \frac{(1-\rho)c}{2a}p_1, \quad (29)$$

$$a_3 = \frac{(1-\rho)(c)_2}{2(a)_2} \left\{ p_2 - \frac{(1-\rho)}{4}p_1^2 \right\}, \quad (30)$$

$$a_4 = \frac{(1-\rho)(c)_3}{2(a)_3} \left\{ p_3 - \frac{(1-\rho)}{2}p_1p_2 + \frac{(1-\rho)^2}{8}p_1^3 \right\}. \quad (31)$$

Thus for any $\mu \in \mathbb{C}$,

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ & = \frac{(c)_2(1-\rho)}{2(a)_2} \left| p_2 - \frac{(1-\rho)\{a(c+1) + 2(a+1)c\mu\}}{4a(c+1)} p_1^2 \right| \end{aligned} \quad (32)$$

and by using (22) in the above expression, we get

$$\begin{aligned} & |a_3 - \mu a_2^2| \\ & \leq \frac{(c)_2(1-\rho)}{(a)_2} \\ & \quad \times \max \left\{ 1, \left| \frac{(1-\rho)}{2a(c+1)} \{a(c+1) + 2(a+1)c\mu\} - 1 \right| \right\} \end{aligned} \quad (33)$$

which, upon simplification, gives the required assertion of Theorem 4.

Equality in (26) holds for the function f_0 defined in $\mathcal{U}$ by

$$\begin{aligned} & f_0(z) \\ & = \begin{cases} \psi(c, a; z) * zh_\rho(z), \\ \frac{|2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)|}{2a(c+1)} \leq 1, \\ \psi(c, a; z) * zh_\rho(z^2), \\ \frac{|2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)|}{2a(c+1)} > 1, \end{cases} \end{aligned} \quad (34)$$

where the function h_ρ is given by (16). This completes the proof of Theorem 4. $\square$

Theorem 5. Let $a > 0$, $c > 0$ and $\mu \in \mathbb{R}$. If the function f , given by (1), belongs to the class $\mathcal{R}(a, c, \rho)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\{a(c+1)(1+\rho) - 2(a+1)c(1-\rho)\mu\}c(1-\rho)}{2a(a)_2}, & \mu < -\frac{a(c+1)}{2(a+1)c} \\ \frac{(c)_2(1-\rho)}{(a)_2}, & -\frac{a(c+1)}{2(a+1)c} \leq \mu \leq \frac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)} \\ \frac{\{2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)\}c(1-\rho)}{2a(a)_2}, & \mu > \frac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)}. \end{cases} \quad (35)$$

The estimates are sharp.

Proof. First, we assume that $\mu < -\{a(c+1)\}/2(a+1)c$. Then

$$\frac{2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)}{2a(c+1)} < -1 \quad (36)$$

so that by (26), we obtain

$$|a_3 - \mu a_2^2| \leq \frac{\{a(c+1)(1+\rho) - 2(a+1)c(1-\rho)\mu\}c(1-\rho)}{2a(a)_2}. \quad (37)$$

Next, let

$$-\frac{a(c+1)}{2(a+1)c} \leq \mu \leq \frac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)}. \quad (38)$$

Then, a routine calculation yields

$$\frac{|2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)|}{2a(c+1)} \leq 1 \quad (39)$$

and by using (26) again, we get

$$|a_3 - \mu a_2^2| \leq \frac{(c)_2(1-\rho)}{(a)_2}. \quad (40)$$

Finally, if $\{a(c+1)(3+\rho)\}/2(a+1)c(1-\rho) > 1$, then

$$\frac{2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)}{2a(c+1)} > 1. \quad (41)$$

Thus, by (26), we have

$$|a_3 - \mu a_2^2| \leq \frac{\{2(a+1)c(1-\rho)\mu - a(c+1)(1+\rho)\}c(1-\rho)}{2a(a)_2}. \quad (42)$$

The estimates are sharp for the function f_0 defined in $\mathcal{U}$ by

$$f_0(z) = \begin{cases} \psi(c, a; z) * zh_p(z), & \mu < -\frac{a(c+1)}{2(a+1)c} \text{ or } \mu > \frac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)} \\ \psi(c, a; z) * zh_p(z^2), & -\frac{a(c+1)}{2(a+1)c} \leq \mu \leq \frac{a(c+1)(3+\rho)}{2(a+1)c(1-\rho)}, \end{cases} \quad (43)$$

where the function h_p is given by (16) and the proof of Theorem 5 is completed. $\square$

Using (21) in (29) and putting $\mu = 0$ and $\mu = 1$, respectively, in Theorem 5, we get the following.

