Research Article

On Harmonic Functions Defined by Differential Operator with Respect to k -Symmetric Points

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Received 29 April 2014; Revised 4 July 2014; Accepted 7 July 2014; Published 23 July 2014

Academic Editor: Heinrich Begehr

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We introduce new classes $MH_k^{\sigma,s}(\lambda, \delta, \alpha)$ and $\overline{MH}_k^{\sigma,s}(\lambda, \delta, \alpha)$ of harmonic univalent functions with respect to k -symmetric points defined by differential operator. We determine a sufficient coefficient condition, representation theorem, and distortion theorem.

1. Introduction

A continuous function $f = u + iv$ is a complex valued harmonic function in a complex domain C if both u and v are real harmonic in C . In any simply connected domain $D \subset C$ we can write $f(z) = h + \overline{g}$, where h and g are analytic in D . We call h the analytic part and g the coanalytic part of f . A necessary and sufficient condition for f to be locally univalent and sense preserving in D is that $|h'(z)| > |g'(z)|$ in D . See Clunie and Shell-Small (see [1]).

Thus for $f = h + \overline{g} \in S^*H$, we may write

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \quad (1)$$

Note that S^*H reduces to S^* , the class of normalized analytic univalent functions if the coanalytic part of $f = h + \overline{g}$ is identically zero. Also, denote by SH the subclasses of S^*H consisting of functions f that map U onto starlike domain.

A function f is said to be starlike of order α in U denoted by $SH(\alpha)$ (see [2]) if

$$\begin{aligned} \frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) &= \operatorname{Im} \left\{ \frac{(\partial/\partial \theta) f(re^{i\theta})}{f(re^{i\theta})} \right\} \\ &= \Re \left\{ \frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right\} \geq \alpha, \quad (2) \\ |z| &= r < 1. \end{aligned}$$

A function f of normalized univalent analytic functions is said to be starlike with respect to symmetrical points in U if it satisfies

$$\Re \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in U; \quad (3)$$

this class was introduced and studied by Sakaguchi in 1959 [3]. Some related classes are studied by Shanmugam et al. [4].

In 1979, Chand and Singh [5] defined the class of starlike functions with respect to k -symmetric points of order α ($0 \leq \alpha < 1$). Related classes are also studied by das and Singh [6]. Ahuja and Jahangiri [7] discussed the class $SH(\alpha)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1) and satisfying the condition

$$\operatorname{Im} \left\{ \frac{2(\partial/\partial \theta) f(re^{i\theta})}{f(re^{i\theta}) - f(-re^{i\theta})} \right\} \geq \alpha. \quad (4)$$

In [8], the authors introduced and studied the class $SH_k(\alpha)$ which denotes the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (1) and

$$\begin{aligned} h_k(z) &= z + \sum_{n=2}^{\infty} \phi_n a_n z^n, \\ g_k(z) &= \sum_{n=1}^{\infty} \phi_n b_n z^n, \quad |b_1| < 1, \end{aligned} \quad (5)$$

where

$$\phi_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}, \quad (k \geq 1, \varepsilon^k = 1). \quad (6)$$

From the definition of ϕ_n we know

$$\phi_n = \begin{cases} 1, & n = ik + 1, \\ 0, & n \neq ik + 1, \end{cases} \quad (n \geq 2, i, k \geq 1). \quad (7)$$

The differential operator $D_{\lambda, \delta}^{\sigma, s}$ was introduced by Ali Abubaker and Darus [9]. We define the differential operator of the harmonic function $f = h + \bar{g}$ given by (5) as

$$D_{\lambda, \delta}^{\sigma, s} f(z) = D_{\lambda, \delta}^{\sigma, s} h(z) + (-1)^s \overline{D_{\lambda, \delta}^{\sigma, s} g(z)}, \quad (8)$$

where

$$\begin{aligned} D_{\lambda, \delta}^{\sigma, s} h(z) &= z + \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) a_n z^n, \\ D_{\lambda, \delta}^{\sigma, s} g(z) &= \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) b_n z^n, \end{aligned} \quad (9)$$

and also $\psi_n(\lambda, \delta, \sigma, s) = n^s (C(\delta, n)[1 + \lambda(n-1)])^\sigma$, $\lambda \geq 0$, $C(\delta, n) = (\delta+1)_{n-1}/(n-1)!$, for $\delta, \sigma, s \in N_0 = \{0, 1, 2, \dots\}$, and $(x)_n$ is the Pochhammer symbol defined by

$$\begin{aligned} (x)_n &= \frac{\Gamma(x+n)}{\Gamma(x)} \\ &= \begin{cases} 1, & n = 0 \\ x(x+1) \cdots (x+n-1), & n = \{1, 2, 3, \dots\}. \end{cases} \end{aligned} \quad (10)$$

