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Research Article

SEL Series Expansion and Generalized Model Construction for the Real Number System via Series of Rationals

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We present an algorithm for constructing infinite series expansion for real numbers, which yields generalized versions of three famous series expansions, namely, Sylvester series, Engel series, and Lüroth series expansions. Using series of rationals, a generalized model for the real number system is also constructed.

1. Introduction

According to [1, 2], it is well known that each $A \in \mathbb{R}$ is uniquely representable as an infinite series expansion called *Sylvester series expansion*, which is of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n}, \tag{1}$$

where

$$a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A - 1 & \text{if } A \in \mathbb{Z}, \end{cases} \qquad a_1 \geq 2, \qquad a_{n+1} \geq a_n (a_n - 1) + 1 \quad \forall n \geq 1. \tag{2}$$

Moreover, $A \in \mathbb{Q}$ if and only if $a_{n+1} = a_n(a_n - 1) + 1$ for all sufficiently large n. An analogous representation (see [1–3]) also states that every real number A has a unique representation as an infinite series expansion called *Engel series expansion*, which is of the form

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}, \tag{3}$$

where

$$a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A - 1 & \text{if } A \in \mathbb{Z}, \end{cases} \qquad a_1 \geq 2, \qquad a_{n+1} \geq a_n \quad \forall n \geq 1. \tag{4}$$

Moreover, $A \in \mathbb{Q}$ if and only if $a_{n+1} = a_n$ for all sufficiently large n. For the last representation (see [1, 2]), it is also known that each $A \in \mathbb{R}$ is uniquely representable as an infinite series expansion called *Lüroth series expansion*, which is of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 (a_1 - 1) \cdots a_n (a_n - 1) a_{n+1}}, \tag{5}$$

where

$$a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A - 1 & \text{if } A \in \mathbb{Z}, \end{cases} \qquad a_n \geq 2 \quad \forall n \geq 1. \tag{6}$$

Moreover, $A \in \mathbb{Q}$ if and only if (a_n) is periodic.

In 1988, A. Knopfmacher and J. Knopfmacher [4] further derived some elementary properties of the Engel series expansion and Sylvester series expansion and then developed two new methods for constructing new models for the real

number system from the ordered field of rational numbers. These methods are partly similar to the one introduced by Rieger [5] for constructing the real numbers via continued fractions.

In the present work, we will first introduce an algorithm for constructing an infinite series expansion for real numbers called *Sylvester-Engel-Lüroth series expansion* or *SEL series expansion* for short which yields generalized versions of three series expansions, namely, Sylvester series expansion, Engel series expansion, and Lüroth series expansion. Then we will establish some elementary properties of the SEL series expansion and develop a method for constructing a generalized model for the real number system using series of rationals, which yields generalized versions of Knopfmachers' models.

2. SEL Series Expansion

Given any real number A, write it as $A = a_0 + A_1$, where

$$a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A - 1 & \text{if } A \in \mathbb{Z} \end{cases} \tag{7}$$

and $0 < A_1 \leq 1$. Then recursively define

$$a_n = 1 + \left\lfloor \frac{1}{A_n} \right\rfloor, \tag{8}$$

$$A_{n+1} = \left(a_n A_n - 1\right) e_n, \tag{9}$$

where $e_n = e_n(a_n)$ is a positive rational number, which may depend on a_n, for all $n \geq 1$.

Using this algorithm and the same proof as in [1, 2], we have the following.

Theorem 1. *Let $A \in \mathbb{R}$ and assume that*

$$\frac{a_n - 1}{e_n} \in \mathbb{N} \tag{10}$$

for all $n \geq 1$. Then A is uniquely representable as an infinite series expansion called SEL series expansion, which is of the form

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 e_1 \cdots a_n e_n a_{n+1}}, \tag{11}$$

where $a_1 \geq 2$ and $a_{n+1} \geq ((a_n - 1)/e_n) + 1$ for all $n \geq 1$.

Lemma 2. *Any series*

$$\frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1 f_1 \cdots b_n f_n b_{n+1}}, \tag{12}$$

where

$$\begin{gathered} f_n = f_n\left(b_n\right) \in \mathbb{Q}^+, \qquad b_1 \geq 2, \\ b_{n+1} \geq \frac{b_n - 1}{f_n} + 1 \geq 2 \quad \forall n \geq 1, \end{gathered} \tag{13}$$

converges to a real number B_1 such that $b_1 - 1 \leq (1/B_1) < b_1$. Furthermore, $b_1 = 1 + \lfloor 1/B_1 \rfloor$.

By setting $e_n = 1/a_n$, $e_n = 1$, and $e_n = a_n - 1$, for all $n \geq 1$ in Theorem 1, and by setting $f_n = 1/b_n$, $f_n = 1$, and $f_n = b_n - 1$, for all $n \geq 1$ in Lemma 2, we obtain the following well-known expansions for real numbers (see [1–3]), namely, Sylvester series expansion, Engel series expansion, and Lüroth series expansion, respectively, as we now record.

Corollary 3. *Each $A \in \mathbb{R}$ is uniquely representable as a Sylvester series expansion; that is,*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n}, \tag{14}$$

where $a_1 \geq 2$ and $a_{n+1} \geq (a_n - 1)a_n + 1$ for all $n \geq 1$.

Conversely, each series of the form (14) *converges.*

Corollary 4. *Each $A \in \mathbb{R}$ is uniquely representable as an Engel series expansion; that is,*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}, \tag{15}$$

where $a_1 \geq 2$ and $a_{n+1} \geq a_n$ for all $n \geq 1$.

Conversely, each series of the form (15) *converges.*

Corollary 5. *Each $A \in \mathbb{R}$ is uniquely representable as a Lüroth series expansion; that is,*

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1\left(a_1 - 1\right) \cdots a_n\left(a_n - 1\right) a_{n+1}}, \tag{16}$$

where $a_n \geq 2$ for all $n \geq 1$.