Corollary 6. Let $a \geq c > 0$. If the function f , given by (1), belongs to the class $\mathcal{R}(a, c, \rho)$, then

$$|a_2| \leq \frac{c(1-\rho)}{a}, \quad (44)$$

$$|a_3| \leq \frac{(c)_2(1-\rho)}{(a)_2}, \quad (45)$$

$$|a_3 - a_2^2| \leq \frac{(c)_2(1-\rho)}{(a)_2}. \quad (46)$$

The estimates in (44) and (46) are sharp for the function f_0 defined by

$$f_0(z) = \psi(c, a; z) * zh_p(z) \quad (z \in \mathcal{U}), \quad (47)$$

whereas the estimate in (45) is sharp for the function f_0 given by

$$f_0(z) = \psi(c, a; z) * zh_p(z^2) \quad (z \in \mathcal{U}), \quad (48)$$

where the function h_p is given by (16).

Letting $a = 2$ and $c = 1$ in Theorem 8, we obtain the following.

Corollary 7. If the function f , given by (1), belongs to the class $\widetilde{\mathcal{R}}(\rho)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\{2(1+\rho) - 3\mu(1-\rho)\}(1-\rho)}{12}, & \mu < -\frac{2}{3} \\ \frac{(1-\rho)}{3}, & -\frac{2}{3} \leq \mu \leq \frac{2(3+\rho)}{3(1-\rho)} \\ \frac{\{3\mu(1-\rho) - 2(1+\rho)\}(1-\rho)}{12}, & \mu > \frac{2(3+\rho)}{3(1-\rho)}. \end{cases} \quad (49)$$

The estimates are sharp for the function f_0 defined in $\mathcal{U}$ by

$$f_0(z) = \begin{cases} \int_0^z \frac{dt}{1-t} * zh_p(z), \\ \mu < -\frac{2}{3} \quad \text{or} \quad \mu > \frac{2(3+\rho)}{3(1-\rho)} \\ \int_0^z \frac{dt}{1-t} * zh_p(z^2), \\ -\frac{2}{3} \leq \mu \leq \frac{2(3+\rho)}{3(1-\rho)}, \end{cases} \quad (50)$$

where h_p is given by (16).

Next, we find the sharp upper bound for the fourth coefficient of functions in the class $\widetilde{\mathcal{R}}(a, c, \rho)$.

Theorem 8. Let the function f , given by (1), belong to the class $\widetilde{\mathcal{R}}(a, c, \rho)$. Then

$$|a_4| \leq \frac{(c)_3(1-\rho)}{(a)_3} \quad (51)$$

and the estimate in (51) is sharp.

Proof. From (31), we have

$$|a_4| = \frac{(c)_3(1-\rho)}{2(a)_3} \left| p_3 - \frac{1-\rho}{2} p_1 p_2 + \frac{(1-\rho)^2}{8} p_1^3 \right|. \quad (52)$$

Since the functions $\phi(z)$ and $\phi(e^{i\theta}z)$ ($\theta \in \mathbb{R}$) are in the class $\mathcal{P}$ simultaneously, we assume without loss of generality that $p_1 > 0$. For convenience of notation, we write $p_1 = p$ ($0 \leq p \leq 2$). Now, by using (23) and (24) in (52), we deduce that

$$\begin{aligned} |a_4| &= \frac{(c)_3(1-\rho)}{2(a)_3} \left| \frac{1+\rho^2}{8} p^3 + \frac{1+\rho}{4} (4-p^2) px \right. \\ &\quad \left. - \frac{1}{4} (4-p^2) px^2 + \frac{1}{2} (4-p^2) (1-|x|^2) z \right| \end{aligned} \quad (53)$$

for some complex numbers x ($|x| \leq 1$) and z ($|z| \leq 1$).