We note that when $s = 0$, $\sigma = 1$, and $\lambda = 0$ we obtain the Ruscheweyh derivative for harmonic functions (see [7]), when $\sigma = 0$ we obtain the Salagean operator for harmonic functions (see [10]), and when $\sigma = 1$, $s = 0$ we obtain the operator for harmonic functions given by Al-Shaqsi and Darus [11].

Let $MH_k^{\sigma, s}(\lambda, \delta, \alpha)$ denote the class of complex-valued, sense-preserving, harmonic univalent functions f of the form (5) which satisfy the condition

$$\begin{aligned} &\operatorname{Im} \left\{ \frac{(\partial/\partial\theta) D_{\lambda, \delta}^{\sigma, s} f(re^{i\theta})}{D_{\lambda, \delta}^{\sigma, s} f_k(re^{i\theta})} \right\} \\ &= \Re \left\{ \frac{z(D_{\lambda, \delta}^{\sigma, s} h(z))' - (-1)^s \overline{z(D_{\lambda, \delta}^{\sigma, s} g(z))'}}{D_{\lambda, \delta}^{\sigma, s} h_k(z) + (-1)^s \overline{D_{\lambda, \delta}^{\sigma, s} g_k(z)}} \right\} \geq \alpha, \end{aligned} \quad (11)$$

where $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta < \pi$, $0 \leq \alpha < 1$ and the functions $D_{\lambda, \delta}^{\sigma, s} h_k$ and $D_{\lambda, \delta}^{\sigma, s} g_k$ are of the form

$$\begin{aligned} D_{\lambda, \delta}^{\sigma, s} h_k(z) &= z + \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \phi_n a_n z^n, \\ D_{\lambda, \delta}^{\sigma, s} g_k(z) &= \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \phi_n b_n z^n. \end{aligned} \quad (12)$$

Further, denote by $\overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$ the subclasses of $MH_k^{\sigma, s}(\lambda, \delta, \alpha)$, such that the functions h and g in $f = h + \bar{g}$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad (13)$$

$$g(z) = \sum_{n=1}^{\infty} |b_n| z^n, \quad |b_1| < 1,$$

and the functions h_k and g_k in $f_k = h_k + \bar{g}_k$ are of the form

$$h_k(z) = z - \sum_{n=2}^{\infty} \phi_n |a_n| z^n, \quad (14)$$

$$g_k(z) = \sum_{n=1}^{\infty} \phi_n |b_n| z^n, \quad |b_1| < 1.$$

In this paper, we obtain inclusion properties and coefficient conditions for the class $MH_k^{\sigma, s}(\lambda, \delta, \alpha)$. A representation theorem and distortion bounds for the class $\overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$ are also established.

Lemma 1 (see [12]). Let $f = h + \bar{g} \in SH$; if

$$\sum_{n=2}^{\infty} \frac{1-\alpha}{2-\alpha} |a_n| + \sum_{n=1}^{\infty} \frac{1+\alpha}{2-\alpha} |b_n| \leq 1, \quad (15)$$

where $0 \leq \alpha < 1$, then f is harmonic, sense-preserving, univalent in U and f is starlike harmonic of order α .

2. Main Results

First, we give a meaningful conclusion about the class $MH_k^{\sigma, s}(\lambda, \delta, \alpha)$.

Theorem 2. Let $f \in MH_k^{\sigma, s}(\lambda, \delta, \alpha)$, where f is given by (1); then f_k defined by (5) is in $MH_1^{\sigma, s}(\lambda, \delta, \alpha) = MH^{\sigma, s}(\lambda, \delta, \alpha)$.

