Research Article

SEL Series Expansion and Generalized Model Construction for the Real Number System via Series of Rationals

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1. Introduction

According to [1, 2], it is well known that each \( A \in \mathbb{R} \) is uniquely representable as an infinite series expansion called Sylvester series expansion, which is of the form

\[
A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n}, \quad (1)
\]

where

\[
a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A - 1 & \text{if } A \in \mathbb{Z} \end{cases}, \quad a_1 \geq 2, \quad (2)
\]

\[
a_{n+1} \geq a_n (a_n - 1) + 1 \quad \forall n \geq 1.
\]

Moreover, \( A \in \mathbb{Q} \) if and only if \( a_{n+1} = a_n (a_n - 1) + 1 \) for all sufficiently large \( n \). An analogous representation (see [1–3]) also states that every real number \( A \) has a unique representation as an infinite series expansion called Engel series expansion, which is of the form

\[
A = a_0 + \frac{1}{a_1 \cdots a_n}, \quad (3)
\]

where

\[
a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A - 1 & \text{if } A \in \mathbb{Z} \end{cases}, \quad a_1 \geq 2, \quad (4)
\]

\[
a_{n+1} \geq a_n \quad \forall n \geq 1.
\]

Moreover, \( A \in \mathbb{Q} \) if and only if \( (a_n) \) is periodic.

In 1988, A. Knopfmacher and J. Knopfmacher [4] further derived some elementary properties of the Engel series expansion and Sylvester series expansion and then developed two new methods for constructing new models for the real
number system from the ordered field of rational numbers. These methods are partly similar to the one introduced by Rieger [5] for constructing the real numbers via continued fractions.

In the present work, we will first introduce an algorithm for constructing an infinite series expansion for real numbers called Sylvester-Engel-Lüroth series expansion or SEL series expansion for short which yields generalized versions of three series expansions, namely, Sylvester series expansion, Engel series expansion, and Lüroth series expansion. Then we will establish some elementary properties of the SEL series expansion and develop a method for constructing a generalized model for the real number system using series of rationals, which yields generalized versions of Knopfmacher’s models.

### 2. SEL Series Expansion

Given any real number \( A \), write it as \( A = a_0 + A_1 \), where

\[
a_0 = \begin{cases} \lfloor A \rfloor & \text{if } A \notin \mathbb{Z} \\ A - 1 & \text{if } A \in \mathbb{Z} \end{cases}
\]

and \( 0 < A_1 \leq 1 \). Then recursively define

\[
a_n = 1 + \left\lfloor \frac{1}{A_n} \right\rfloor,
\]

\[
A_{n+1} = (a_n A_n - 1) e_n,
\]

where \( e_n = e_n(a_n) \) is a positive rational number, which may depend on \( a_n \), for all \( n \geq 1 \).

Using this algorithm and the same proof as in [1, 2], we have the following.

**Theorem 1.** Let \( A \in \mathbb{R} \) and assume that

\[
a_n - 1 < e_n
\]

for all \( n \geq 1 \). Then \( A \) is uniquely representable as an infinite series expansion called SEL series expansion, which is of the form

\[
A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n a_{n+1}},
\]

where \( a_1 \geq 2 \) and \( a_{n+1} \geq (a_n - 1)/e_n + 1 \) for all \( n \geq 1 \).

**Lemma 2.** Any series

\[
\frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \cdots b_n b_{n+1}},
\]

where

\[
f_n = f_n(b_n) \in \mathbb{Q}^+, \quad b_1 \geq 2,
\]

\[
b_{n+1} \geq \frac{b_n - 1}{f_n} + 1 \geq 2 \quad \forall n \geq 1,
\]

converges to a real number \( B_1 \) such that \( b_1 - 1 \leq (1/B_1) < b_1 \). Furthermore, \( b_1 = 1 + [1/B_1] \).

By setting \( e_n = 1/a_n, e_n = 1, \) and \( e_n = a_n - 1, \) for all \( n \geq 1 \) in Theorem 1, and by setting \( f_n = 1/b_n, f_n = 1, \) and \( f_n = b_n - 1, \) for all \( n \geq 1 \) in Lemma 2, we obtain the following well-known expansions for real numbers (see [1–3]), namely, Sylvester series expansion, Engel series expansion, and Lüroth series expansion, respectively, as we now record.

**Corollary 3.** Each \( A \in \mathbb{R} \) is uniquely representable as a Sylvester series expansion; that is,

\[
A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n},
\]

where \( a_1 \geq 2 \) and \( a_{n+1} \geq (a_n - 1)/a_n + 1 \) for all \( n \geq 1 \).

Conversely, each series of the form (14) converges.

**Corollary 4.** Each \( A \in \mathbb{R} \) is uniquely representable as an Engel series expansion; that is,

\[
A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_1 \cdots a_n},
\]

where \( a_1 \geq 2 \) and \( a_{n+1} \geq a_n \) for all \( n \geq 1 \).

Conversely, each series of the form (15) converges.