Applying the triangle inequality in the above expression followed by the replacement of $|x|$ with y in the resulting equation, we obtain

$$\begin{aligned} |a_4| &\leq \frac{(c)_3(1-\rho)}{2(a)_3} \left\{ \frac{1+\rho^2}{8} p^3 + \frac{1+\rho}{4} (4-p^2) py \right. \\ &\quad \left. + \frac{1}{4} (4-p^2) (p-2) y^2 + \frac{1}{2} (4-p^2) \right\} \\ &= G(p, y) \quad (0 \leq p \leq 2, 0 \leq y \leq 1) \quad (\text{say}). \end{aligned} \quad (54)$$

We next maximize the function $G(p, y)$ on the closed rectangle $[0, 2] \times [0, 1]$. Since

$$\frac{\partial G}{\partial y} = \frac{1}{4} (4-p^2) \{p(1+\rho) - 2(2-p)y\}, \quad (55)$$

we have $\partial G / \partial y < 0$ for $0 < p < 2$ and $0 < y < 1$. Thus, $G(p, y)$ cannot have a maximum in the interior on the closed rectangle $[0, 2] \times [0, 1]$. Therefore, for fixed $p \in [0, 2]$

$$\max_{0 \leq y \leq 1} G(p, y) = G(p, 0) = F(p) \quad (\text{say}), \quad (56)$$

where

$$F(p) = \frac{1+\rho^2}{8} p^3 + \frac{1}{2} (4-p^2) \quad (0 \leq p \leq 2). \quad (57)$$

A routine calculation yields

$$F'(p) = \frac{3(1+\rho^2)}{8} p^2 - p = 0 \quad (58)$$

for $p = 0$ or $p = 8/\{3(1+\rho^2)\}$. Since $F''(0) = -1 < 0$ and $F''(8/\{3(1+\rho^2)\}) = 1 > 0$, we conclude that the maximum of F is attained at $p = 0$. Thus, the upper bound of the function G corresponds to $p = y = 0$. Putting $p = y = 0$ in (54), we get our desired estimate (51).

Equality in (51) holds for the function f_0 defined by

$$f_0(z) = \psi(c, a; z) * zh_p(z^3) \quad (0 \leq \rho < 1; z \in \mathcal{U}), \quad (59)$$

where h_p is given by (16). $\square$

In the following theorem, we find the sharp upper bound to the second Hankel determinant for the class $\widetilde{\mathcal{R}}(a, c, \rho)$.

Theorem 9. Let $a \geq c > 0$ and $(a+2)(c+1) - 3(a-c) > 0$. If the function f , given by (1), belongs to the class $\widetilde{\mathcal{R}}(a, c, \rho)$, then

$$|a_2 a_4 - a_3^2| \leq \left\{ \frac{(c)_2(1-\rho)}{(a)_2} \right\}^2. \quad (60)$$

The estimate in (60) is sharp.

Proof. From (29), (30), and (31), we deduce that

$$\begin{aligned} |a_2 a_4 - a_3^2| &= \frac{c(c)_2(1-\rho)^2}{4a(a)_2} \\ &\times \left| \left(\frac{c+2}{a+2} \right) p_1 p_3 - \left(\frac{c+1}{a+1} \right) p_2^2 - \frac{(a-c)(1-\rho)}{2(a+1)(a+2)} p_1^2 p_2 \right. \\ &\quad \left. + \frac{\{(c+2)(a+1) + (a-c)\}(1-\rho)^2}{16(a+1)(a+2)} p_1^4 \right|. \end{aligned} \quad (61)$$

As in Theorem 8, we assume without loss of generality that $p_1 > 0$ and for convenience of notation, we write $p_1 = p$ ($0 \leq p \leq 2$). By using (23) and (24) in (61), we get

$$\begin{aligned} & \left| a_2 a_4 - a_3^2 \right| \\ &= \frac{c(c)_2(1-\rho)^2}{4a(a)_2} \\ & \times \left| \frac{4(a-c)p + \{(c+2)(a+1) + (a-c)\}(1-\rho)^2}{16(a+1)(a+2)} p^4 \right. \\ & \quad + \frac{(a-c)(1+\rho)}{4(a+1)(a+2)} (4-p^2) p^2 x \\ & \quad - \frac{\{4(a+2)(c+1) + (a-c)p^2\}}{4(a+1)(a+2)} (4-p^2) x^2 \\ & \quad \left. + \frac{c+2}{2(a+2)} (4-p^2) p (1-|x|^2) z \right|. \end{aligned} \quad (62)$$