Proof. Let $f \in MH_k^{\sigma, s}(\lambda, \delta, \alpha)$. Then substituting $re^{i\theta}$ by $\varepsilon^\nu re^{i\theta}$, where $\varepsilon^k = 1$ ($\nu = 0, 1, \dots, k-1$) in (11), respectively, we have

$$\operatorname{Im} \left\{ \frac{(\partial/\partial\theta) D_{\lambda, \delta}^{\sigma, s} f(\varepsilon^\nu re^{i\theta})}{D_{\lambda, \delta}^{\sigma, s} f_k(\varepsilon^\nu re^{i\theta})} \right\} \geq \alpha. \quad (16)$$

According to the definition of f_k and $\varepsilon^k = 1$, we know $f_k(\varepsilon^\nu re^{i\theta}) = \varepsilon^\nu f_k(re^{i\theta})$ for any $\nu = 0, 1, \dots, k-1$ and summing up we can get

$$\begin{aligned} &\operatorname{Im} \left\{ \frac{1}{k} \sum_{\nu=0}^{k-1} \frac{(\partial/\partial\theta) D_{\lambda, \delta}^{\sigma, s} f(\varepsilon^\nu re^{i\theta})}{\varepsilon^\nu D_{\lambda, \delta}^{\sigma, s} f_k(re^{i\theta})} \right\} \\ &= \operatorname{Im} \left\{ \frac{(\partial/\partial\theta) D_{\lambda, \delta}^{\sigma, s} f_k(re^{i\theta})}{D_{\lambda, \delta}^{\sigma, s} f_k(re^{i\theta})} \right\} \geq \alpha; \end{aligned} \quad (17)$$

that is, $f_k \in MH^{\sigma, s}(\lambda, \delta, \alpha)$. $\square$

Next, a sufficient coefficient condition for harmonic functions in $MH^{\sigma,s}(\lambda, \delta, \alpha)$ is given.

Theorem 3. Let $f = h + \bar{g}$ with h and g given by (1) and $f_k = h_k + \bar{g}_k$ with h_k and g_k given by (5). Let

$$\begin{aligned} & \sum_{n=1}^{\infty} \psi_{(n-1)k+1}(\lambda, \delta, \sigma, s) \left[\frac{(n-1)k - \alpha\phi_n + 1}{1 - \alpha} |a_{(n-1)k+1}| \right. \\ & \quad \left. + \frac{(n-1)k + \alpha\phi_n + 1}{1 - \alpha} |b_{(n-1)k+1}| \right] \\ & + \sum_{\substack{n=2 \\ n \neq ik+1}}^{\infty} \frac{n\psi_n(\lambda, \delta, \sigma, s)}{1 - \alpha} [|a_n| + |b_n|] \leq 2, \end{aligned} \quad (18)$$

where $a_1 = 1$, $i \geq 1$, and $\lambda \geq 0$, for $\delta, \sigma, s \in N_0 = \{0, 1, 2, \dots\}$ and $k \geq 1$. Then f is sense-preserving harmonic univalent in U and $f \in MH_k^{\sigma,s}(\lambda, \delta, \alpha)$.

Proof. Since

$$\begin{aligned} & \sum_{n=1}^{\infty} \left[\frac{n - \alpha}{1 - \alpha} |a_n| + \frac{n + \alpha}{1 - \alpha} |b_n| \right] \\ & \leq \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \left[\frac{n - \alpha\phi_n}{1 - \alpha} |a_n| + \frac{n + \alpha\phi_n}{1 - \alpha} |b_n| \right], \\ & \quad \phi_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}, \quad \varepsilon^k = 1, \\ & = \sum_{n=1}^{\infty} \psi_{(n-1)k+1}(\lambda, \delta, \sigma, s) \\ & \quad \times \left[\frac{(n-1)k - \alpha\phi_n + 1}{1 - \alpha} |a_{(n-1)k+1}| \right. \\ & \quad \left. + \frac{(n-1)k + \alpha\phi_n + 1}{1 - \alpha} |b_{(n-1)k+1}| \right] \\ & + \sum_{\substack{n=2 \\ n \neq ik+1}}^{\infty} \frac{n\psi_n(\lambda, \delta, \sigma, s)}{1 - \alpha} [|a_n| + |b_n|] \leq 2, \end{aligned} \quad (19)$$

