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Research Article

Statistical Summability through de la Vallée-Poussin Mean in Probabilistic Normed Spaces

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Two concepts—one of statistical convergence and the other of de la Vallée-Poussin mean—play an important role in recent research on summability theory. In this work we define a new type of summability methods and statistical completeness involving the ideas of de la Vallée-Poussin mean and statistical convergence in the framework of probabilistic normed spaces.

1. Introduction, Definitions, and Preliminaries

Fast [1] presented the following definition of statistical convergence for sequences of real numbers. Let $K \subseteq \mathbb{N}$, the set of natural numbers, and $K_n = \{k \le n : k \in K\}$. The natural density of K is defined by $\delta(K) = \lim_n n^{-1}|K_n|$ if the limit exists, where $|K_n|$ denotes the cardinality of K_n.

The sequence $x = (x_j)$ is said to be statistically convergent to the number ℓ if for every $\epsilon > 0$ the set $K_\varepsilon := \{k \in \mathbb{N} : |x_k - \ell| \ge \varepsilon\}$ has natural density zero; that is, for each $\epsilon > 0$,

$$\lim_n \frac{1}{n} \left|\left\{j \le n : \left|x_j - \ell\right| \ge \epsilon\right\}\right| = 0. \tag{1}$$

Note that every convergent sequence is statistically convergent to the same limit, but its converse need not be true.

In 1985, Fridy [2] has defined the notion of statistically Cauchy sequence and proved that it is equivalent to statistical convergence and since then a large amount of work has appeared. Various extensions, generalizations, variants, and applications have been given by several authors so far, for example, [3–8] and references therein. In the recent past, Mursaleen [9] presented a generalization of statistical convergence by using de la Vallée-Poussin mean which is known λ-statistical convergence and further studied by Çolak and Bektas [10, 11]. For more details related to this concept we refer to [12–18].

Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \le \lambda_n + 1, \qquad \lambda_1 = 0. \tag{2}$$

The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) =: \frac{1}{\lambda_n} \sum_{j \in I_n} x_j, \tag{3}$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_j)$ is said to be (V, λ)-summable to a number ℓ if

$$t_n(x) \longrightarrow \ell \quad \text{as } n \longrightarrow \infty. \tag{4}$$

In this case ℓ is called λ-limit of x.

Let $K \subseteq \mathbb{N}$ be a set of positive integers; then

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} \left|\{n - \lambda_n + 1 \le j \le n : j \in K\}\right| \tag{5}$$

is said to be λ-*density* of K.

In case $\lambda_n = n$, λ-density reduces to the natural density. Also, since $(\lambda_n/n) \le 1$, $\delta(K) \le \delta_\lambda(K)$ for every $K \subseteq \mathbb{N}$.

The number sequence $x = (x_j)$ is said to be λ-statistically convergent to the number ℓ if, for each $\epsilon > 0$, $\delta_\lambda(K_\epsilon) = 0$, where $K_\epsilon = \{j \in \mathbb{N} : |x_j - \ell| > \epsilon\}$; that is,

$$\lim_n \frac{1}{\lambda_n} \left|\left\{j \in I_n : \left|x_j - \ell\right| > \epsilon\right\}\right| = 0. \tag{6}$$

In this case we write $S_\lambda\text{-}\lim_j x_j = \ell$ and we denote the set of all λ-statistically convergent sequences by S_λ.

A distribution function is an element of Δ^+, where $\Delta^+ = \{f : \mathbb{R} \to [0,1]; f$ is left-continuous, nondecreasing, $f(0) = 0$ and $f(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{f \in \Delta^+; l^- f(+\infty) = 1\}$. Here $l^- f(+\infty)$ denotes the left limit of the function f at the point x. The space Δ^+ is partially ordered by the usual pointwise ordering of functions; that is, $f \le g$ if and only if $f(x) \le g(x)$ for all $x \in \mathbb{R}$.

A triangle function is a binary operation on Δ^+, namely, a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative nondecreasing and which has e as unit; that is, for all $f, g, h \in \Delta^+$, we have

(i) $\tau(\tau(f,g),h) = \tau(f,\tau(g,h))$,

(ii) $\tau(f,g) = \tau(g,f)$,

(iii) $\tau(f,h) = \tau(g,h)$ whenever $f \le g$,

(iv) $\tau(f,e) = f$.

Here e is the d.f. defined by

$$e(x) = \begin{cases} 0 & \text{if } x \le 0; \\ 1 & \text{if } x > 0. \end{cases} \tag{7}$$

We remark that the set Δ as well as its subsets can be partially ordered by the usual pointwise order: in this order, e is the maximal element in Δ^+.