**Corollary 5.** Each \( A \in \mathbb{R} \) is uniquely representable as a Lüroth series expansion; that is,

\[
A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n (a_n - 1) a_{n+1}},
\]

where \( a_n \geq 2 \) for all \( n \geq 1 \).

Conversely, each series of the form (16) converges.

**Proposition 6.** Let

\[
a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 a_2 \cdots a_n a_{n+1}},
\]

\[
b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} \frac{1}{b_1 b_2 \cdots b_n b_{n+1}},
\]

be SEL series expansions of distinct real numbers \( A \) and \( B \), respectively. Then the condition \( A < B \) is equivalent to the following:

(i) \( a_0 < b_0 \) if \( a_0 \neq b_0 \);

(ii) \( a_i > b_i \) for the first \( i \geq 1 \) such that \( a_i \neq b_i \) if \( a_0 = b_0 \).

**Proof.** If \( a_0 < b_0 \), then

\[
A = a_0 + A_1 \leq a_0 + 1 \leq b_0 < b_0 + B_1 = B.
\]

Hence (i) follows.
Now assume that \( a_0 = b_0 \) and \( a_i > b_i \) for the first \( i \geq 1 \) such that \( a_i \neq b_i \). Applying (9) repeatedly, we obtain

\[
A_n = \frac{1}{a_n} + \frac{1}{a_n e_n} \left( \frac{1}{a_{n+1}} + \frac{A_{n+1}}{a_{n+1} e_{n+1}} \right)
\]

\[
\vdots
\]

\[
= \frac{1}{a_n} + \frac{1}{a_n e_n a_{n+1}} + \cdots + \frac{1}{a_n e_n \cdots a_{n+k-1} e_{n+k-1} a_{n+k}} + \frac{A_{n+k+1}}{a_n e_n \cdots a_{n+k-1} e_{n+k-1} a_{n+k} e_{n+k}}
\]

for all \( n, k \in \mathbb{N} \). Using (8), we get

\[
\frac{1}{a_n} < A_n \leq \frac{1}{a_n - 1}
\]

for all \( n \geq 1 \). We now prove that \( 0 < A_n \leq 1 \) for all \( n \geq 1 \) by induction on \( n \). If \( n = 1 \), then we have seen that \( 0 < A_1 \leq 1 \). Assume now that \( 0 < A_n \leq 1 \) for \( n \geq 1 \). By (8), we see that \( a_n \geq 2 \). Since

\[
A_{n+1} = (a_n A_n - 1) e_n = \left( A_n - \frac{1}{a_n} \right) a_n e_n
\]

and using (10) and (20), we have that

\[
0 < A_{n+1} = (a_n A_n - 1) e_n \leq \left( \frac{a_n}{a_n - 1} \right) e_n = \frac{e_n}{a_n - 1} \leq 1,
\]

as desired. It follows that \( a_n \geq 2 \) for all \( n \geq 1 \).

Next, we will prove that

\[
A_n = \frac{1}{a_n} + \frac{1}{a_n e_n a_{n+1}} + \cdots
\]

(23)

for all \( n \geq 1 \). For \( n, k \geq 1 \), let

\[
B_k = \frac{1}{a_n} + \frac{1}{a_n e_n a_{n+1}} + \cdots + \frac{1}{a_n e_n \cdots a_{n+k-1} e_{n+k-1} a_{n+k}},
\]

(24)

Since \( A_n, e_n > 0 \) and \( a_n \in \mathbb{N} \), for all \( n \geq 1 \), the sequence of real numbers \( B_k \) is increasing and bounded above. Thus,

\[
\lim_{k \to \infty} B_k \text{ exists and so } \lim_{k \to \infty} \frac{1}{a_n e_n \cdots a_{n+k} e_{n+k} a_{n+k+1}} = 0.
\]

(25)

Since (20) and \( a_{n+k+1} \geq 2 \), we deduce that

\[
0 < \frac{A_{n+k+1}}{a_n e_n \cdots a_{n+k} e_{n+k} a_{n+k+1}} \leq \frac{1}{a_n e_n \cdots a_{n+k} e_{n+k} a_{n+k+1} \cdot a_{n+k+1} - 1} \leq \frac{2}{a_n e_n \cdots a_{n+k} e_{n+k} a_{n+k+1}}
\]

(26)

and so

\[
\lim_{k \to \infty} \frac{A_{n+k+1}}{a_n e_n \cdots a_{n+k} e_{n+k} a_{n+k+1} \cdot a_{n+k+1} - 1} = 0,
\]

(27)

showing that (23) holds as desired.

Using (23), (20), and \( a_i - 1 \geq b_i \), we have

\[
A = a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_i e_i \cdots a_{i-1} e_{i-1} a_i} + \frac{1}{a_i e_i \cdots a_{i+1} a_{i+1}} + \cdots
\]

\[
= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_i e_i \cdots a_{i-1} e_{i-1} a_i} \left( \frac{1}{a_i} + \frac{1}{a_i e_i} \right) + \cdots
\]

\[
= a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_i e_i \cdots a_{i-1} e_{i-1} a_i} \left( \frac{1}{a_i - 1} \right)
\]

\[
\leq a_0 + \frac{1}{a_1} + \cdots + \frac{1}{a_i e_i \cdots a_{i-1} e_{i-1} a_i} \left( \frac{1}{b_i} \right)
\]

\[
= b_0 + \frac{1}{b_1} + \cdots + \frac{1}{b_i e_i \cdots b_{i-1} e_{i-1} b_i}
\]

(28)

and the assertion follows.

