Researc h Article
Statistical Summability through de la Vallée-Poussin Mean in Probabilistic Normed Spaces

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Two concepts—one of statistical convergence and the other of de la Vallée-Poussin mean—play an important role in recent research on summability theory. In this work we define a new type of summability methods and statistical completeness involving the ideas of de la Vallée-Poussin mean and statistical convergence in the framework of probabilistic normed spaces.

1. Introduction, Definitions, and Preliminaries

Fast [1] presented the following definition of statistical convergence for sequences of real numbers. Let \( K \subseteq \mathbb{N} \), the set of natural numbers, and \( K_n = \{ k \leq n : k \in K \} \). The natural density of \( K \) is defined by \( \delta(K) = \lim_{n \to \infty} \frac{1}{n} |K_n| \) if the limit exists, where \( |K_n| \) denotes the cardinality of \( K_n \).

The sequence \( x = (x_j) \) is said to be statistically convergent to the number \( \ell \) if for every \( \epsilon > 0 \) the set \( K_\epsilon := \{ k \in \mathbb{N} : |x_k - \ell| \geq \epsilon \} \) has natural density zero; that is, for each \( \epsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n} \left| j \leq n : |x_j - \ell| \geq \epsilon \right| = 0.
\]

(1)

Note that every convergent sequence is statistically convergent to the same limit, but its converse need not be true.

In 1985, Fridy [2] has defined the notion of statistically Cauchy sequence and proved that it is equivalent to statistical convergence and since then a large amount of work has appeared. Various extensions, generalizations, variants, and applications have been given by several authors so far, for example, [3–8] and references therein. In the recent past, Mursaleen [9] presented a generalization of statistical convergence by using de la Vallée-Poussin mean which is known \( \lambda \)-statistical convergence and further studied by Çolak and Bektas [10,11]. For more details related to this concept we refer to [12–18].

Let \( \lambda = (\lambda_n) \) be a nondecreasing sequence of positive numbers tending to \( \infty \) such that

\[
\lambda_{n+1} \leq \lambda_n + 1, \quad \lambda_1 = 0.
\]

(2)

The generalized de la Vallée-Poussin mean is defined by

\[
t_n(x) := \frac{1}{\lambda_n} \sum_{j \in \lambda_n} x_j,
\]

(3)

where \( I_n = [n - \lambda_n + 1, n] \).

A sequence \( x = (x_j) \) is said to be \( (V, \lambda) \)-summable to a number \( \ell \) if

\[
t_n(x) \longrightarrow \ell \quad \text{as} \quad n \longrightarrow \infty.
\]

(4)

In this case \( \ell \) is called \( \lambda \)-limit of \( x \).

Let \( K \subseteq \mathbb{N} \) be a set of positive integers; then

\[
\delta_\lambda(K) = \lim_{n \to \infty} \frac{1}{\lambda_n} \left| \{ n - \lambda_n + 1 \leq j \leq n : j \in K \} \right|
\]

(5)

is said to be \( \lambda \)-density of \( K \).

In case \( \lambda_n = n \), \( \lambda \)-density reduces to the natural density. Also, since \( (\lambda_n/n) \leq 1 \), \( \delta(K) \leq \delta_\lambda(K) \) for every \( K \subseteq \mathbb{N} \).

The number sequence \( x = (x_j) \) is said to be \( \lambda \)-statistically convergent to the number \( \ell \) if, for each \( \epsilon > 0 \), \( \delta_\lambda(K_\epsilon) = 0 \), where \( K_\epsilon = \{ j \in \mathbb{N} : |x_j - \ell| > \epsilon \} \); that is,

\[
\lim_{n \to \infty} \frac{1}{\lambda_n} \left| j \in I_n : |x_j - \ell| > \epsilon \right| = 0.
\]

(6)
In this case we write $S_\lambda\lim_j x_j = \ell$ and we denote the set of all $\lambda$-statistically convergent sequences by $S_\lambda$.

A distribution function is an element of $\Delta^+$, where $\Delta^+ = \{ f: \mathbb{R} \to [0,1]; f \text{ is left-continuous, nondecreasing, } f(0) = 0 \text{ and } f(+\infty) = 1 \}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{ f \in \Delta^+; f(+\infty) = 1 \}$. Here $I^+ f(+\infty)$ denotes the left limit of the function $f$ at the point $x$. The space $\Delta^+$ is partially ordered by the usual pointwise ordering of functions; that is, $f \leq g$ if and only if $f(x) \leq g(x)$ for all $x \in \mathbb{R}$.