Conversely, each series of the form (16) *converges.*

Proposition 6. *Let*

$$\begin{gathered} a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 e_1 \cdots a_n e_n a_{n+1}}, \\ b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1 e_1 \cdots b_n e_n b_{n+1}} \end{gathered} \tag{17}$$

be SEL series expansions of distinct real numbers A and B, respectively. Then the condition $A < B$ is equivalent to the following:

(i) *$a_0 < b_0$ if $a_0 \neq b_0$;*

(ii) *$a_i > b_i$ for the first $i \geq 1$ such that $a_i \neq b_i$ if $a_0 = b_0$.*

Proof. If $a_0 < b_0$, then

$$A = a_0 + A_1 \leq a_0 + 1 \leq b_0 < b_0 + B_1 = B. \tag{18}$$

Hence (i) follows.

Now assume that $a_0 = b_0$ and $a_i > b_i$ for the first $i \geq 1$ such that $a_i \neq b_i$. Applying (9) repeatedly, we obtain

$$\begin{aligned} A_n &= \frac{1}{a_n} + \frac{A_{n+1}}{a_n e_n} \\ &= \frac{1}{a_n} + \frac{1}{a_n e_n}\left(\frac{1}{a_{n+1}} + \frac{A_{n+2}}{a_{n+1}e_{n+1}}\right) \\ &\vdots \\ &= \frac{1}{a_n} + \frac{1}{a_n e_n a_{n+1}} \\ &\quad + \cdots + \frac{1}{a_n e_n \cdots a_{n+k-1} e_{n+k-1} a_{n+k}} \\ &\quad + \frac{A_{n+k+1}}{a_n e_n \cdots a_{n+k-1} e_{n+k-1} a_{n+k} e_{n+k}} \end{aligned} \tag{19}$$

for all $n, k \in \mathbb{N}$. Using (8), we get

$$\frac{1}{a_n} < A_n \leq \frac{1}{a_n - 1} \tag{20}$$

for all $n \geq 1$. We now prove $0 < A_n \leq 1$ for all $n \geq 1$ by induction on n. If $n = 1$, then we have seen that $0 < A_1 \leq 1$. Assume now that $0 < A_n \leq 1$ for $n \geq 1$. By (8), we see that $a_n \geq 2$. Since

$$A_{n+1} = (a_n A_n - 1) e_n = \left(A_n - \frac{1}{a_n}\right) a_n e_n \tag{21}$$

and using (10) and (20), we have that

$$0 < A_{n+1} = (a_n A_n - 1) e_n \leq \left(\frac{a_n}{a_n - 1} - 1\right) e_n = \frac{e_n}{a_n - 1} \leq 1, \tag{22}$$

as desired. It follows that $a_n \geq 2$ for all $n \geq 1$.

Next, we will prove that

$$A_n = \frac{1}{a_n} + \frac{1}{a_n e_n a_{n+1}} + \cdots \tag{23}$$

for all $n \geq 1$. For $n, k \geq 1$, let

$$B_k = \frac{1}{a_n} + \frac{1}{a_n e_n a_{n+1}} + \cdots + \frac{1}{a_n e_n \cdots a_{n+k-2} e_{n+k-2} a_{n+k-1}}. \tag{24}$$

Since $A_n, e_n > 0$ and $a_n \in \mathbb{N}$, for all $n \geq 1$, the sequence of real numbers (B_k) is increasing and bounded above. Thus, $\lim_{k\to\infty} B_k$ exists and so

$$\lim_{k\to\infty} \frac{1}{a_n e_n \cdots a_{n+k} e_{n+k} a_{n+k+1}} = 0. \tag{25}$$

Since (20) and $a_{n+k+1} \geq 2$, we deduce that

$$\begin{aligned} 0 &< \frac{A_{n+k+1}}{a_n e_n \cdots a_{n+k} e_{n+k}} \leq \frac{1}{a_n e_n \cdots a_{n+k} e_{n+k}} \cdot \frac{a_{n+k+1}}{a_{n+k+1}} \cdot \frac{1}{a_{n+k+1} - 1} \\ &\leq \frac{2}{a_n e_n \cdots a_{n+k} e_{n+k} a_{n+k+1}}, \end{aligned} \tag{26}$$

and so

$$\lim_{k\to\infty} \frac{A_{n+k+1}}{a_n e_n \cdots a_{n+k} e_{n+k}} = 0, \tag{27}$$

showing that (23) holds as desired.

Using (23), (20), and $a_i - 1 \geq b_i$, we have

$$\begin{aligned} A &= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_1 e_1 \cdots a_{i-1} e_{i-1} a_i} + \frac{1}{a_1 e_1 \cdots a_i e_i a_{i+1}} + \cdots \\ &= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_1 e_1 \cdots a_{i-1} e_{i-1}}\left(\frac{1}{a_i} + \frac{1}{a_i e_i a_{i+1}} + \cdots\right) \\ &= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_1 e_1 \cdots a_{i-1} e_{i-1}} A_i \\ &\leq a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_1 e_1 \cdots a_{i-1} e_{i-1}}\left(\frac{1}{a_i - 1}\right) \\ &\leq a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_1 e_1 \cdots a_{i-1} e_{i-1}}\left(\frac{1}{b_i}\right) \\ &= b_0 + \frac{1}{b_1} + \cdots + \frac{1}{b_1 e_1 \cdots b_{i-1} e_{i-1} b_i}, \\ &< B, \end{aligned} \tag{28}$$

and the assertion follows. □

The following corollaries follow immediately from Proposition 6 by setting $e_n = 1/a_n$, $e_n = 1$, and $e_n = a_n - 1$, for all $n \geq 1$, respectively; the first two corollaries readily appear in [4].