The following corollaries follow immediately from Proposition 6 by setting \( e_n = 1/a_n, e_n = 1 \), and \( e_n = a_n - 1 \), for all \( n \geq 1 \), respectively; the first two corollaries readily appear in [4].

Corollary 7. Let

\[
a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n}
\]

(29)

\[
b_0 + \sum_{n=1}^{\infty} \frac{1}{b_n}
\]

be the Sylvester series expansions of real numbers \( A \) and \( B \), respectively. Then the condition \( A < B \) is equivalent to the following:

(i) \( a_0 < b_0 \) if \( a_0 \neq b_0 \);

(ii) \( a_i > b_i \) for the first \( i \geq 1 \) such that \( a_i \neq b_i \) if \( a_0 = b_0 \).

Corollary 8. Let

\[
a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n}
\]

(30)

\[
b_0 + \sum_{n=1}^{\infty} \frac{1}{b_n}
\]

be the Engel series expansions of real numbers \( A \) and \( B \), respectively. Then the condition \( A < B \) is equivalent to the following:
(i) \( a_0 < b_0 \) if \( a_0 \neq b_0 \);
(ii) \( a_i > b_i \) for the first \( i \geq 1 \) such that \( a_i \neq b_i \) if \( a_0 = b_0 \).

**Corollary 9.** Let
\[
\begin{align*}
a_0 &< b_0 \text{ if } a_0 \neq b_0; \\
\frac{1}{a_1} + \frac{1}{a_1} + \sum_{n=1}^{\infty} a_n (a_1 - 1) \cdots (a_n - 1) a_{n+1}, \\
b_0 &< b_0 + \frac{1}{b_1} + \sum_{n=1}^{\infty} b_n (b_1 - 1) \cdots (b_n - 1) b_{n+1}
\end{align*}
\]
be the Lüroth series expansions of real numbers \( A \) and \( B \), respectively. Then the condition \( A < B \) is equivalent to the following:
(i) \( a_0 < b_0 \) if \( a_0 \neq b_0 \);
(ii) \( a_i > b_i \) for the first \( i \geq 1 \) such that \( a_i \neq b_i \) if \( a_0 = b_0 \).

### 3. Constructions and Ordered Properties

In this section, we construct a generalized model for the real number system using series of rationals, which yields generalized versions of Knopmachers' models in \([4]\) as follows: let \( f : [2, \infty) \rightarrow \mathbb{R} \) be a nondecreasing function such that \( f(x) \geq 2 \) for all \( x \geq 2 \) and let \( \mathcal{U}_f \) be the set of all formal infinite sequences \( \mathcal{A} = (a_0, a_1, a_2, \ldots) \) of integers \( a_n \) such that \( a_1 \geq 2 \) and \( a_{n+1} \geq f(a_n) \) for all \( n \geq 1 \). In other words,
\[
\mathcal{U}_f = \left\{ (a_0, a_1, a_2, \ldots) \mid a_0, a_n \in \mathbb{Z}, a_1 \geq 2, a_{n+1} \geq f(a_n) \quad \forall n \geq 1 \right\}.
\]
As an analogue to Proposition 6, we define order relation on \( \mathcal{U}_f \) as follows.
For \( \mathcal{A} = (a_0, a_1, a_2, \ldots), \mathcal{B} = (b_0, b_1, b_2, \ldots) \in \mathcal{U}_f \), we say that \( \mathcal{A} < \mathcal{B} \) if and only if
(i) \( a_0 < b_0 \) if \( a_0 \neq b_0 \), or
(ii) \( a_i > b_i \) for the first \( i \geq 1 \) such that \( a_i \neq b_i \) if \( a_0 = b_0 \).

Note that \( \mathcal{A} \leq \mathcal{B} \) if and only if \( \mathcal{A} < \mathcal{B} \) or \( \mathcal{A} = \mathcal{B} \).

**Lemma 10.** \( \leq \) is a total ordering relation on \( \mathcal{U}_f \).

**Proof.** It is clear that \( \leq \) is reflexive and antisymmetric and any two elements in \( \mathcal{U}_f \) are comparable. It remains to show that \( \leq \) is transitive. Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{U}_f \) be such that \( \mathcal{A} \leq \mathcal{B} \) and \( \mathcal{B} \leq \mathcal{C} \). If \( \mathcal{A} = \mathcal{B} = \mathcal{C} \), \( \mathcal{A} = \mathcal{B} < \mathcal{C} \), or \( \mathcal{B} \leq \mathcal{C} \), then \( \mathcal{A} \leq \mathcal{C} \). Assume that \( \mathcal{A} < \mathcal{B} \) and \( \mathcal{B} < \mathcal{C} \), so \( a_0 \leq b_0 \leq c_0 \). If \( a_0 < b_0 < c_0, a_0 = b_0 < c_0, \) or \( a_0 < b_0 = c_0, \) then it is clear that \( \mathcal{A} < \mathcal{C} \).