by Lemma 1, we conclude that f is sense-preserving, harmonic univalent, and starlike in U . To prove $f \in MH_k^{\sigma,s}(\lambda, \delta, \alpha)$, according to the condition (11), we need to show that

$$\begin{aligned} & \operatorname{Im} \left\{ \frac{(\partial/\partial\theta) D_{\lambda,\delta}^{\sigma,s} f(re^{i\theta})}{D_{\lambda,\delta}^{\sigma,s} f_k(re^{i\theta})} \right\} \\ & = \Re \left\{ \frac{z(D_{\lambda,\delta}^{\sigma,s} h(z))' - (-1)^s \overline{z(D_{\lambda,\delta}^{\sigma,s} g(z))'}}{D_{\lambda,\delta}^{\sigma,s} h_k(z) + (-1)^s \overline{D_{\lambda,\delta}^{\sigma,s} g_k(z)}} \right\} \\ & = \Re \frac{A(z)}{B(z)} \geq \alpha, \end{aligned}$$

$$\begin{aligned} A(z) &= z(D_{\lambda,\delta}^{\sigma,s} h(z))' - (-1)^s \overline{z(D_{\lambda,\delta}^{\sigma,s} g(z))'} \\ &= z + \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) a_n z^n - (-1)^s \\ & \quad \times \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) b_n z^n, \end{aligned} \quad (20)$$

$$\begin{aligned} B(z) &= D_{\lambda,\delta}^{\sigma,s} h_k(z) + (-1)^s \overline{D_{\lambda,\delta}^{\sigma,s} g_k(z)} \\ &= z + \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \phi_n a_n z^n \\ & \quad + (-1)^s \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \phi_n b_n z^n, \end{aligned} \quad (21)$$

where

$$\phi_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{(n-1)\nu}, \quad \varepsilon^k = 1. \quad (22)$$

Using the fact that $\Re(\omega) \geq \alpha$ if and only if $|1 - \alpha + \omega| \geq |1 + \alpha - \omega|$, it suffices to show that

$$|A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \geq 0. \quad (23)$$

On the other hand, for $A(z)$ and $B(z)$ as given in (20) and (21), respectively, we have

$$\begin{aligned} & |A(z) + (1 - \alpha)B(z)| - |A(z) - (1 + \alpha)B(z)| \\ &= \left| (1 - \alpha) D_{\lambda,\delta}^{\sigma,s} h_k(z) + z(D_{\lambda,\delta}^{\sigma,s} h(z))' \right. \\ & \quad \left. + (-1)^s \left[(1 - \alpha) D_{\lambda,\delta}^{\sigma,s} g_k(z) - z(D_{\lambda,\delta}^{\sigma,s} g(z))' \right] \right| \\ & \quad - \left| (1 + \alpha) D_{\lambda,\delta}^{\sigma,s} h_k(z) - z(D_{\lambda,\delta}^{\sigma,s} h(z))' \right. \\ & \quad \left. + (-1)^s \left[(1 + \alpha) D_{\lambda,\delta}^{\sigma,s} g_k(z) + z(D_{\lambda,\delta}^{\sigma,s} g(z))' \right] \right| \\ &= \left| (2 - \alpha)z + \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) [n + (1 - \alpha)\phi_n] a_n z^n \right. \\ & \quad \left. - (-1)^s \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) [n - (1 - \alpha)\phi_n] b_n z^n \right| \\ & \quad - \left| -\alpha z + \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) [n - (1 + \alpha)\phi_n] a_n z^n \right. \\ & \quad \left. - (-1)^s \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) [n + (1 + \alpha)\phi_n] b_n z^n \right| \end{aligned}$$

$$\begin{aligned}
&\geq (2 - \alpha) |z| - \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) [n + (1 - \alpha) \phi_n] |a_n| |z|^n \\
&\quad - \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) [n + (1 - \alpha) \phi_n] \|b_n\| |z|^n \\
&\quad - \alpha |z| - \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) [n - (1 + \alpha) \phi_n] |a_n| |z|^n \\
&\quad - \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) [n + (1 + \alpha) \phi_n] |b_n| |z|^n \\
&= (2 - \alpha) |z| \left\{ 1 - \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \left[\frac{n - \alpha \phi_n}{1 - \alpha} \right] |a_n| |z|^{n-1} \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \left[\frac{n + \alpha \phi_n}{1 - \alpha} \right] |b_n| |z|^{n-1} \right\} \\
&\geq 2(1 - \alpha) \left\{ 1 - \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \left[\frac{n - \alpha \phi_n}{1 - \alpha} \right] |a_n| \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \left[\frac{n + \alpha \phi_n}{1 - \alpha} \right] |b_n| \right\} \geq 0.
\end{aligned} \tag{24}$$