Corollary 7. *Let*

$$\begin{aligned} &a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n}, \\ &b_0 + \sum_{n=1}^{\infty} \frac{1}{b_n} \end{aligned} \tag{29}$$

be the Sylvester series expansions of real numbers A and B, respectively. Then the condition $A < B$ is equivalent to the following:

(i) *$a_0 < b_0$ if $a_0 \neq b_0$;*

(ii) *$a_i > b_i$ for the first $i \geq 1$ such that $a_i \neq b_i$ if $a_0 = b_0$.*

Corollary 8. *Let*

$$\begin{aligned} &a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n}, \\ &b_0 + \sum_{n=1}^{\infty} \frac{1}{b_1 \cdots b_n} \end{aligned} \tag{30}$$

be the Engel series expansions of real numbers A and B, respectively. Then the condition $A < B$ is equivalent to the following:

(i) $a_0 < b_0$ *if* $a_0 \neq b_0$;

(ii) $a_i > b_i$ *for the first* $i \geq 1$ *such that* $a_i \neq b_i$ *if* $a_0 = b_0$.

Corollary 9. *Let*

$$a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1(a_1-1)\cdots a_n(a_n-1)a_{n+1}},$$
$$b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1(b_1-1)\cdots b_n(b_n-1)b_{n+1}} \tag{31}$$

be the Lüroth series expansions of real numbers A and B, respectively. Then the condition $A < B$ *is equivalent to the following:*

(i) $a_0 < b_0$ *if* $a_0 \neq b_0$;

(ii) $a_i > b_i$ *for the first* $i \geq 1$ *such that* $a_i \neq b_i$ *if* $a_0 = b_0$.

3. Constructions and Ordered Properties

In this section, we construct a generalized model for the real number system using series of rationals, which yields generalized versions of Knopmachers' models in [4] as follows: let $f : [2, \infty) \to \mathbb{R}$ be a nondecreasing function such that $f(x) \geq 2$ for all $x \geq 2$ and let $\mathcal{U}_f$ be the set of all formal infinite sequences $\mathcal{A} = (a_0, a_1, a_2, \ldots)$ of integers a_n such that $a_1 \geq 2$ and $a_{n+1} \geq f(a_n)$ for all $n \geq 1$. In other words,

$$\mathcal{U}_f = \{(a_0, a_1, a_2, \ldots) \mid a_0, a_n \in \mathbb{Z}, a_1 \geq 2, \quad a_{n+1} \geq f(a_n) \quad \forall n \geq 1\}. \tag{32}$$

As an analogue to Proposition 6, we define order relation on $\mathcal{U}_f$ as follows.

For $\mathcal{A} = (a_0, a_1, a_2, \ldots), \mathcal{B} = (b_0, b_1, b_2, \ldots) \in \mathcal{U}_f$, we say that

$\mathcal{A} < \mathcal{B}$ if and only if

(i) $a_0 < b_0$ if $a_0 \neq b_0$, or

(ii) $a_i > b_i$ for the first $i \geq 1$ such that $a_i \neq b_i$ if $a_0 = b_0$.

Note that $\mathcal{A} \leq \mathcal{B}$ if and only if $\mathcal{A} < \mathcal{B}$ or $\mathcal{A} = \mathcal{B}$.

Lemma 10. $\leq$ *is a total ordering relation on* $\mathcal{U}_f$.

Proof. It is clear that $\leq$ is reflexive and antisymmetric and any two elements in $\mathcal{U}_f$ are comparable. It remains to show that $\leq$ is transitive. Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{U}_f$ be such that $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{B} \leq \mathcal{C}$. If $\mathcal{A} = \mathcal{B} = \mathcal{C}, \mathcal{A} = \mathcal{B} < \mathcal{C}$, or $\mathcal{A} < \mathcal{B} = \mathcal{C}$, then $\mathcal{A} \leq \mathcal{C}$. Assume that $\mathcal{A} < \mathcal{B}$ and $\mathcal{B} < \mathcal{C}$, so $a_0 \leq b_0 \leq c_0$. If $a_0 < b_0 < c_0, a_0 = b_0 < c_0$, or $a_0 < b_0 = c_0$, then it is clear that $\mathcal{A} < \mathcal{C}$.

Now assume that $a_0 = b_0 = c_0$. Then $a_i > b_i$ for the first $i \geq 1$ such that $a_i \neq b_i$ and $b_j > c_j$ for the first $j \geq 1$ such that $b_j \neq c_j$. Thus,

(i) if $i < j$, then $a_r = b_r = c_r$, for $r < i$ and $a_i > b_i = c_i$;

(ii) if $i = j$, then $a_r = b_r = c_r$, for $r < i$ and $a_i > b_i > c_i$;

(iii) if $j < i$, then $a_r = b_r = c_r$, for $r < j$ and $a_j = b_j > c_j$.

Hence $\mathcal{A} < \mathcal{C}$ in each case, and so $\mathcal{A} \leq \mathcal{C}$ as desired. □

For convenience, we will denote $f(\dot{a})$ by the infinite sequence

$$f(a), f^2(a), f^3(a), \ldots, \tag{33}$$

where $a \in \mathbb{N}$ with $a \geq 2$ and f^j is the jth composite iteration of f.

Theorem 11. *Every nonempty subset of* $\mathcal{U}_f$ *which is bounded above has a least upper bound (supremum).*

Proof. Let $\mathcal{X}$ be a nonempty subset of $\mathcal{U}_f$ which is bounded above by a sequence $\mathcal{B} = (b_0, b_1, b_2, \ldots) \in \mathcal{U}_f$. Then $\mathcal{A} \leq \mathcal{B}$, and so $a_0 \leq b_0$ for all $\mathcal{A} \in \mathcal{X}$. Let m_0 be the maximum value of a_0 for all $\mathcal{A} \in \mathcal{X}$. Let

$$\mathcal{M} = \left(m_0, 2, f(\dot{2})\right). \tag{34}$$

We will first show that $\mathcal{M}$ is an upper bound of $\mathcal{X}$. For $\mathcal{A} = (a_0, a_1, a_2, \ldots) \in \mathcal{X}$, if $a_0 < m_0$, then it is clear that $\mathcal{A} < \mathcal{M}$. If $a_0 = m_0$, then we prove that