Now assume that \( a_0 = b_0 = c_0 \). Then \( a_i > b_i \) for the first \( i \geq 1 \) such that \( a_i \neq b_i \) and \( b_j > c_j \) for the first \( j \geq 1 \) such that \( b_j \neq c_j \). Thus,
(i) if \( i < j \), then \( a_r = b_r = c_r \), for \( r < i \) and \( a_i > b_i \).

(ii) if \( i = j \), then \( a_r = b_r = c_r \), for \( r < i \) and \( a_i > b_i \).

(iii) if \( j < i \), then \( a_r = b_r = c_r \), for \( r < j \) and \( a_j > b_j \).

Hence \( \mathcal{A} < \mathcal{C} \) in each case, and so \( \mathcal{A} \leq \mathcal{C} \) as desired. \( \square \)

For convenience, we will denote \( f(a) \) by the infinite sequence
\[
f(a), f^2(a), f^3(a), \ldots,
\]
where \( a \in \mathbb{N} \) with \( a \geq 2 \) and \( f^j \) is the \( j \)th composite iteration of \( f \).

**Theorem 11.** Every nonempty subset of \( \mathcal{U}_f \) which is bounded above has a least upper bound (supremum).

**Proof.** Let \( \mathcal{X} \) be a nonempty subset of \( \mathcal{U}_f \) which is bounded above by a sequence \( \mathcal{B} = (b_0, b_1, b_2, \ldots) \in \mathcal{U}_f \). Then \( \mathcal{A} \leq \mathcal{B} \), and so \( a_0 \leq b_0 \) for all \( \mathcal{A} \in \mathcal{X} \). Let \( m_0 \) be the maximum value of \( a_0 \) for all \( \mathcal{A} \in \mathcal{X} \). Let
\[
\mathcal{M} = \left( m_0, 2, f(2) \right).
\]
We will first show that \( \mathcal{M} \) is an upper bound of \( \mathcal{X} \). For \( \mathcal{A} = (a_0, a_1, a_2, \ldots) \in \mathcal{X} \), if \( a_0 < m_0 \), then it is clear that \( \mathcal{A} < \mathcal{M} \). If \( a_0 = m_0 \), then we prove that
\[
a_n \geq f^{n-1}(2)
\]
for all \( n \geq 1 \) by induction on \( n \), where \( f^0 \) is the identity map. If \( n = 1 \), we have \( a_1 \geq 2 \geq f^0(2) \). Assume now that \( a_n \geq f^{n-1}(2) \) for \( n \geq 1 \). Since \( f \) is nondecreasing, we obtain
\[
a_{n+1} \geq f(a_n) \geq f(f^{n-1}(2)) = f^n(2),
\]
as desired. It then follows that \( \mathcal{A} < \mathcal{M} \); that is, \( \mathcal{M} \) is an upper bound of \( \mathcal{X} \). Hence we may assume that \( b_0 = m_0 \), and so \( b_i = a_i \) for some \( \mathcal{A} = (a_0, a_1, a_2, \ldots) \in \mathcal{X} \). Moreover, we may assume that \( \mathcal{B} \neq \mathcal{X} \), since otherwise \( \mathcal{B} = \sup \mathcal{X} \). Then \( \mathcal{A} \leq \mathcal{B} \) for all \( \mathcal{A} \in \mathcal{X} \), and there is the largest index \( k \geq 0 \) such that
\[
a_0 = b_0, \quad a_i = b_i, \ldots, a_k = b_k
\]
for every \( \mathcal{A} \in \mathcal{X} \) with \( a_0 = b_0 \).

Next, we define a sequence \( \mathcal{C} = (c_0, c_1, c_2, \ldots) \in \mathcal{U}_f \) and then show that \( \mathcal{C} = \sup \mathcal{X} \). Let \( c_0 = b_0, c_1 = b_1, \ldots, c_k = b_k \), and let \( c_{k+1} \) be the least possible value for \( a_{k+1} \) of any element \( \mathcal{A} \in \mathcal{X} \) with \( a_0 = b_0 \). Let \( c_{k+2} \) be the least possible value for \( a_{k+2} \) of any element \( \mathcal{X} \) of the form
\[
(c_0, \ldots, c_{k+1}, a_{k+2}, a_{k+3}, \ldots).
\]
Continuing inductively to define \( c_{k+2} \) as the least possible value for \( a_{k+2} \) of any element \( \mathcal{X} \) of the form
\[
(c_0, \ldots, c_{k+1}, a_{k+1}, a_{k+2}, \ldots).
\]
this process eventually yields a sequence \( \mathcal{C} = (c_0, c_1, c_2, \ldots) \) with \( c_i \geq f(c_j) \) for all \( i = 1, 2, \ldots \).
Finally, we prove that $C = \sup X$. It is clear that $A < C$ for every $A \in X$ with $a_0 < b_0 = c_0$. By the construction of $C$, we see that $A \leq C$ for every $A \in X$ with $a_0 = b_0$, so $C$ is an upper bound of $X$. If $X$ has an upper bound $D < C$, then $d_0 \leq C$. If $d_0 < c_0$, $C < A$ for all $A \in X$ with $a_0 = b_0$, which is impossible. Thus $d_0 = c_0$ and $d_j > c_j$ for the first $j \geq 1$ such that $d_j \neq c_j$. Hence every element of the form