A triangle function is a binary operation on $\Delta^+$, namely, a function $\tau: \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative nondecreasing and which has $e$ as unit; that is, for all $f, g, h \in \Delta^+$, we have

\begin{enumerate}[(i)]  
  \item $\tau(\tau(f, g), h) = \tau(f, \tau(g, h))$,  
  \item $\tau(f, g) = \tau(g, f)$,  
  \item $\tau(f, h) = \tau(g, h)$ whenever $f \leq g$,  
  \item $\tau(f, e) = f$.
\end{enumerate}

Here $e$ is the d.f. defined by

$$
e(x) = \begin{cases} 0 & \text{if } x \leq 0; \\ 1 & \text{if } x > 0. \end{cases} \quad (7)$$

We remark that the set $\Delta$ as well as its subsets can be partially ordered by the usual pointwise order: in this order, $e$ is the maximal element in $\Delta^+$.

There are two definitions of probabilistic normed space, the original one by Šerstnev [19] who used the idea of Menger [20] to define such space and the other one by Alsina et al. [21].

According to Šerstnev, a probabilistic normed space (for short, PN-space) is a triple $(X, \nu, \tau)$, where $X$ is a real linear space, $\tau$ is a triangle function, and $\nu$ is the probabilistic norm; that is, $\nu$ is a map from $X$ into $\Delta^+$ that satisfies the following conditions:

\begin{enumerate}[(i)]  
  \item $\nu_x = e$ if and only if $x = \theta$, where $\theta$ is the null vector of $X$,  
  \item $\nu_{\alpha x}(t) = \nu_x(t/|\alpha|)$ for all $t > 0, \alpha \in \mathbb{R}$ with $\alpha \neq 0$, and $x \in X$,  
  \item $\nu_{x+y} \geq \tau(\nu_x, \nu_y)$ whenever $x, y \in X$.
\end{enumerate}

Here $\nu_x(t)$ denotes the value of $\nu_x$ at $t \in \mathbb{R}$.

In this paper, using the notions of statistical convergence and de la Vallée-Poussin mean, we define and study a new type of summability methods in the setting of probabilistic normed spaces. We also introduce a new type of statistical completeness through de la Vallée-Poussin mean in this framework.

2. Statistical Summability through de la Vallée-Poussin Mean

Here we introduce the notions of $\lambda$-summable and statistically $\lambda$-summable in PN-space and give some of its properties. We will assume throughout this paper that $(X, \mathcal{F}, \tau)$ is a probabilistic normed space.

**Definition 1.** A sequence $x = (x_k)$ is said to be $\lambda$-summable in PN-space $(X, \mathcal{F}, \tau)$ or simply $S(\lambda)$-summable to $\xi$ if for each $\epsilon > 0, \theta \in (0, 1)$ there exists a positive integer $j_0$ such that $\nu_{x_j}(\theta) > 1 - \theta$ for all $j \geq j_0$. In this case one writes $\nu(\lambda)\lim x_k = \xi$ and $\xi$ is called the $\nu(\lambda)$-limit of the sequence $x = (x_k)$.

**Definition 2.** A sequence $x = (x_k)$ is said to be statistically $\lambda$-summable in $(X, \nu, \tau)$ or simply $S(\lambda)$-summable to $\xi$ if for each $\epsilon > 0, \theta \in (0, 1)$ the set $K_{\lambda}(\xi) = \{ j \in \mathbb{N}; \nu_{x_j}(\theta) < 1 - \theta \}$ has natural density zero (briefly, $\delta(K_{\lambda}(\xi)) = 0$); that is,

$$\lim_{n \to \infty} \frac{1}{n} \left| \left\{ j \leq n : \nu_{x_j}(\theta) < 1 - \theta \right\} \right| = 0. \quad (8)$$

In this case one writes $\nu(S(\lambda))\lim x_k = \xi$, and $\xi$ is called the $\nu(S(\lambda))$-limit of $x$.

**Theorem 3.** If a sequence $x = (x_k)$ is statistically $\lambda$-summable in PN-space, that is, $\nu(S(\lambda))\lim x_k = \xi$ exists, then it is unique.