$$a_n \geq f^{n-1}(2) \tag{35}$$

for all $n \geq 1$ by induction on n, where f^0 is the identity map. If $n = 1$, we have $a_1 \geq 2 = f^0(2)$. Assume now that $a_n \geq f^{n-1}(2)$ for $n \geq 1$. Since f is nondecreasing, we obtain

$$a_{n+1} \geq f(a_n) \geq f\left(f^{n-1}(2)\right) = f^n(2), \tag{36}$$

as desired. It then follows that $\mathcal{A} \leq \mathcal{M}$; that is, $\mathcal{M}$ is an upper bound of $\mathcal{X}$. Hence we may assume that $b_0 = m_0$, and so $b_0 = a_0$ for some $\mathcal{A} = (a_0, a_1, a_2, \ldots) \in \mathcal{X}$. Moreover, we may assume that $\mathcal{B} \notin \mathcal{X}$, since otherwise $\mathcal{B} = \sup \mathcal{X}$. Then $\mathcal{A} < \mathcal{B}$ for all $\mathcal{A} \in \mathcal{X}$, and there is the largest index $k \geq 0$ such that

$$a_0 = b_0, \qquad a_1 = b_1, \ldots, a_k = b_k \tag{37}$$

for every $\mathcal{A} \in \mathcal{X}$ with $a_0 = b_0$.

Next, we define a sequence $\mathcal{C} = (c_0, c_1, c_2, \ldots) \in \mathcal{U}_f$ and then show that $\mathcal{C} = \sup \mathcal{X}$. Let $c_0 = b_0, c_1 = b_1, \ldots, c_k = b_k$, and let c_{k+1} be the least possible value for a_{k+1} of any $\mathcal{A} \in \mathcal{X}$ with $a_0 = b_0$. Let c_{k+2} be the least possible value for a_{k+2} of any element of $\mathcal{X}$ of the form

$$(c_0, \ldots, c_{k+1}, a_{k+2}, a_{k+3}, \ldots). \tag{38}$$

Continuing inductively to define c_{k+i+1} as the least possible value for a_{k+i+1} of any element of $\mathcal{X}$ of the form

$$(c_0, \ldots, c_{k+i}, a_{k+i+1}, a_{k+i+2}, \ldots), \tag{39}$$

this process eventually yields a sequence $\mathcal{C} = (c_0, c_1, c_2, \ldots)$ with $c_{i+1} \geq f(c_i)$ for all $i = 1, 2, \ldots$.

Finally, we prove that $\mathscr{C} = \sup \mathscr{X}$. It is clear that $\mathscr{A} < \mathscr{C}$ for every $\mathscr{A} \in \mathscr{X}$ with $a_0 < b_0 = c_0$. By the construction of $\mathscr{C}$, we see that $\mathscr{A} \le \mathscr{C}$ for every $\mathscr{A} \in \mathscr{X}$ with $a_0 = b_0$, so $\mathscr{C}$ is an upper bound of $\mathscr{X}$. If $\mathscr{X}$ has an upper bound $\mathscr{D} < \mathscr{C}$, then $d_0 \le c_0$. If $d_0 < c_0 = a_0$, then $\mathscr{D} < \mathscr{A}$ for all $\mathscr{A} \in \mathscr{X}$ with $a_0 = b_0$, which is impossible. Thus $d_0 = c_0$ and $d_j > c_j$ for the first $j \ge 1$ such that $d_j \ne c_j$. Hence every element of the form

$$\mathscr{A} = \left(c_0, c_1, \ldots, c_j, a_{j+1}, a_{j+2}, \ldots\right) \tag{40}$$

in $\mathscr{X}$ satisfies $\mathscr{D} < \mathscr{A} \le \mathscr{D}$, a contradiction. □

Theorem 12. *Given any element $\mathscr{A}$ of $\mathscr{U}_f$, there exist $\mathscr{A}^{(n)}, \mathscr{A}_{(n)} \in \mathscr{U}_f$ for $n \ge 1$ such that*

(i) $\mathscr{A}_{(m)} < \mathscr{A}_{(n)} < \mathscr{A} \le \mathscr{A}^{(n)} \le \mathscr{A}^{(m)}$ *for* $m < n$,

(ii) $\mathscr{A} = \sup \mathscr{A}_{(n)} = \inf \mathscr{A}^{(n)}$.

Proof. Let $\mathscr{A} = (a_0, a_1, \ldots)$ be any sequence in $\mathscr{U}_f$. We define $\mathscr{A}^{(n)}$ and $\mathscr{A}_{(n)}$ for $n \ge 1$ as follows:

$$\begin{aligned} \mathscr{A}^{(n)} &= \left(a_0, a_1, \ldots, a_{n-1}, a_n, f\left(\dot{a}_n\right)\right), \\ \mathscr{A}_{(n)} &= \left(a_0, a_1, \ldots, a_{n-1}, a_n', f\left(\dot{a}_n'\right)\right), \end{aligned} \tag{41}$$

where $a_n' = a_n + 1$. It is clear that $\mathscr{A}^{(n)}, \mathscr{A}_{(n)} \in \mathscr{U}_f$ for all $n \ge 1$.

(i) For positive integers m, n, with $m < n$, we have

$$\begin{aligned} \mathscr{A}^{(m)} &= \left(a_0, a_1, \ldots, a_{m-1}, a_m, f\left(a_m\right), f^2\left(a_m\right), \ldots\right), \\ \mathscr{A}^{(n)} &= \left(a_0, a_1, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots, \right. \\ &\qquad \left. a_{n-1}, a_n, f\left(a_n\right), f^2\left(a_n\right), \ldots\right), \\ \mathscr{A} &= \left(a_0, a_1, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots, \right. \\ &\qquad \left. a_{n-1}, a_n, a_{n+1}, a_{n+2}, \ldots\right), \\ \mathscr{A}_{(n)} &= \left(a_0, a_1, \ldots, a_{m-1}, a_m, a_{m+1}, \ldots, \right. \\ &\qquad \left. a_{n-1}, a_n', f\left(a_n'\right), f^2\left(a_n'\right), \ldots\right), \\ \mathscr{A}_{(m)} &= \left(a_0, a_1, \ldots, a_{m-1}, a_m', f\left(a_m'\right), f^2\left(a_m'\right), \ldots\right). \end{aligned} \tag{42}$$

It is clear that $\mathscr{A}_{(m)} < \mathscr{A}_{(n)} < \mathscr{A}$. Since $a_{i+1} \ge f(a_i)$ for all $i \ge 1$ and the property of f, we have $\mathscr{A} \le \mathscr{A}^{(n)} \le \mathscr{A}^{(m)}$. Thus part (i) follows.