$$a = (a_0, c_1, \ldots, c_{j-1}, a_{j+1}, a_{j+2}, \ldots)$$

in $X$ satisfies $D < a \leq D$, a contradiction. \hfill \square

**Theorem 12.** Given any element $A$ of $U_f$, there exist $A^{(n)}, A^{(m)} \in U_f$ for $n \geq 1$ such that

(i) $A^{(n)} < A^{(m)} < A \leq A^{(m)} \leq A^{(n)}$ for $m < n$,

(ii) $A = \sup A^{(n)} = \inf A^{(m)}$.

**Proof.** Let $A = (a_0, a_1, \ldots)$ be any sequence in $U_f$. We define $A^{(n)}$ and $A^{(m)}$ for $n \geq 1$ as follows:

$$A^{(n)} = (a_0, a_1, \ldots, a_{n-1}, a_n, f(a_n)), \quad A^{(m)} = (a_0, a_1, \ldots, a_{m-1}, a_m, f(a_m), f(a_{m-1}), \ldots),$$

where $a_i' = a_i + 1$. It is clear that $A^{(n)}, A^{(m)} \in U_f$ for all $n \geq 1$.

(i) For positive integers $m, n$, with $m < n$, we have

$$A^{(m)} = (a_0, a_1, \ldots, a_{m-1}, a_m, f(a_m), f^2(a_m), \ldots),$$

$$A^{(n)} = (a_0, a_1, \ldots, a_{n-1}, a_n, f(a_n), f^2(a_n), \ldots),$$

$$A = (a_0, a_1, \ldots, a_{n-1}, a_n, f(a_n), f^2(a_n), \ldots).$$

(ii) Suppose that there exists $B \in U_f$ such that $A^{(n)} \leq B < A$ for all $n \geq 1$. Then we must have $a_0 = b_0$ and $b_k > a_k$ for the first $k$ such that $a_k \neq b_k$. Thus

$$B < A^{(k+1)} = (a_0, a_1, \ldots, a_k, a_{k+1}', f(a_{k+1}')), \quad (43)$$

a contradiction. Hence $A = \sup A^{(n)}$. Similarly, suppose that there is $C \in U_f$ such that $A < C \leq A^{(n)}$ for all $n \geq 1$. Then we must have $a_0 = c_0$ and $a_i > c_i$ for the first $i$ such that $a_i \neq c_i$. This gives the contradiction

$$A^{(n)} = (a_0, a_1, \ldots, a_{n-1}, a_n, f(a_n), f(a_{n-1}), \ldots) \leq C.$$

Thus $A = \inf A^{(n)}$ and part (ii) follows. \hfill \square

### 4. Algebraic Operations in $U_f$

For a nondecreasing function $f : [2, \infty) \to \mathbb{R}$ such that $f(x) \geq 2$ for all $x \geq 2$, let $\rho_f$ be any 1-1 order-preserving map from $\mathbb{R}$ onto $U_f$. To show that $\rho_f^{-1}$ is order-preserving, let $A, B \in U_f$ be such that $A \leq B$. Then $A = \rho_f(A)$ and $B = \rho_f(B)$ for some $A, B \in \mathbb{R}$. Since $\rho_f$ is order-preserving, we obtain

$$\rho_f^{-1}(A) = A \leq B = \rho_f^{-1}(B),$$

as desired.

For $A \in U_f$, let $A^{(n)}$ and $A^{(m)}$ be real numbers such that

$$\rho_f(A^{(n)}) = A^{(n)} = \rho_f(A^{(m)}),$$

for $n \geq 1$, where $A^{(n)}$ and $A^{(m)}$ are defined as in Theorem 12. Since $\rho_f$ is order-preserving, we deduce that

$$A^{(m)} < A^{(n)} < A \leq A^{(n)} \leq A^{(m)}$$

for all positive numbers $m, n$ with $m < n$, where $A = \rho_f^{-1}(A)$.

Now, for any $A, B \in U_f$, define

$$A \oplus B = \rho_f\left(\sup(A^{(n)} + B^{(n)})\right),$$

$$-A = \rho_f\left(\sup(-A^{(n)})\right).$$

Note that both operators are well defined in $U_f$ because

$$A^{(n)} + B^{(n)} \leq A + B, \quad -A^{(n)} \leq -A \quad (n \geq 1)$$

and $(A^{(n)} + B^{(n)}), (-A^{(n)})$ are increasing sequences of real numbers. It follows that

$$A + B = \sup(A^{(n)} + B^{(n)}),$$

$$-A = \sup(-A^{(n)}).$$

Using (48), we deduce that

$$\rho_f(A + B) = \rho_f(A) \oplus \rho_f(B),$$

$$\rho_f(-A) = -\rho_f(A)$$

for all $A, B \in \mathbb{R}$.