Now, by applying the triangle inequality in (62) and replacing $|x|$ by y in the resulting equation, we get

$$\begin{aligned} & \left| a_2 a_4 - a_3^2 \right| \\ & \leq \frac{c(c)_2(1-\rho)^2}{4a(a)_2} \\ & \times \left\{ \frac{4(a-c)p + \{(a+1)(c+2) + (a-c)\}(1-\rho)^2}{16(a+1)(c+2)} p^4 \right. \\ & \quad + \frac{c+2}{2(a+2)(4-p^2)p} \\ & \quad + \frac{(a-c)(1+\rho)}{4(a+1)(a+2)} (4-p^2) p^2 y \\ & \quad + \frac{\{(a-c)p^2 - 2(a+1)(c+2)p + 4(a+2)(c+1)\}}{4(a+1)(a+2)} \\ & \quad \left. \times (4-p^2) y^2 \right\} \\ & = \mathcal{G}(p, y) \quad (0 \leq p \leq 2, 0 \leq y \leq 1) \quad (\text{say}). \end{aligned} \quad (63)$$

We next maximize the function $\mathcal{G}(p, y)$ on the closed rectangle $[0, 2] \times [0, 1]$. Since

$$\begin{aligned} \frac{\partial \mathcal{G}}{\partial y} &= \frac{(a-c)(1+\rho)}{4(a+1)(a+2)} (4-p^2) p^2 \\ & \quad + \frac{\{2(a+1)(c+2) - (2+p)(a-c)\}}{2(a+1)(a+2)} \\ & \quad \times (4-p^2)(2-p)y > 0 \end{aligned} \quad (64)$$

for $0 < p < 2$ and $0 < y < 1$, it follows that $\mathcal{G}(p, y)$ cannot have a maximum in the interior on the closed rectangle $[0, 2] \times [0, 1]$. Thus, for fixed $p \in [0, 2]$

$$\max_{0 \leq y \leq 1} \mathcal{G}(p, y) = \mathcal{G}(p, 1) = \mathcal{F}(p) \quad (\text{say}), \quad (65)$$

where

$$\begin{aligned} & \mathcal{F}(p) \\ &= \frac{c(c)_2(1-\rho)^2}{4a(a)_2} \\ & \times \left\{ \frac{4(a-c)p + \{(a+1)(c+2) + (a-c)\}(1-\rho)^2}{16(a+1)(c+2)} p^4 \right. \\ & \quad + \frac{c+2}{2(a+2)(4-p^2)p} \\ & \quad + \frac{(a-c)(1+\rho)}{4(a+1)(a+2)} (4-p^2) p^2 \\ & \quad + \frac{\{(a-c)p^2 - 2(a+1)(c+2)p + 4(a+2)(c+1)\}}{4(a+1)(a+2)} \\ & \quad \left. \times (4-p^2) \right\}, \end{aligned} \quad (66)$$

$0 \leq \rho < 1$ and $0 \leq p \leq 2$. Differentiating $\mathcal{F}$ with respect to p , we deduce that

$$\begin{aligned} & \mathcal{F}'(p) \\ &= \frac{c(c)_2(1-\rho)^2}{4a(a)_2} \\ & \times \left[\frac{\{(a+1)(c+2) + (a-c)\}(1-\rho)^2 - 8(a-c)}{4(a+1)(a+2)} p^3 \right. \\ & \quad \left. - \frac{2\{(a+2)(c+1) - (a-c)(2+\rho)\}}{(a+1)(a+2)} p \right] = 0 \end{aligned} \quad (67)$$

for $p = 0$ or

$$p^2 = \frac{8\{(a+2)(c+1) - (a-c)(2+\rho)\}}{\{(a+1)(c+2) + (a-c)\}(1-\rho)^2 - 8(a-c)}. \quad (68)$$

Since $p^2 > 4$ and

$$\mathcal{F}''(0) = \frac{2\{(a+2)(c+1) - (a-c)(2+\rho)\}}{(a+1)(a+2)} < 0 \quad (69)$$

by the hypothesis, we conclude that the maximum value of $\mathcal{F}$ is attained at $p = 0$ so that the upper bound of the function $\mathcal{G}$ corresponds to $p = 0$ and $y = 1$. Thus, by letting $p = 0$ and $y = 1$ in (63), we get the estimate (60).

The estimate in (60) is sharp for the function f_0 given by (48). This completes the proof of Theorem 9. $\square$

Putting $a = 2$ and $c = 1$ in Theorem 9, we get the following.