There are two definitions of probabilistic normed space, the original one by Šerstnev [19] who used the idea of Menger [20] to define such space and the other one by Alsina et al. [21].

According to Šerstnev, a probabilistic normed space (for short, PN-space) is a triple (X, ν, τ), where X is a real linear space, τ is a triangle function, and ν is the probabilistic norm; that is, ν is a map from X into Δ^+ that satisfies the following conditions:

(i) $\nu_x = e$ if and only if $x = \theta$, where θ is the null vector of X,

(ii) $\nu_{\alpha x}(t) = \nu_x(t/|\alpha|)$ for all $t > 0$, $\alpha \in \mathbb{R}$ with $\alpha \ne 0$, and $x \in X$,

(iii) $\nu_{x+y} \ge \tau(\nu_x, \nu_y)$ whenever $x, y \in X$.

Here $\nu_x(t)$ denotes the value of ν_x at $t \in \mathbb{R}$.

In this paper, using the notions of statistical convergence and de la Vallée-Poussin mean, we define and study a new type of summability methods in the setting of probabilistic normed spaces. We also introduce a new type of statistical completeness through de la Vallée-Poussin mean in this framework.

2. Statistical Summability through de la Vallée-Poussin Mean

Here we introduce the notions of λ-summable and statistically λ-summable in PN-space and give some of its properties. We will assume throughout this paper that $(X, \mathcal{F}, \tau)$ is a probabilistic normed space.

Definition 1. A sequence $x = (x_k)$ is said to be λ-summable in PN-space (X, ν, τ) or simply $(\lambda)_\nu$-summable to ξ if for each $\varepsilon > 0$, $\theta \in (0,1)$ there exists a positive integer j_0 such that $\nu_{t_j(x)-\xi}(\varepsilon) > 1 - \theta$ for all $j \ge j_0$. In this case one writes $\nu(\lambda)\text{-}\lim x_k = \xi$ and ξ is called the $\nu(\lambda)$-limit of the sequence $x = (x_k)$.

Definition 2. A sequence $x = (x_k)$ is said to be statistically *λ-summable* in (X, ν, τ) or simply $S(\lambda)_\nu$-summable to ξ if for each $\varepsilon > 0$, $\theta \in (0,1)$ the set $K_\varepsilon(\lambda) = \{j \in \mathbb{N} : \nu_{t_j(x)-\xi}(\varepsilon) \le 1 - \theta\}$ has natural density zero (briefly, $\delta(K_\varepsilon(\lambda)) = 0$); that is,

$$\lim_n \frac{1}{n} \left| \left\{ j \le n : \nu_{t_j(x)-\xi}(\varepsilon) \le 1 - \theta \right\} \right| = 0. \tag{8}$$

In this case one writes $\nu(S_\lambda)\text{-}\lim x_k = \xi$, and ξ is called the $\nu(S_\lambda)$-limit of x. One may write (8) in the alternative form as

$$\lim_n \frac{1}{n} \left| \left\{ j \le n : \nu_{t_j(x)-\xi}(\varepsilon) > 1 - \theta \right\} \right| = 1. \tag{9}$$

Theorem 3. *If a sequence $x = (x_k)$ is statistically λ-summable in PN-space, that is, $\nu(S_\lambda)\text{-}\lim x_k = \xi$ exists, then it is unique.*

Proof. Suppose that there exist two elements $\xi_1, \xi_2 \in X$ with $\xi_1 \ne \xi_2$ such that $\nu(S_\lambda)\text{-}\lim x_k = \xi_1$ and $\nu(S_\lambda)\text{-}\lim x_k = \xi_2$. Let $\epsilon > 0$ be given. Choose $q > 0$ such that

$$\tau((1-q),(1-q)) > 1 - \epsilon. \tag{10}$$

Then, for any $t > 0$, we define

$$\begin{aligned} M_q'(\lambda) &= \left\{ j \in \mathbb{N} : \nu_{t_j(x)-\xi_1}(t) \le 1 - q \right\}, \\ M_q''(\lambda) &= \left\{ j \in \mathbb{N} : \nu_{t_j(x)-\xi_2}(t) \le 1 - q \right\}. \end{aligned} \tag{11}$$