**Theorem 13.** $(U_f, \oplus)$ is an abelian group. Furthermore, for $A, B, C \in U_f$, one has the following:

(i) If $A < B$, then $A \oplus C < B \oplus C$;
(ii) \( A < B \) if and only if \(-A > -B\).

Proof. First, we will show that \( (\mathcal{U}, \varnothing) \) is an abelian group. Let \( A \neq \rho_f(A), B = \rho_f(B), C = \rho_f(C) \in \mathbf{U}_f \) for some \( A, B, C \in \mathbb{R} \). Using (51), we obtain the following:

1. \( A \oplus B = \rho_f(A) \oplus \rho_f(B) = \rho_f(A + B) \in \mathbf{U}_f \);
2. \( A \oplus (B \oplus C) = \rho_f(A + (B + C)) = \rho_f((A + B) + C) = (A \oplus B) \oplus C \);
3. let \( \varnothing_f := \rho_f(0) \in \mathbf{U}_f \); then \( A \oplus \varnothing_f = \rho_f(A) \oplus \rho_f(0) = \rho_f(A + 0) = A \) for all \( A \in \mathbf{U}_f \);
4. \( A \oplus (-A) = \rho_f(A) \oplus \rho_f(-A) = \rho_f(0) = \varnothing_f \) for all \( A \in \mathbf{U}_f \); and
5. \( A \oplus B = \rho_f(A) \oplus \rho_f(B) = \rho_f(A + B) = \rho_f(B + A) = \rho_f(B) \oplus \rho_f(A) = B \oplus A \).

From (1)–(5), we conclude that \( (\mathbf{U}_f, \varnothing) \) is an abelian group.

Lastly, assume that \( A < B \). Since \( \rho_f^{-1} \) is order-preserving, we have \( A < B \). Using (51), we have

\[
A \oplus C = \rho_f(A + C) < \rho_f(B + C) = A \oplus C,
\]

\[
A = -\rho_f(A) = \rho_f(-A) < \rho_f(B) = B.
\]

Hence (ii) follows since \(-A < B\).

Next, we define the binary operation \( \circ \) on \( \mathbf{U}_f \) as follows: for any \( A, B \in \mathbf{U}_f \), let

\[
A \circ B = \begin{cases} 
\rho_f\left(\sup \left(\left(A_{(n)} \cdot B_{(n)}\right)\right)\right) & \text{if } \mathcal{A} \circ \mathcal{B} \circ \mathcal{C} \circ \mathcal{D}, \\
-\left(\mathcal{A} \circ \mathcal{B}\right) & \text{if } \mathcal{A} \circ \mathcal{B} = \mathcal{C} \circ \mathcal{D}, \\
\rho_f(A) & \text{if } \mathcal{A} \circ \mathcal{B} = \mathcal{C} \circ \mathcal{D}.
\end{cases}
\]

Also define

\[
A^{-1} = \begin{cases} 
\rho_f\left(\sup \left(\left(A_{(n)}^{-1}\right)\right)\right) & \text{if } \mathcal{A} > \mathcal{B}, \\
-\left(\mathcal{A} \circ \mathcal{B}\right) & \text{if } \mathcal{A} > \mathcal{B}.
\end{cases}
\]

5. Examples

In this section, we will give three models for the real number system which are special examples of the ordered field \( (\mathbf{U}_f, \circ, \varnothing) \), where \( f \) is chosen from three different nondecreasing functions from \([2, \infty) \) into \( \mathbb{R} \) such that \( f(x) \geq 2 \) for all \( x \geq 2 \). The first two propositions were readily mentioned in Knopmacher’s results [4].

Proposition 15. Let \( f_5: [2, \infty) \to \mathbb{R} \) be defined by

\[
f_5(x) = x(x - 1) + 1,
\]

Theorem 14. \( (\mathbf{U}_f, \circ, \varnothing) \) is an ordered field. Furthermore, if

\[
A < B \text{ and } C > D,
\]

then \( A \circ C < B \circ D \).

Proof. By Theorem 13, we know that \( (\mathbf{U}_f, \varnothing) \) is an abelian group. To prove that \( (\mathbf{U}_f, \circ, \varnothing) \) is a field, it remains to show that \( (\mathbf{U}_f \setminus \{ \mathcal{F}_f \}, \circ) \) is an abelian group. Let \( A = \rho_f(A), B = \rho_f(B), \mathcal{C} = \rho_f(C) \in \mathbf{U}_f \). Using (56) and (57), we obtain the following:

1. \( A \circ B = \rho_f(A) \circ \rho_f(B) = \rho_f(A \cdot B) \in \mathbf{U}_f \);
2. \( A \circ B = \rho_f(A) \circ \rho_f(B) = \rho_f((A \cdot B) \circ C) \in \mathbf{U}_f \);
3. let \( \mathcal{F} := \rho_f(1) \in \mathbf{U}_f \); then \( A \circ \mathcal{F} = \rho_f(A \cdot 1) = A \) for all \( A \in \mathbf{U}_f \); and
4. \( A \circ \mathcal{F} = \rho_f(A) \circ \rho_f(A^{-1}) = \mathcal{F} \) for all \( A \in \mathbf{U}_f \).