Corollary 10. *If the function f , given by (1) belongs to the class $\widetilde{\mathcal{R}}(\rho)$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{(1-\rho)^2}{9} \quad (70)$$

and the estimate is sharp for the function f_0 defined by

$$f_0(z) = \int_0^z \frac{dt}{1-t} * zh_p(z^2) \quad (z \in \mathcal{U}), \quad (71)$$

where the function h_p is given by (16).

Theorem 11. *Let $\gamma > 0$, $a \geq 1/(2\gamma)$, $c > 0$ and $1/2 \leq \rho < 1$. If $f \in \mathcal{A}$ satisfies the following inequality*

$$\operatorname{Re} \left\{ \frac{\mathcal{L}(a+1, c) f(z)}{\mathcal{L}(a, c) f(z)} \right\} > \frac{(2a\gamma + 1)\rho - 1}{2a\gamma\rho} \quad (z \in \mathcal{U}), \quad (72)$$

then

$$\frac{\mathcal{L}(a, c) f(z)}{z} < \left\{ \frac{1 + (1-2\rho)z}{1-z} \right\}^{1/\gamma} \quad (z \in \mathcal{U}). \quad (73)$$

The result is the best possible.

Proof. We define the function w by

$$\frac{\mathcal{L}(a, c) f(z)}{z} = \left\{ \frac{1 + (1-2\rho)w(z)}{1-w(z)} \right\}^{1/\gamma} \quad (z \in \mathcal{U}). \quad (74)$$

Choosing the principal branch in the right hand side in (74), we note that w is analytic in $\mathcal{U}$ with $w(0) = 0$. Furthermore, logarithmically differentiating (74) and using the identity (12) in the resulting equation, we find that

$$\begin{aligned} \frac{\mathcal{L}(a+1, c) f(z)}{\mathcal{L}(a, c) f(z)} &= 1 + \frac{1-2\rho}{a\gamma} \frac{zw'(z)}{1 + (1-2\rho)w(z)} \\ &\quad + \frac{1}{a\gamma} \frac{zw'(z)}{1-w(z)} \quad (z \in \mathcal{U}). \end{aligned} \quad (75)$$

We claim that $|w(z)| < 1$ for all $z \in \mathcal{U}$. If not, then there exists a point $z_0 \in \mathcal{U}$ such that

$$\max \{|w(z)| : |z| \leq |z_0|\} = |w(z_0)| = 1 \quad (w(z_0) \neq 1), \quad (76)$$

and let $w(z_0) = e^{i\theta}$. Now, by applying Jack's lemma [25], we have

$$z_0 w'(z_0) = k w(z_0) \quad (k \geq 1). \quad (77)$$

From (75) and (77), we obtain

$$\begin{aligned} &\operatorname{Re} \left\{ \frac{\mathcal{L}(a+1, c) f(z)}{\mathcal{L}(a, c) f(z)} \right\} \\ &= 1 + \frac{k}{a\gamma} \operatorname{Re} \left(\frac{e^{i\theta}}{1 - e^{i\theta}} \right) \\ &\quad + \frac{(1-2\rho)k}{a\gamma} \operatorname{Re} \left\{ \frac{e^{i\theta}}{1 + (1-2\rho)e^{i\theta}} \right\} \\ &= 1 - \frac{c}{2a\gamma} + \frac{(1-2\rho)k}{a\gamma} \\ &\quad \times \frac{1-2\rho + \cos\theta}{1 + 2(1-2\rho)\cos\theta + (1-2\rho)^2} \\ &\leq 1 - \frac{c}{2a\gamma} + \frac{(2\rho-1)k}{2a\gamma\rho} \\ &\leq \frac{(2a\gamma+1)\rho-1}{2a\gamma\rho}, \end{aligned} \quad (78)$$

which contradicts the hypothesis (72). Thus, we conclude that $|w(z)| < 1$ for all $z \in \mathcal{U}$ and (74) yields the required subordination relation (73).