(ii) Suppose that there exists $\mathscr{B} \in \mathscr{U}_f$ such that $\mathscr{A}_{(n)} \le \mathscr{B} < \mathscr{A}$ for all $n \ge 1$. Then we must have $a_0 = b_0$ and $b_k > a_k$ for the first k such that $a_k \ne b_k$. Thus

$$\mathscr{B} < \mathscr{A}_{(k+1)} = \left(a_0, a_1, \ldots, a_k, a_{k+1}', f\left(\dot{a}_{k+1}'\right)\right), \tag{43}$$

a contradiction. Hence $\mathscr{A} = \sup \mathscr{A}_{(n)}$. Similarly, suppose that there is $\mathscr{C} \in \mathscr{U}_f$ such that $\mathscr{A} < \mathscr{C} \le \mathscr{A}^{(n)}$ for all $n \ge 1$. Then we must have $a_0 = c_0$ and $a_l > c_l$ for the first l such that $a_l \ne c_l$. This gives the contradiction

$$\mathscr{A}^{(l)} = \left(a_0, a_1, \ldots, a_{l-1}, a_l, f\left(\dot{a}_l\right)\right) < \mathscr{C}. \tag{44}$$

Thus $\mathscr{A} = \inf \mathscr{A}^{(n)}$ and part (ii) follows. □

4. Algebraic Operations in $\mathscr{U}_f$

For a nondecreasing function $f : [2, \infty) \to \mathbb{R}$ such that $f(x) \ge 2$ for all $x \ge 2$, let ρ_f be any 1-1 order-preserving map from $\mathbb{R}$ onto $\mathscr{U}_f$. To show that ρ_f^{-1} is order-preserving, let $\mathscr{A}, \mathscr{B} \in \mathscr{U}_f$ be such that $\mathscr{A} \le \mathscr{B}$. Then $\mathscr{A} = \rho_f(A)$ and $\mathscr{B} = \rho_f(B)$ for some $A, B \in \mathbb{R}$. Since ρ_f is order-preserving, we obtain

$$\rho_f^{-1}(\mathscr{A}) = A \le B = \rho_f^{-1}(\mathscr{B}), \tag{45}$$

as desired.

For $\mathscr{A} \in \mathscr{U}_f$, let $A_{(n)}$ and $A^{(n)}$ be real numbers such that

$$\rho_f\left(A_{(n)}\right) = \mathscr{A}_{(n)}, \qquad \rho_f\left(A^{(n)}\right) = \mathscr{A}^{(n)}, \tag{46}$$

for $n \ge 1$, where $\mathscr{A}_{(n)}$ and $\mathscr{A}^{(n)}$ are defined as in Theorem 12. Since ρ_f is order-preserving, we deduce that

$$A_{(m)} < A_{(n)} < A \le A^{(n)} \le A_{(m)} \tag{47}$$

for all positive numbers m, n with $m < n$, where $A = \rho_f^{-1}(\mathscr{A})$.

Now, for any $\mathscr{A}, \mathscr{B} \in \mathscr{U}_f$, define

$$\begin{aligned} \mathscr{A} \oplus \mathscr{B} &= \rho_f\left(\sup\left(A_{(n)} + B_{(n)}\right)\right) \\ -\mathscr{A} &= \rho_f\left(\sup\left(-A^{(n)}\right)\right). \end{aligned} \tag{48}$$

Note that both operators are well defined in $\mathscr{U}_f$ because

$$A_{(n)} + B_{(n)} \le A + B, \quad -A^{(n)} \le -A \quad (n \ge 1) \tag{49}$$

and $(A_{(n)} + B_{(n)}), (-A^{(n)})$ are increasing sequences of real numbers. It follows that

$$\begin{aligned} A + B &= \sup\left(A_{(n)} + B_{(n)}\right), \\ -A &= \sup\left(-A^{(n)}\right). \end{aligned} \tag{50}$$

Using (48), we deduce that

$$\begin{aligned} \rho_f(A + B) &= \rho_f(A) \oplus \rho_f(B), \\ \rho_f(-A) &= -\rho_f(A) \end{aligned} \tag{51}$$

for all $A, B \in \mathbb{R}$.

Theorem 13. *$(\mathscr{U}_f, \oplus)$ is an abelian group. Furthermore, for $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathscr{U}_f$, one has the following:*

(i) *if $\mathscr{A} < \mathscr{B}$, then $\mathscr{A} \oplus \mathscr{C} < \mathscr{B} \oplus \mathscr{C}$;*

(ii) $\mathscr{A} < \mathscr{B}$ *if and only if* $-\mathscr{A} > -\mathscr{B}$.

Proof. First, we will show that $(\mathscr{U}_f, \oplus)$ is an abelian group. Let $\mathscr{A} = \rho_f(A), \mathscr{B} = \rho_f(B), \mathscr{C} = \rho_f(C) \in \mathscr{U}_f$ for some $A, B, C \in \mathbb{R}$. Using (51), we obtain the following:

(1) $\mathscr{A} \oplus \mathscr{B} = \rho_f(A) \oplus \rho_f(B) = \rho_f(A+B) \in \mathscr{U}_f$;

(2) $\mathscr{A} \oplus (\mathscr{B} \oplus \mathscr{C}) = \rho_f(A + (B+C)) = \rho_f((A+B)+C) = (\mathscr{A} \oplus \mathscr{B}) \oplus \mathscr{C}$;

(3) let $\mathscr{O}_f := \rho_f(0) \in \mathscr{U}_f$; then $\mathscr{A} \oplus \mathscr{O}_f = \rho_f(A) \oplus \rho_f(0) = \rho_f(A+0) = \mathscr{A}$ for all $\mathscr{A} \in \mathscr{U}_f$;

(4) $\mathscr{A} \oplus (-\mathscr{A}) = \rho_f(A) \oplus \rho_f(-A) = \rho_f(0) = \mathscr{O}_f$ for all $\mathscr{A} \in \mathscr{U}_f$; and

(5) $\mathscr{A} \oplus \mathscr{B} = \rho_f(A) \oplus \rho_f(B) = \rho_f(A+B) = \rho_f(B+A) = \rho_f(B) \oplus \rho_f(A) = \mathscr{B} \oplus \mathscr{A}$.