From (1)–(5), we conclude that \( (\mathbf{U}_f \setminus \{ \mathcal{F}_f \}, \circ) \) is an abelian group; thus \( (\mathbf{U}_f, \circ, \varnothing) \) is a field.

Next, we will show that \( (\mathbf{U}_f, \circ, \varnothing) \) is an ordered field (see [6]). Let

\[
\mathcal{P}_f = \left\{ A \in \mathbf{U}_f \mid A > \mathcal{F}_f \right\}
\]

and \( A, B \in \mathcal{P}_f \). Then \( A = \rho_f(A) \) and \( B = \rho_f(B) \) for some positive real numbers \( A \) and \( B \).

Finally, assume that \( A < B \) and \( C > D \). Since \( \rho_f^{-1} \) is order-preserving, then \( A < B \). Using (56), we have

\[
A \circ C = \rho_f(A \cdot C) < \rho_f(B \cdot C) = B \circ C
\]

since \( \rho_f \) is order-preserving. This completes the proof of the theorem. 

5. Examples

In this section, we will give three models for the real number system which are special examples of the ordered field \( (\mathbf{U}_f, \circ, \varnothing) \), where \( f \) is chosen from three different nondecreasing functions from \([2, \infty) \) into \( \mathbb{R} \) such that \( f(x) \geq 2 \) for all \( x \geq 2 \). The first two propositions were readily mentioned in Knopmacher’s results [4].

Proposition 15. Let \( f_5: [2, \infty) \to \mathbb{R} \) be defined by

\[
f_5(x) = x(x - 1) + 1,
\]
for all \( x \geq 2 \),
\[
\mathcal{U}_{f} = \{ (a_0, a_1, a_2, \ldots ) \mid a_n \in \mathbb{Z}, a_1 \geq 2, \ a_{n+1} \geq a_n (a_n - 1) + 1 \ \forall n \geq 1 \} \tag{61}
\]
and \( \rho_{f} : \mathbb{R} \to \mathcal{U}_{f} \) defined by
\[
\rho_{f} (A) = (a_0, a_1, a_2, \ldots ), \tag{62}
\]
for all \( A \in \mathbb{R} \), where
\[
A = a_0 + \sum_{n=1}^{\infty} \frac{1}{a_n} \tag{63}
\]
is the Sylvester series expansion of \( A \). Then \( (\mathcal{U}_{f}, \oplus, \odot) \) is an ordered field containing \( (\rho_{f}(\mathbb{Q}), \oplus, \odot) \) as a dense subfield. Furthermore, for \( A, B, C \in \mathcal{U}_{f} \), one has the following:
(i) if \( A < B \), then \( A \oplus C < B \oplus C \);
(ii) \( A < B \) if and only if \( -A > -B \);
(iii) \( A < B \) and \( C > 0_{f} \), then \( A \odot C < B \odot C \),
where \( 0_{f} \) is the zero element in \( \mathcal{U}_{f} \).

**Proof.** It is clear that \( f_{E} \) is a nondecreasing function such that \( f(x) \geq 2 \) for all \( x \geq 2 \). It is an immediate consequence of Corollary 3 that \( \rho_{f} \) is 1-1 map from \( \mathbb{R} \) onto \( \mathcal{U}_{f} \). By Corollary 7 and the definition of order in \( \mathcal{U}_{f} \), this map is order-preserving. Using Theorems 13 and 14, we can conclude that \( (\mathcal{U}_{f}, \oplus, \odot) \) is an ordered field containing \( \rho_{f}(\mathbb{Q}) \) and satisfies properties (i), (ii), and (iii). To complete the proof of this proposition, it remains to show that \( \rho_{f}(\mathbb{Q}) \) is a dense subfield of \( \mathcal{U}_{f} \).

It is clear that \( \rho_{f}(\mathbb{Q}) \) is a subfield of \( \mathcal{U}_{f} \). Now, let \( A, B \in \mathcal{U}_{f} \) be such that \( A < B \). If \( a_0 < b_0 \), let
\[
C = (b_0, b_1, f_{E}(b_{1})), \tag{64}
\]
then it is easy to see that \( A < C < B \) and \( C \in \rho_{f}(\mathbb{Q}) \). On the other hand, if \( a_0 = b_0 \), then \( a_m > b_m \) for the first \( m \) such that \( a_m \neq b_m \). In that case, we have \( A < D < B \), where
\[
D = (b_0 \ldots, b_m, f_{E}(b_{m+1})). \tag{65}
\]
Then \( D \in \rho_{f}(\mathbb{Q}) \). This shows that \( \rho_{f}(\mathbb{Q}) \) is dense in \( \mathcal{U}_{f} \), as desired. \( \square \)