To see that the result is the best possible, we consider the function $f_0 \in \mathcal{A}$ defined by

$$\begin{aligned} f_0(z) &= \psi(c, a; z) * z \left\{ \frac{1 + (1-2\rho)z}{1-z} \right\}^{1/\gamma} \\ &\quad \left(\gamma > 0, a \geq \frac{1}{2\gamma}, c > 0, \frac{1}{2} \leq \rho < 1; z \in \mathcal{U} \right) \end{aligned} \quad (79)$$

from which it follows that

$$\frac{\mathcal{L}(a, c) f_0(z)}{z} = \left\{ \frac{1 + (1-2\rho)z}{1-z} \right\}^{1/\gamma} \quad (z \in \mathcal{U}). \quad (80)$$

Thus, f_0 satisfies the subordination relation (73). On differentiating the expression in (80) followed by the use of the identity (12) in the resulting equation, we deduce that

$$\begin{aligned} &\frac{\mathcal{L}(a+1, c) f_0(z)}{\mathcal{L}(a, c) f_0(z)} \\ &= 1 + \left(\frac{1-2\rho}{a\gamma} \right) \frac{z}{1 + (1-2\rho)z} + \left(\frac{1}{a\gamma} \right) \frac{z}{1-z} \quad (z \in \mathcal{U}). \end{aligned} \quad (81)$$

This implies that

$$\frac{\mathcal{L}(a+1, c) f_0(z)}{\mathcal{L}(a, c) f_0(z)} \rightarrow \frac{(2a\gamma+1)\rho-1}{2a\gamma\rho} \quad \text{as } z \rightarrow -1. \quad (82)$$

and the proof of Theorem 11 is completed. $\square$

In the special case $\gamma = 2$, we get the following sufficient condition for the class $\widetilde{\mathcal{R}}(a, c, \rho)$.

Corollary 12. Let $a \geq 1/4$, $c > 0$ and $1/2 \leq \rho < 1$. If $f \in \mathcal{A}$ satisfies the following inequality:

$$\operatorname{Re} \left\{ \frac{\mathcal{L}(a+1, c) f(z)}{\mathcal{L}(a, c) f(z)} \right\} > \frac{(4a+1)\rho-1}{4a} \quad \left(\frac{1}{2} \leq \rho < 1; z \in \mathcal{U} \right), \quad (83)$$

then $f \in \widetilde{\mathcal{R}}(a, c, \rho)$. The result is the best possible for the function f_0 given by (47).

Letting $a = 2$, $c = 1$ and $\gamma = 2$ in Theorem 11, we obtain the following.

Corollary 13. If $1/2 \leq \rho < 1$ and $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} \left\{ 1 + \frac{f''(z)}{f'(z)} \right\} > \frac{5\rho-1}{4\rho} \quad \left(\frac{1}{2} \leq \rho < 1; z \in \mathcal{U} \right), \quad (84)$$

then $f \in \widetilde{\mathcal{R}}(\rho)$. The result is the best possible for the function f_0 defined by

$$f_0(z) = \int_0^z \frac{dt}{1-t} * zh_\rho(z) \quad \left(\frac{1}{2} \leq \rho < 1; z \in \mathcal{U} \right), \quad (85)$$

where the function h_ρ is given by (16).

Theorem 14. Let $a > 0$, $c > 0$ and $\gamma > 0$. If $f \in \mathcal{A}$ satisfies the following subordination relation:

$$\frac{\mathcal{L}(a, c) f(z)}{z} < \left\{ \frac{1 + (1-2\rho)z}{1-z} \right\}^{1/\gamma} \quad \left(\frac{1}{2} \leq \rho < 1; z \in \mathcal{U} \right), \quad (86)$$

then

$$\operatorname{Re} \left(\frac{\mathcal{L}(a+1, c) f(z)}{\mathcal{L}(a, c) f(z)} \right) > \rho \quad (|z| < r_0(a, \gamma, \rho)), \quad (87)$$

where

$$r_0(a, \gamma, \rho) = \begin{cases} \frac{(1+a\gamma\rho) - \sqrt{(1+a\gamma\rho)^2 - (a\gamma)^2(2\rho-1)}}{a\gamma(2\rho-1)}, & \frac{1}{2} < \rho < 1 \\ \frac{a\gamma}{2+a\gamma}, & \rho = \frac{1}{2}. \end{cases} \quad (88)$$

The bound $r_0(a, \gamma, \rho)$ in (88) is the best possible.