From (1)–(5), we conclude that $(\mathscr{U}_f, \oplus)$ is an abelian group.

Lastly, assume that $\mathscr{A} < \mathscr{B}$. Since ρ_f^{-1} is order-preserving, we have $A < B$. Using (51), we have

$$\begin{gathered} \mathscr{A} \oplus \mathscr{C} = \rho_f(A+C) < \rho_f(B+C) = \mathscr{B} \oplus \mathscr{C}, \\ -\mathscr{A} = -\rho_f(A) = \rho_f(-A) > \rho_f(-B) = -\rho_f(B) = -\mathscr{B}. \end{gathered} \tag{52}$$

Hence (ii) follows since $-(-\mathscr{A}) = -(-\rho_f(A)) = -(\rho_f(-A)) = \rho_f(A) = \mathscr{A}$. □

Next, we define the binary operation $\odot$ on $\mathscr{U}_f$ as follows: for any $\mathscr{A}, \mathscr{B} \in \mathscr{U}_f$, let

$$\mathscr{A} \odot \mathscr{B} = \begin{cases} \rho_f(\sup(A_{(n)} \cdot B_{(n)})) & \text{if } \mathscr{A} > \mathscr{O}_f, \mathscr{B} > \mathscr{O}_f, \\ -(\mathscr{A} \odot (-\mathscr{B})) & \text{if } \mathscr{A} > \mathscr{O}_f, \mathscr{B} < \mathscr{O}_f, \\ -((-\mathscr{A}) \odot \mathscr{B}) & \text{if } \mathscr{A} < \mathscr{O}_f, \mathscr{B} > \mathscr{O}_f, \\ (-\mathscr{A}) \odot (-\mathscr{B}) & \text{if } \mathscr{A} < \mathscr{O}_f, \mathscr{B} < \mathscr{O}_f, \\ \mathscr{O}_f & \text{if } \mathscr{A} = \mathscr{O}_f \text{ or } \mathscr{B} = \mathscr{O}_f. \end{cases} \tag{53}$$

Also define

$$\mathscr{A}^{-1} = \begin{cases} \rho_f\left(\sup\left(\left(A^{(n)}\right)^{-1}\right)\right) & \text{if } \mathscr{A} > \mathscr{O}_f \\ -(-\mathscr{A})^{-1} & \text{if } \mathscr{A} < \mathscr{O}_f. \end{cases} \tag{54}$$

To see that these definitions are unambiguous, first we note that

$$A_{(n)} \cdot B_{(n)} \le A \cdot B, \qquad \left(A^{(n)}\right)^{-1} \le A^{-1} \tag{55}$$

for all sufficiently large n, where $\mathscr{A}, \mathscr{B} > \mathscr{O}_f$, since $A_{(n)}$ and $B_{(n)}$ are positive for all sufficiently large n. Secondly, in order to cover all cases, we use the fact from Theorem 13 (ii) that $\mathscr{A} < \mathscr{O}_f$ if and only if $-\mathscr{A} > \mathscr{O}_f$. Using (53), we deduce that

$$\rho_f(A \cdot B) = \rho_f(A) \odot \rho_f(B) \tag{56}$$

for all real numbers A and B. In addition, using (54), we have

$$\rho_f(A)^{-1} = \rho_f\left(A^{-1}\right) \tag{57}$$

for all nonzero real number A.

Theorem 14. $(\mathscr{U}_f, \oplus, \odot)$ *is an ordered field. Furthermore, if* $\mathscr{A} < \mathscr{B}$ *and* $\mathscr{C} > \mathscr{O}_f$, *then* $\mathscr{A} \odot \mathscr{C} < \mathscr{B} \odot \mathscr{C}$.

Proof. By Theorem 13, we know that $(\mathscr{U}_f, \oplus)$ is an abelian group. To prove that $(\mathscr{U}_f, \oplus, \odot)$ is a field, it remains to show that $(\mathscr{U}_f \setminus \{\mathscr{O}_f\}, \odot)$ is an abelian group. Let $\mathscr{A} = \rho_f(A), \mathscr{B} = \rho_f(B), \mathscr{C} = \rho_f(C) \in \mathscr{U}_f$ for some $A, B, C \in \mathbb{R}$. Using (56) and (57), we obtain the following:

(1) $\mathscr{A} \odot \mathscr{B} = \rho_f(A) \odot \rho_f(B) = \rho_f(A \cdot B) \in \mathscr{U}_f$;

(2) $\mathscr{A} \odot (\mathscr{B} \odot \mathscr{C}) = \rho_f(A \cdot (B \cdot C)) = \rho_f((A \cdot B) \cdot C) = (\mathscr{A} \odot \mathscr{B}) \odot \mathscr{C}$;

(3) let $\mathscr{I} := \rho_f(1) \in \mathscr{U}_f$; then $\mathscr{A} \odot \mathscr{I} = \rho_f(A \cdot 1) = \mathscr{A}$ for all $\mathscr{A} = \rho_f(A) \in \mathscr{U}_f$;

(4) $\mathscr{A} \odot \mathscr{A}^{-1} = \rho_f(A) \odot \rho_f(A^{-1}) = \rho_f(1) = \mathscr{I}$ for all $\mathscr{A} = \rho_f(A) \in \mathscr{U}_f$; and

(5) $\mathscr{A} \odot \mathscr{B} = \rho_f(A \cdot B) = \rho_f(B \cdot A) = \mathscr{B} \odot \mathscr{A}$.