**Proposition 16.** Let \( f_{E} : [2, \infty) \to \mathbb{R} \) be defined by
\[
f_{E}(x) = x, \tag{66}
\]
for all \( x \geq 2 \),
\[
\mathcal{U}_{f_{E}} = \{ (a_0, a_1, a_2, \ldots ) \mid a_n \in \mathbb{Z}, a_1 \geq 2, \ a_{n+1} \geq a_n \ \forall n \geq 1 \} \tag{67}
\]
and \( \rho_{f_{E}} : \mathbb{R} \to \mathcal{U}_{f_{E}} \) defined by
\[
\rho_{f_{E}} (A) = (a_0, a_1, a_2, \ldots ), \tag{68}
\]
for all \( A \in \mathbb{R} \), where
\[
A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 (a_1 - 1) \cdots a_n (a_n - 1) a_{m+1}} \tag{69}
\]
is the Engel series expansion of \( A \). Then \( (\mathcal{U}_{f_{E}}, \oplus, \odot) \) is an ordered field containing \( (\rho_{f_{E}}(\mathbb{Q}), \oplus, \odot) \) as a dense subfield. Furthermore, for \( A, B, C \in \mathcal{U}_{f_{E}} \), one has the following:
(i) if \( A < B \), then \( A \oplus C < B \oplus C \);
(ii) \( A < B \) if and only if \( -A > -B \);
(iii) \( A < B \) and \( C > 0_{f_{E}} \), then \( A \odot C < B \odot C \),
where \( 0_{f_{E}} \) is the zero element in \( \mathcal{U}_{f_{E}} \).

**Proof.** It is clear that \( f_{E} \) is a nondecreasing function such that \( f(x) \geq 2 \) for all \( x \geq 2 \). It is an immediate consequence of Corollary 4 that \( \rho_{f_{E}} \) is 1-1 map from \( \mathbb{R} \) onto \( \mathcal{U}_{f_{E}} \). By Corollary 8 and the definition of order in \( \mathcal{U}_{f_{E}} \), this map is order-preserving. Using Theorems 13 and 14, we can conclude that \( (\mathcal{U}_{f_{E}}, \oplus, \odot) \) is an ordered field containing \( \rho_{f_{E}}(\mathbb{Q}) \) and satisfies properties (i), (ii), and (iii). For density of rationals, we can prove in a similar way to the proof of Proposition 15 that \( \rho_{f_{E}}(\mathbb{Q}) \) is a dense subfield of \( \mathcal{U}_{f_{E}} \). \( \square \)

**Proposition 17.** Let \( f_{L} : [2, \infty) \to \mathbb{R} \) be defined by
\[
f_{L}(x) = x, \tag{70}
\]
for all \( x \geq 2 \),
\[
\mathcal{U}_{f_{L}} = \{ (a_0, a_1, a_2, \ldots ) \mid a_n \in \mathbb{Z}, a_1 \geq 2 \ \forall n \geq 1 \} \tag{71}
\]
and \( \rho_{f_{L}} : \mathbb{R} \to \mathcal{U}_{f_{L}} \) defined by
\[
\rho_{f_{L}} (A) = (a_0, a_1, a_2, \ldots ), \tag{72}
\]
for all \( A \in \mathbb{R} \), where
\[
A = a_0 + \frac{1}{a_1} + \sum_{n=1}^{\infty} \frac{1}{a_1 (a_1 - 1) \cdots a_n (a_n - 1) a_{m+1}} \tag{73}
\]
is the Lüroth series expansion of \( A \). Then \( (\mathcal{U}_{f_{L}}, \oplus, \odot) \) is an ordered field containing \( (\rho_{f_{L}}(\mathbb{Q}), \oplus, \odot) \) as a dense subfield. Furthermore, for \( A, B, C \in \mathcal{U}_{f_{L}} \), one has the following:
(i) if \( A < B \), then \( A \oplus C < B \oplus C \);
(ii) \( A < B \) if and only if \( -A > -B \);
(iii) \( A < B \) and \( C > 0_{f_{L}} \), then \( A \odot C < B \odot C \),
where \( 0_{f_{L}} \) is the zero element in \( \mathcal{U}_{f_{L}} \).

**Proof.** It is clear that \( f_{L} \) is a nondecreasing function such that \( f(x) \geq 2 \) for all \( x \geq 2 \). It is an immediate consequence of Corollary 5 that \( \rho_{f_{L}} \) is 1-1 map from \( \mathbb{R} \) onto \( \mathcal{U}_{f_{L}} \). By Corollary 9 and the definition of order in \( \mathcal{U}_{f_{L}} \), this map is order-preserving. Using Theorems 13 and 14, we can conclude that \( (\mathcal{U}_{f_{L}}, \oplus, \odot) \) is an ordered field containing \( \rho_{f_{L}}(\mathbb{Q}) \) and satisfies properties (i), (ii), and (iii). For density of rationals, we can prove in a similar way to the proof of Proposition 15 that \( \rho_{f_{L}}(\mathbb{Q}) \) is a dense subfield of \( \mathcal{U}_{f_{L}} \). \( \square \)
Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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