**Proof.** Suppose that there exist two elements $\xi_1, \xi_2 \in X$ with $\xi_1 \neq \xi_2$ such that $\nu(S(\lambda))\lim x_k = \xi_1$ and $\nu(S(\lambda))\lim x_k = \xi_2$. Let $\epsilon > 0$ be given. Choose $q > 0$ such that

$$\tau\left((1 - q), (1 - q)\right) > 1 - \epsilon. \quad (10)$$

Then, for any $t > 0$, we define

$$M_q(\lambda) = \left\{ j \in \mathbb{N} : \nu_{x_j}(\lambda) \leq 1 - q \right\},$$

$$M_q''(\lambda) = \left\{ j \in \mathbb{N} : \nu_{x_j}(\lambda) \leq 1 - q \right\}. \quad (11)$$

Since $\nu(S(\lambda))\lim x_k = \xi_1$ implies $\delta(M_q(\lambda)) = 0$ and, similarly, we have $\delta(M_q''(\lambda)) = 0$. Now, let $M_q(\lambda) = M_q'(\lambda) \cap M_q''(\lambda)$. It follows that $\delta(M_q(\lambda)) = 0$ and hence the complement $M_q^c(\lambda)$ is nonempty set and $\delta(M_q^c(\lambda)) = 1$. Now, if $k \in \mathbb{N} \setminus M_q(\lambda)$, then

$$\nu(x_t, \xi_1, \xi_2)(t) \geq \tau\left((1 - q), (1 - q)\right) > 1 - \epsilon. \quad (12)$$

Since $\epsilon > 0$ was arbitrary, we get $\nu(x_t, \xi_1, \xi_2)(t) = 1$ for all $t > 0$. Hence $\xi_1 = \xi_2$. This means that $\nu(S(\lambda))$-limit is unique. \(\square\)

The following theorem gives the algebraic properties of statistically $\lambda$-summable sequences in PN-spaces.

**Theorem 4.** Let $x = (x_k)$ and $y = (y_k)$ be two sequences. If $\nu(S(\lambda))\lim x_k = \xi_1$ and $\nu(S(\lambda))\lim y_k = \xi_2$, then

\begin{enumerate}[(i)]  
  \item $\nu(S(\lambda))\lim (x_k \pm y_k) = \xi_1 \pm \xi_2$,  
  \item $\nu(S(\lambda))\lim ax_k = a\xi_k, a(\neq 0) \in \mathbb{R}$.
\end{enumerate}

Proof of the theorem is straightforward and so omitted.
Theorem 5. If a sequence \( x = (x_k) \) is \( \lambda \)-summable to \( \xi \) in PN-space; then it is statistically \( \lambda \)-summable to the same limit.

Proof. Let \( \nu(\lambda)\text{-lim}x_k = \xi \). Then for every \( \epsilon > 0 \) and \( t > 0 \), there is a positive integer \( j_0 \) such that

\[
\nu_{t,\lambda}(x) - \xi(t) > 1 - \epsilon
\]

for all \( j \geq j_0 \). Since the set

\[
K_\epsilon(\lambda) := \left\{ j \in \mathbb{N} : \nu_{t,\lambda}(x) - \xi(t) \leq 1 - \epsilon \right\}
\]

is contained in \( \{1, 2, 3, \ldots, j_0 - 1\} \). As we know, every finite subset of \( \mathbb{N} \) has natural density zero; that is, \( \delta(K_\epsilon(\lambda)) = 0 \). Hence, a sequence \( x = (x_k) \) is \( S(\lambda)_\epsilon \)-summable to \( \xi \).

Remark 6. The converse of the above theorem is not true in general, which is verified by the following example.

Example 7. Let \( (\mathbb{R}, | \cdot |) \) denote the space of all real numbers with the usual norm and \( r(a, b) = |a - b| \) for all \( a, b \in [0, 1] \). Let \( \nu_\epsilon(t) = t/(t + |x|) \) for all \( x \in X \) and \( t > 0 \). In this case, we observe that \( (\mathbb{R}, \nu, r) \) is a PN-space. Define a sequence \( x = (x_k) \) by

\[
t_j(\epsilon) = \begin{cases} j & \text{if } j = n^2, \ n \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}
\]

For \( \epsilon > 0 \) and \( t > 0 \), write

\[
K_\epsilon(\lambda) = \left\{ j \in \mathbb{N} : \nu_{t,\lambda}(x) - \xi(t) \leq 1 - \epsilon \right\}.
\]

It is easy to see that

\[
\nu_{t,\lambda}(x) - \xi(t) = \begin{cases} t & \text{for } j = n^2, \ n \in \mathbb{N} \\ t + j & \text{otherwise;} \end{cases}
\]

hence

\[
\lim \nu_{t,\lambda}(x) = \begin{cases} 0 & \text{for if } j = n^2, \ n \in \mathbb{N} \\ 1 & \text{otherwise.} \end{cases}
\]

Therefore, the sequence \( (x_k) \) is not \( (\lambda)_\epsilon \)-summable. But the set \( K_\epsilon(\lambda) \) has natural density zero since \( K_\epsilon(\lambda) \subset \{1, 4, 9, 16, \ldots\} \). From here, we conclude that the converse of Theorem 5 need not be true.