Note that, by substituting the value of ϕ_n given by (7) in the previous inequality above, then

$$\begin{aligned}
&|A(z) + (1 - \alpha) B(z)| - |A(z) - (1 + \alpha) B(z)| \\
&\geq 2(1 - \alpha) \left\{ 1 - \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \right. \\
&\quad \times \left[\frac{(n-1)k - \alpha + 1}{1 - \alpha} |a_{(n-1)k+1}| \right. \\
&\quad \left. - \frac{(n-1)k + \alpha + 1}{1 - \alpha} |b_{(n-1)k+1}| \right] \\
&\quad \left. + \sum_{\substack{n=2 \\ n \neq ik+1}}^{\infty} \frac{n\psi_n(\lambda, \delta, \sigma, s)}{1 - \alpha} [|a_n| + |b_n|] \right\} \geq 0,
\end{aligned} \tag{25}$$

by (18). Thus concludes the proof of the theorem. $\square$

Next, the condition (18) is also necessary for functions in $\overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$, which is clarified.

Theorem 4. Let $f = h + \bar{g}$ with h and g given by (13) and $f_k = h_k + \bar{g}_k$ with h_k and g_k given by (14). Then $f \in \overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$ if and only if

$$\begin{aligned}
&\sum_{n=1}^{\infty} \psi_{(n-1)k+1}(\lambda, \delta, \rho, s) \\
&\quad \times \left[\frac{(n-1)k - \alpha \phi_n + 1}{1 - \alpha} |a_{(n-1)k+1}| \right.
\end{aligned}$$

$$\begin{aligned}
&\quad \left. + \frac{(n-1)k + \alpha \phi_n + 1}{1 - \alpha} |b_{(n-1)k+1}| \right] \\
&\quad + \sum_{\substack{n=2 \\ n \neq ik+1}}^{\infty} \frac{n\psi_n(\lambda, \delta, \sigma, s)}{1 - \alpha} [|a_n| + |b_n|] \leq 2,
\end{aligned} \tag{26}$$

where $a_1 = 1$, $\iota \geq 1$, and $\lambda \geq 0$, for $\delta, \sigma, s \in N_0 = \{0, 1, 2, \dots\}$ and $k \geq 1$ given by (6).

Proof. The if part follows from Theorem 3 upon noting that if the analytic and coanalytic parts of $f = h + \bar{g} \in \overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$ are of the form (13), then $f \in \overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$. For the only if part, we show that $f \notin \overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$, if the condition (26) does not hold. Thus we can write

$$\Re \left\{ \frac{z(D_{\lambda, \delta}^{\sigma, s} h(z))' - (-1)^s \overline{z(D_{\lambda, \delta}^{\sigma, s} g(z))'}}{D_{\lambda, \delta}^{\sigma, s} h_k(z) + (-1)^s \overline{D_{\lambda, \delta}^{\sigma, s} g_k(z)}} \right\} \geq \alpha; \tag{27}$$

this is equivalent to

$$0 \leq \Re \left\{ \frac{z(D_{\lambda, \delta}^{\sigma, s} h(z))' - (-1)^s \overline{z(D_{\lambda, \delta}^{\sigma, s} g(z))'}}{D_{\lambda, \delta}^{\sigma, s} h_k(z) + (-1)^s \overline{D_{\lambda, \delta}^{\sigma, s} g_k(z)}} \right\} - \alpha. \tag{28}$$

That is,

$$\begin{aligned}
&\Re \left\{ \left((1 - \alpha) z - \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) (n - \alpha \phi_n) |a_n| z^n \right. \right. \\
&\quad \left. \left. - (-1)^s \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) (n + \alpha \phi_n) |b_n| z^n \right) \right. \\
&\quad \times \left(z - \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \phi_n |a_n| z^n \right. \\
&\quad \left. \left. + (-1)^s \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \phi_n |b_n| z^n \right)^{-1} \right\} \\
&\geq 0.
\end{aligned} \tag{29}$$