From (1)–(5), we conclude that $(\mathscr{U}_f \setminus \{\mathscr{O}_f\}, \odot)$ is an abelian group; thus $(\mathscr{U}_f, \oplus, \odot)$ is a field.

Next, we will show that $(\mathscr{U}_f, \oplus, \odot)$ is an ordered field (see [6]). Let

$$\mathscr{P}_f = \left\{\mathscr{A} \in \mathscr{U}_f \mid \mathscr{A} > \mathscr{O}_f\right\} \tag{58}$$

and $\mathscr{A}, \mathscr{B} \in \mathscr{P}_f$. Then $\mathscr{A} = \rho_f(A)$ and $\mathscr{B} = \rho_f(B)$ for some positive real numbers A and B.

(1) Since $A+B > 0$, we have $\mathscr{A} \oplus \mathscr{B} = \rho_f(A+B) > \rho_f(0) = \mathscr{O}_f$, and so $\mathscr{A} \oplus \mathscr{B} \in \mathscr{P}_f$.

(2) Since $A \cdot B > 0$, we have $\mathscr{A} \odot \mathscr{B} = \rho_f(A \cdot B) > \rho_f(0) = \mathscr{O}_f$, and so $\mathscr{A} \odot \mathscr{B} \in \mathscr{P}_f$.

(3) Since $A > 0$, we have $-A < 0$ and $-\mathscr{A} = -\rho_f(A) = \rho_f(-A) < \rho_f(0) = \mathscr{O}_f$. Hence $-\mathscr{A} \notin \mathscr{P}_f$.

(4) For $\mathscr{A} = \rho_f(A) \in \mathscr{U}_f$, we have $A = 0$ or $A > 0$ or $-A > 0$. It follows that $\mathscr{A} = \mathscr{O}_f, \mathscr{A} > \mathscr{O}_f$, or $-\mathscr{A} > \mathscr{O}_f$. Hence $\mathscr{A} = \mathscr{O}_f, \mathscr{A} \in \mathscr{P}_f$, or $-\mathscr{A} \in \mathscr{P}_f$.

Finally, assume that $\mathscr{A} < \mathscr{B}$ and $\mathscr{C} > \mathscr{O}_f$. Since ρ_f^{-1} is order-preserving, then $A < B$ and $C > 0$, and so $A \cdot C < B \cdot C$. Thus

$$\mathscr{A} \odot \mathscr{C} = \rho_f(A \cdot C) < \rho_f(B \cdot C) = \mathscr{B} \odot \mathscr{C} \tag{59}$$

since ρ_f is order-preserving. This completes the proof of the theorem. □

5. Examples

In this section, we will give three models for the real number system which are special examples of the ordered field $(\mathscr{U}_f, \oplus, \odot)$, where f is chosen from three different nondecreasing functions from $[2, \infty)$ into $\mathbb{R}$ such that $f(x) \ge 2$ for all $x \ge 2$. The first two propositions were readily mentioned in Knopmachers' results [4].

Proposition 15. *Let* $f_S : [2, \infty) \to \mathbb{R}$ *be defined by*

$$f_S(x) = x(x-1) + 1, \tag{60}$$

for all $x \geq 2$,

$$\mathcal{U}_{f_S} = \{(a_0, a_1, a_2, \ldots) \mid a_n \in \mathbb{Z}, a_1 \geq 2, \\ a_{n+1} \geq a_n (a_n - 1) + 1 \ \forall n \geq 1\} \tag{61}$$

and $\rho_{f_S} : \mathbb{R} \to \mathcal{U}_{f_S}$ *defined by*

$$\rho_{f_S}(A) = (a_0, a_1, a_2, \ldots), \tag{62}$$

for all $A \in \mathbb{R}$, *where*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n} \tag{63}$$

is the Sylvester series expansion of A. Then $(\mathcal{U}_{f_S}, \oplus, \odot)$ *is an ordered field containing* $(\rho_{f_S}(\mathbb{Q}), \oplus, \odot)$ *as a dense subfield. Furthermore, for* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{U}_{f_S}$, *one has the following:*

(i) *if* $\mathcal{A} < \mathcal{B}$, *then* $\mathcal{A} \oplus \mathcal{C} < \mathcal{B} \oplus \mathcal{C}$;

(ii) $\mathcal{A} < \mathcal{B}$ *if and only if* $-\mathcal{A} > -\mathcal{B}$;

(iii) *if* $\mathcal{A} < \mathcal{B}$ *and* $\mathcal{C} > \mathcal{O}_{f_S}$, *then* $\mathcal{A} \odot \mathcal{C} < \mathcal{B} \odot \mathcal{C}$,

where $\mathcal{O}_{f_S}$ *is the zero element in* $\mathcal{U}_{f_S}$.

Proof. It is clear that f_S is a nondecreasing function such that $f(x) \geq 2$ for all $x \geq 2$. It is an immediate consequence of Corollary 3 that ρ_{f_S} is 1-1 map from $\mathbb{R}$ onto $\mathcal{U}_{f_S}$. By Corollary 7 and the definition of order in $\mathcal{U}_{f_S}$, this map is order-preserving. Using Theorems 13 and 14, we can conclude that $(\mathcal{U}_{f_S}, \oplus, \odot)$ is an ordered field containing $\rho_{f_S}(\mathbb{Q})$ and satisfies properties (i), (ii), and (iii). To complete the proof of this proposition, it remains to show that $\rho_{f_S}(\mathbb{Q})$ is a dense subfield of $\mathcal{U}_{f_S}$.