Theorem 8. A sequence \( x = (x_k) \) is statistically \( \lambda \)-summable in PN-space to \( \xi \) if and only if there exists a subset \( K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N} \) such that \( \delta(K) = 1 \) and \( \nu(\lambda)\text{-lim}x_k = \xi \).

Proof. Suppose that there exists a subset \( K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N} \) such that \( \delta_K(\lambda) = 1 \) and \( \nu(\lambda)\text{-lim}x_k = \xi \). Then there exists a positive integer \( N \in \mathbb{N} \) such that for \( n \geq N \)

\[
\nu_{t,\lambda}(x) - \xi(t) > 1 - \epsilon.
\]

Put \( K_\epsilon(\lambda) = \{n \in \mathbb{N} : \nu_{t,\lambda}(x) - \xi(t) \leq 1 - \epsilon\} \) and \( K' = \{k_{N+1}, k_{N+2}, \ldots\} \). Then \( \delta(K') = 1 \) and \( K_\epsilon(\lambda) \subseteq \mathbb{N} - K' \) which implies that \( \delta(K_\epsilon(\lambda)) = 0 \). Hence \( x = (x_k) \) is statistically \( \lambda \)-summable to \( \xi \) in PN-space.

Conversely, let sequence \( x = (x_k) \) is statistically \( \lambda \)-summable to \( \xi \). For \( q = 1, 2, 3, \ldots \) and \( t > 0 \), write

\[
K_q(\lambda) = \left\{ j \in \mathbb{N} : \nu_{t,\lambda}(x) - \xi(t) \geq 1 - \frac{1}{q} \right\},
\]

\[
M_q(\lambda) = \left\{ j \in \mathbb{N} : \nu_{t,\lambda}(x) - \xi(t) < 1 - \frac{1}{q} \right\}.
\]

Then \( \delta(K_q(\lambda)) = 0 \) and

\[
M_1(\lambda) \supset M_2(\lambda) \supset \cdots \supset M_q(\lambda) \supset M_{q+1}(\lambda) \supset \cdots,
\]

\[
\delta(M_q(\lambda)) = 1, \quad q = 1, 2, \ldots
\]

Now we have to show that for \( j \in M_q(\lambda), x = (x_k) \) is \( (\lambda)_\epsilon \)-summable to \( \xi \). Suppose that \( x = (x_k) \) is not \( (\lambda)_\epsilon \)-summable to \( \xi \). Therefore there is \( \epsilon > 0 \) such that \( \nu_{t,\lambda}(x) - \xi(t) \geq \epsilon \) for infinitely many terms. Let

\[
M_\epsilon(\lambda) = \left\{ j \in \mathbb{N} : \nu_{t,\lambda}(x) - \xi(t) < \epsilon \right\},
\]

and \( \epsilon > 1/q \) with \( q = 1, 2, 3, \ldots \) Then

\[
\delta(M_\epsilon(\lambda)) = 0,
\]

and, by (21), \( M_\epsilon(\lambda) \subset M_q(\lambda) \). Hence \( \delta(M_\epsilon(\lambda)) = 0 \), which contradicts (22) and therefore \( x = (x_k) \) is \( (\lambda)_\epsilon \)-summable to \( \xi \).

Similarly we can prove the following dual statement.

Theorem 9. A sequence \( x = (x_k) \) is \( \lambda \)-statistically summable in PN-space to \( \xi \) if and only if there exists a subset \( K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N} \) such that \( \delta_\lambda(K) = 1 \) and \( \nu(\lambda)\text{-lim}x_k = \xi \).

In this section, we define the notions of statistically \( \lambda \)-Cauchy and statistically \( \lambda \)-complete with respect to probabilistic normed space and prove related results.

Definition 10. A sequence \( x = (x_k) \) is said to be statistically \( \lambda \)-Cauchy in \((X, \nu, r)\) or simply \( S(\lambda)_\epsilon \)-Cauchy if, for every \( \epsilon > 0 \) and \( \theta \in (0, 1) \), there exists a number \( N = N(\epsilon, \theta) \) such that, for all \( j, h \geq N \), the set \( S_\epsilon(\lambda) = \{j \in \mathbb{N} : \nu_{t,\lambda}(x) - \xi(t) \leq 1 - \theta\} \) has natural density zero (briefly, \( \delta(S_\epsilon(\lambda)) = 0 \)); that is,

\[
\lim_{n \to \infty} \frac{1}{n} \left| \left\{ j \leq n : \nu_{t,\lambda}(x) - \xi(t) \leq 1 - \theta \right\} \right| = 0.
\]

Theorem 11. A sequence \( x = (x_k) \) is \( \lambda \)-summable in PN-space; then it is statistically \( \lambda \)-Cauchy.