The above-required condition must hold for all values of z , $|z| = r < 1$. Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\begin{aligned}
&\left((1 - \alpha) - \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) (n - \alpha \phi_n) |a_n| r^{n-1} \right. \\
&\quad \left. - \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) (n + \alpha \phi_n) |b_n| r^{n-1} \right)
\end{aligned}$$

$$\begin{aligned} & \times \left(1 - \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \phi_n |a_n| r^{n-1} \right. \\ & \quad \left. + \sum_{n=1}^{\infty} \psi_n(\lambda, \delta, \sigma, s) \phi_n |b_n| r^{n-1} \right)^{-1} \\ & \geq 0. \end{aligned} \quad (30)$$

If the condition (26) does not hold, then the numerator in (30) is negative for r sufficiently close to 1. Hence there exists a $z_0 = r_0$ in $(0, 1)$ for which the quotient in (30) is negative. This contradicts the required condition for $f \in \overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$ and the proof is complete. $\square$

Now, the distortion result is given.

Theorem 5. If $f \in \overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$, then

$$\begin{aligned} |f(z)| & \geq (1 - |b_1|) - \frac{1}{\psi_2(\lambda, \delta, \sigma, s)} \left[\frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_1| \right] r^2, \\ |f(z)| & \leq (1 + |b_1|) + \frac{1}{\psi_2(\lambda, \delta, \sigma, s)} \left[\frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_1| \right] r^2. \end{aligned} \quad (31)$$

Proof. We will only prove the left-hand inequality of the above theorem. The arguments for the right-hand inequality are similar and so we omit it. Let $f \in \overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$. Taking the absolute value of $f(z)$ we obtain

$$\begin{aligned} |f(z)| & \geq (1 - |b_1|) r - \sum_{n=2}^{\infty} [|a_n| + |b_n|] r^n \\ & \geq (1 - |b_1|) r - \sum_{n=2}^{\infty} [|a_n| + |b_n|] r^2 \\ & \geq (1 - |b_1|) r - \frac{1 - \alpha}{\psi_2(\lambda, \delta, \rho, s)(2 - \alpha\phi_2)} \\ & \quad \times \sum_{n=2}^{\infty} \frac{\psi_2(\lambda, \delta, \rho, s)(2 - \alpha\phi_2)}{1 - \alpha} [|a_n| + |b_n|] r^2 \\ & \geq (1 - |b_1|) r - \frac{1 - \alpha}{2 - \alpha} \sum_{n=2}^{\infty} \psi_n(\lambda, \delta, \rho, s) \\ & \quad \times \left[\frac{(n - \alpha\phi_n)}{1 - \alpha} |a_n| + \frac{(n + \alpha\phi_n)}{1 - \alpha} |b_n| \right] r^2 \end{aligned} \quad (32)$$

by (26):

$$= (1 - |b_1|) - \frac{1}{\psi_2(\lambda, \delta, \sigma, s)} \left[\frac{1 - \alpha}{2 - \alpha} - \frac{1 + \alpha}{2 - \alpha} |b_1| \right] r^2. \quad (33)$$

The following covering result follows from the left-hand inequality in Theorem 5. $\square$

Corollary 6. If $f \in \overline{MH}_k^{\sigma, s}(\lambda, \delta, \alpha)$ then

$$\begin{aligned} \left\{ \omega : |\omega| < \frac{2\psi_2 - 1 - (\psi_2(\lambda, \delta, \rho, s) - 1)\alpha}{\psi_2(\lambda, \delta, \rho, s)(2 - \alpha)} \right. \\ \left. - \frac{2\psi_2(\lambda, \delta, \rho, s) - 1 - (\psi_2(\lambda, \delta, \rho, s) + 1)\alpha}{\psi_2(\lambda, \delta, \rho, s)(2 - \alpha)} |b_1| \right\} \\ \subset f(U). \end{aligned} \quad (34)$$

Note that other work related to Sakaguchi and classes of functions with respect to symmetric points can be found in [13–16].

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

Both authors read and approved the final paper.

Acknowledgments

The work presented here was partially supported by AP-2013-009 and DIP-2013-001. The authors also would like to thank the referees for the comments given to improve the paper.

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