It is clear that $\rho_{f_S}(\mathbb{Q})$ is a subfield of $\mathcal{U}_{f_S}$. Now, let $\mathcal{A}, \mathcal{B} \in \mathcal{U}_{f_S}$ be such that $\mathcal{A} < \mathcal{B}$. If $a_0 < b_0$, let

$$\mathcal{C} = \left(b_0, b_1', f_S(\dot{b}_1')\right); \tag{64}$$

then it is easy to see that $\mathcal{A} < \mathcal{C} < \mathcal{B}$ and $\mathcal{C} \in \rho_{f_S}(\mathbb{Q})$. On the other hand, if $a_0 = b_0$, then $a_m > b_m$ for the first m such that $a_m \neq b_m$. In that case, we have $\mathcal{A} < \mathcal{D} < \mathcal{B}$, where

$$\mathcal{D} = \left(b_0, \ldots, b_m, b_{m+1}', f_S(\dot{b}_{m+1}')\right). \tag{65}$$

Then $\mathcal{D} \in \rho_{f_S}(\mathbb{Q})$. This shows that $\rho_{f_S}(\mathbb{Q})$ is dense in $\mathcal{U}_{f_S}$ as desired. □

Proposition 16. *Let* $f_E : [2, \infty) \to \mathbb{R}$ *be defined by*

$$f_E(x) = x, \tag{66}$$

for all $x \geq 2$,

$$\mathcal{U}_{f_E} = \{(a_0, a_1, a_2, \ldots) \mid a_n \in \mathbb{Z}, a_1 \geq 2, \\ a_{n+1} \geq a_n \ \forall n \geq 1\} \tag{67}$$

and $\rho_{f_E} : \mathbb{R} \to \mathcal{U}_{f_E}$ *defined by*

$$\rho_{f_E}(A) = (a_0, a_1, a_2, \ldots), \tag{68}$$

for all $A \in \mathbb{R}$, *where*

$$A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n} \tag{69}$$

is the Engel series expansion of A. Then $(\mathcal{U}_{f_E}, \oplus, \odot)$ *is an ordered field containing* $(\rho_{f_E}(\mathbb{Q}), \oplus, \odot)$ *as a dense subfield. Furthermore, for* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{U}_{f_E}$, *one has the following:*

(i) *if* $\mathcal{A} < \mathcal{B}$, *then* $\mathcal{A} \oplus \mathcal{C} < \mathcal{B} \oplus \mathcal{C}$;

(ii) $\mathcal{A} < \mathcal{B}$ *if and only if* $-\mathcal{A} > -\mathcal{B}$;

(iii) *if* $\mathcal{A} < \mathcal{B}$ *and* $\mathcal{C} > \mathcal{O}_{f_E}$, *then* $\mathcal{A} \odot \mathcal{C} < \mathcal{B} \odot \mathcal{C}$,

where $\mathcal{O}_{f_E}$ *is the zero element in* $\mathcal{U}_{f_E}$.

Proof. It is clear that f_E is a nondecreasing function such that $f(x) \geq 2$ for all $x \geq 2$. It is an immediate consequence of Corollary 4 that ρ_{f_E} is 1-1 map from $\mathbb{R}$ onto $\mathcal{U}_{f_E}$. By Corollary 8 and the definition of order in $\mathcal{U}_{f_E}$, this map is order-preserving. Using Theorems 13 and 14, we can conclude that $(\mathcal{U}_{f_E}, \oplus, \odot)$ is an ordered field containing $\rho_{f_E}(\mathbb{Q})$ and satisfies properties (i), (ii), and (iii). For density of rationals, we can prove in a similar way to the proof of Proposition 15 that $\rho_{f_E}(\mathbb{Q})$ is a dense subfield of $\mathcal{U}_{f_E}$. □

Proposition 17. *Let* $f_L : [2, \infty) \to \mathbb{R}$ *be defined by*

$$f_L(x) = 2, \tag{70}$$

for all $x \geq 2$,

$$\mathcal{U}_{f_L} = \{(a_0, a_1, a_2, \ldots) \mid a_n \in \mathbb{Z}, a_n \geq 2 \ \forall n \geq 1\} \tag{71}$$

and $\rho_{f_L} : \mathbb{R} \to \mathcal{U}_{f_L}$ *defined by*

$$\rho_{f_L}(A) = (a_0, a_1, a_2, \ldots), \tag{72}$$

for all $A \in \mathbb{R}$, *where*

$$A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1(a_1 - 1) \cdots a_n(a_n - 1) a_{n+1}} \tag{73}$$

is the Lüroth series expansion of A. Then $(\mathcal{U}_{f_L}, \oplus, \odot)$ *is an ordered field containing* $(\rho_{f_L}(\mathbb{Q}), \oplus, \odot)$ *as a dense subfield. Furthermore, for* $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{U}_{f_L}$, *one has the following:*

(i) *if* $\mathcal{A} < \mathcal{B}$, *then* $\mathcal{A} \oplus \mathcal{C} < \mathcal{B} \oplus \mathcal{C}$;

(ii) $\mathcal{A} < \mathcal{B}$ *if and only if* $-\mathcal{A} > -\mathcal{B}$;

(iii) *if* $\mathcal{A} < \mathcal{B}$ *and* $\mathcal{C} > \mathcal{O}_{f_L}$, *then* $\mathcal{A} \odot \mathcal{C} < \mathcal{B} \odot \mathcal{C}$,

where $\mathcal{O}_{f_L}$ *is the zero element in* $\mathcal{U}_{f_L}$.

Proof. It is clear that f_L is a nondecreasing function such that $f(x) \geq 2$ for all $x \geq 2$. It is an immediate consequence of Corollary 5 that ρ_{f_L} is 1-1 map from $\mathbb{R}$ onto $\mathcal{U}_{f_L}$. By Corollary 9 and the definition of order in $\mathcal{U}_{f_L}$, this map is order-preserving. Using Theorems 13 and 14, we conclude that $(\mathcal{U}_{f_L}, \oplus, \odot)$ is an ordered field containing $\rho_{f_L}(\mathbb{Q})$ and satisfies properties (i), (ii), and (iii). For density of rationals, we can prove in a similar way to the proof of Proposition 15 that $\rho_{f_L}(\mathbb{Q})$ is a dense subfield of $\mathcal{U}_{f_L}$. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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