Proof. Suppose that \( \nu(S_\epsilon(\lambda))\text{-lim}x_k = \xi \). Let \( \epsilon > 0 \) be a given number and choose \( q > 0 \) such that

\[
\tau((1 - q), (1 - q)) > 1 - \epsilon.
\]
Then, for $t > 0$, we have $\delta(A_q(\lambda)) = 0$, where $A_q(\lambda) = \{ j \in \mathbb{N} : \gamma_j(x(t/2) \leq 1 - q) \}$ which implies that

$$\delta \left( A_q^\delta(\lambda) \right) = \delta \left( \left\{ j \in \mathbb{N} : \gamma_j(x(t/2) \geq \left( \frac{t}{2} \right) > 1 - q \right\} \right) = 1. \quad (27)$$

Let $m \in A_q^\delta(\lambda)$. Then $\gamma_{n,(x(t/2))} > 1 - q$. Now, let

$$B_\varepsilon(\lambda) = \left\{ j \in \mathbb{N} : \gamma_j(x(t/2)) \leq 1 - \varepsilon \right\}. \quad (28)$$

We need to show that $B_\varepsilon(\lambda) \subset A_q(\lambda)$. Let $j \in B_\varepsilon(\lambda)$. Then $\gamma_j(x(t/2)) \leq 1 - \varepsilon$ and hence $\gamma_j(x(t/2)) \leq 1 - q$; that is, $j \in A_q(\lambda)$. Otherwise, if $\gamma_j(x(t/2)) > 1 - q$, then

$$1 - \varepsilon \geq \gamma_j(x(t/2)) \geq \tau \left( \gamma_j(x(t/2)) \geq \left( \frac{t}{2} \right) \right) \geq \tau \left( (1 - q) \right) > 1 - \varepsilon,$$

which is not possible. Therefore $B_\varepsilon(\lambda) \subset A_q(\lambda)$ and hence a sequence $x$ is statistically $\lambda$-Cauchy in PN-space.

**Definition 12.** Let $(X, \nu, \tau)$ be a PN-space. Then,

(i) PN-space is said to be complete if every Cauchy sequence is convergent in $(X, \nu, \tau)$;

(ii) PN-space is said to be statistically $\lambda$-complete or simply $S(\lambda)$-complete if every statistically $\lambda$-Cauchy sequence is convergent in $(X, \nu, \tau)$ is statistically $\lambda$-summable.

**Theorem 13.** Every probabilistic normed space $(X, \nu, \tau)$ is statistically $\lambda$-complete but not complete in general.

**Proof.** Suppose that $x = (x_k)$ is statistically $\lambda$-Cauchy in PN-space but not statistically $\lambda$-summable. Then there exists $M \in \mathbb{N}$ such that

$$\delta \left( E_\nu(\lambda) \right) = \delta \left( \left\{ j \in \mathbb{N} : \gamma_j(x(t/2)) \leq 1 - \varepsilon \right\} \right) = 0,$$

$$\delta \left( F_\nu(\lambda) \right) = \delta \left( \left\{ j \in \mathbb{N} : \gamma_j(x(t/2)) \geq \left( \frac{t}{2} \right) > 1 - \varepsilon \right\} \right) = 0. \quad (30)$$

This implies that $\delta(F_\nu(\lambda)) = 1$. Since

$$\gamma_j(x(t/2)) \geq 2 \gamma_j(x(t/2)) \geq 1 - \varepsilon,$$

if $\gamma_j(x(t/2)) > (1 - \varepsilon)/2$, then $\delta(E_\nu(\lambda)) = 0$; that is, $\delta(E_\nu(\lambda)) = 1$, which leads to a contradiction, since $x = (x_k)$ was statistically $\lambda$-Cauchy. Hence $x = (x_k)$ must be statistically $\lambda$-summable in PN-space.

A probabilistic normed space is not complete in general; we verify this by the following example.

**Example 14.** Let $X = (0, 1)$ and $x(t) = t/(t + |x|)$ for $t > 0$. Then $(X, \nu, \tau)$ is a probabilistic normed space but not complete, since the sequence $(1/(n + 1))$ is Cauchy with respect to $(X, \nu, \tau)$ but not convergent with respect to the present PN-space.

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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**References**