Proof. From (86), we get

$$\left(\frac{\mathcal{L}(a, c) f(z)}{z} \right) = \{\rho + (1-\rho)\phi(z)\}^{1/\gamma} \quad (\phi \in \mathcal{P}; z \in \mathcal{U}), \quad (89)$$

where we choose the principal branch in (89). Taking logarithmic differentiation in (89) and using the identity (12) in the resulting equation, we deduce that

$$\begin{aligned} \operatorname{Re} \left\{ \frac{\mathcal{L}(a+1, c) f(z)}{\mathcal{L}(a, c) f(z)} \right\} - \rho \\ \geq (1-\rho) \left[1 - \frac{|z\phi'(z)|}{a\gamma\{|\rho + (1-\rho)\phi(z)|\}} \right] \quad (z \in \mathcal{U}). \end{aligned} \quad (90)$$

Using the following well-known estimates [21]

$$\frac{|z\phi'(z)|}{\operatorname{Re}\{\phi(z)\}} \leq \frac{2r}{1-r^2}, \quad |\phi(z)| \leq \frac{1+r}{1-r} \quad (|z| = r < 1) \quad (91)$$

in (90), we get

$$\begin{aligned} \operatorname{Re} \left\{ \frac{\mathcal{L}(a+1, c) f(z)}{\mathcal{L}(a, c) f(z)} \right\} - \rho \\ \geq (1-\rho) \left[1 - \frac{2r}{a\gamma\{\rho(1-r)^2 + (1-\rho)(1-r^2)\}} \right] \\ \geq (1-\rho) \left[1 - \frac{2r}{a\gamma\{(2\rho-1)r^2 - 2\rho r + 1\}} \right] \end{aligned} \quad (92)$$

which is certainly positive for $|z| < r_0(a, \gamma, \rho)$, where $r_0(a, \gamma, \rho)$ is given by (88).

To show that the result is the best possible, we consider the function f_0 defined by

$$f_0(z) = \psi(c, a; z) * z \left\{ \rho + (1-\rho) \frac{1+z}{1-z} \right\}^{1/\gamma} \quad \left(\frac{1}{2} \leq \rho < 1, 0 < \gamma; z \in \mathcal{U} \right). \quad (93)$$

Noting that

$$\begin{aligned} \left\{ \frac{\mathcal{L}(a+1, c) f_0(z)}{\mathcal{L}(a, c) f_0(z)} \right\} - \rho \\ = (1-\rho) \left[1 + \frac{2z}{a\gamma\{\rho(1-z)^2 + (1-\rho)(1-z^2)\}} \right] = 0 \end{aligned} \quad (94)$$

for $z = -r_0(a, \gamma, \rho)$, we conclude that the bound is the best possible. This proves Theorem 14. $\square$

Taking $\gamma = 2$ in Theorem 14, we get the following.

Corollary 15. If $a > 0$, $c > 0$, $1/2 \leq \rho < 1$ and $f \in \widetilde{\mathcal{R}}(a, c, \rho)$, then

$$\operatorname{Re} \left(\frac{\mathcal{L}(a+1, c) f(z)}{\mathcal{L}(a, c) f(z)} \right) > \rho \quad (|z| < \kappa(a, \rho)), \quad (95)$$

where

$$\kappa(a, \rho) = \begin{cases} \frac{(1 + 2a\rho) - \sqrt{4a^2(1 - \rho)^2 + 4a\rho + 1}}{2a(2\rho - 1)}, & \frac{1}{2} < \rho < 1 \\ \frac{a}{1 + a}, & \rho = \frac{1}{2}. \end{cases} \quad (96)$$

The bound $\kappa(a, \rho)$ is the best possible for the function f_0 , given by (47).

Setting $a = 2, c = 1$ and $\gamma = 1$ in Theorem 14, we get the following.

Corollary 16. If $f \in \mathcal{A}$ satisfies

$$\operatorname{Re} \{f'(z)\} > \rho \quad \left(\frac{1}{2} \leq \rho < 1; z \in \mathcal{U}\right), \quad (97)$$

then

$$\operatorname{Re} \left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 2\rho - 1 \quad (|z| < \kappa(\rho)), \quad (98)$$

where

$$\kappa(\rho) = \begin{cases} \frac{(1 + 2\rho) - \sqrt{4\rho^2 - 4\rho + 5}}{2(2\rho - 1)}, & \frac{1}{2} < \rho < 1 \\ \frac{1}{2}, & \rho = \frac{1}{2}. \end{cases} \quad (99)$$

The bound $\kappa(\rho)$ is the best possible for the function f_0 , given in Corollary 13.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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