Since $\nu(S_\lambda)\text{-}\lim x = \xi_1$ implies $\delta(M_q'(\lambda)) = 0$ and, similarly, we have $\delta(M_q''(\lambda)) = 0$. Now, let $M_q(\lambda) = M_q'(\lambda) \cap M_q''(\lambda)$. It follows that $\delta(M_q(\lambda)) = 0$ and hence the complement $M_q^c(\lambda)$ is nonempty set and $\delta(M_q^c(\lambda)) = 1$. Now, if $k \in \mathbb{N} \setminus M_q(\lambda)$, then

$$\begin{aligned} \nu_{\xi_1-\xi_2}(t) &\ge \tau\left(\nu_{t_j(x)-\xi_1}\left(\frac{t}{2}\right), \nu_{t_j(x)-\xi_2}\left(\frac{t}{2}\right)\right) \\ &> \tau((1-q),(1-q)) > 1 - \epsilon. \end{aligned} \tag{12}$$

Since $\epsilon > 0$ was arbitrary, we get $\nu_{\xi_1-\xi_2}(t) = 1$ for all $t > 0$. Hence $\xi_1 = \xi_2$. This means that $\nu(S_\lambda)$-limit is unique. □

The following theorem gives the algebraic properties of statistically λ-summable sequences in PN-spaces.

Theorem 4. *Let $x = (x_k)$ and $y = (y_k)$ be two sequences. If $\nu(S_\lambda)\text{-}\lim x_k = \xi_1$ and $\nu(S_\lambda)\text{-}\lim y_k = \xi_2$, then*

(i) $\nu(S_\lambda)\text{-}\lim (x_k \pm y_k) = \xi_1 \pm \xi_2$,

(ii) $\nu(S_\lambda)\text{-}\lim \alpha x_k = \alpha\xi$, $\alpha(\ne 0) \in \mathbb{R}$.

Proof of the theorem is straightforward and so omitted.

Theorem 5. *If a sequence $x = (x_k)$ is λ-summable to ξ in PN-space; then it is statistically λ-summable to the same limit.*

Proof. Let $\nu(\lambda)$-$\lim x_k = \xi$. Then for every $\epsilon > 0$ and $t > 0$, there is a positive integer j_0 such that

$$\nu_{t_j(x)-\xi}(t) > 1 - \epsilon \tag{13}$$

for all $j \geq j_0$. Since the set

$$K_\epsilon(\lambda) := \left\{ j \in \mathbb{N} : \nu_{t_j(x)-\xi}(t) \leq 1 - \epsilon \right\} \tag{14}$$

is contained in $\{1, 2, 3, \ldots, j_0 - 1\}$. As we know, every finite subset of $\mathbb{N}$ has natural density zero; that is, $\delta(K_\epsilon(\lambda)) = 0$. Hence, a sequence $x = (x_k)$ is $S(\lambda)_\nu$-summable to ξ. □

Remark 6. The converse of the above theorem is not true in general, which is verified by the following example.

Example 7. Let $(\mathbb{R}, |\cdot|)$ denote the space of all real numbers with the usual norm and $\tau(a, b) = ab$ for all $a, b \in [0, 1]$. Let $\nu_x(t) = t/(t + |x|)$ for all $x \in X$ and $t > 0$. In this case, we observe that $(\mathbb{R}, \nu, \tau)$ is a PN-space. Define a sequence $x = (x_k)$ by

$$t_j(x) = \begin{cases} j; & \text{if } j = n^2,\ n \in \mathbb{N} \\ 0; & \text{otherwise.} \end{cases} \tag{15}$$

For $\epsilon > 0$ and $t > 0$, write

$$K_\epsilon(\lambda) = \left\{ j \in \mathbb{N} : \nu_{t_j(x)}(t) \leq 1 - \epsilon \right\}. \tag{16}$$

It is easy to see that

$$\nu_{t_j(x)}(t) = \frac{t}{t + \left| t_j(x) \right|} = \begin{cases} \dfrac{t}{t+j}, & \text{for } j = n^2,\ n \in \mathbb{N}; \\ 1, & \text{otherwise;} \end{cases} \tag{17}$$

hence

$$\lim \nu_{t_j(x)}(t) = \begin{cases} 0, & \text{for if } j = n^2, n \in \mathbb{N}; \\ 1, & \text{otherwise.} \end{cases} \tag{18}$$

Therefore, the sequence (x_k) is not $(\lambda)_\nu$-summable. But the set $K_\epsilon(\lambda)$ has natural density zero since $K_\epsilon(\lambda) \subset \{1, 4, 9, 16, \ldots\}$. From here, we conclude that the converse of Theorem 5 need not be true.

Theorem 8. *A sequence $x = (x_k)$ is statistically λ-summable in PN-space to ξ if and only if there exists a subset $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and $\nu(\lambda)$-$\lim x_{k_n} = \xi$.*

Proof. Suppose that there exists a subset $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\theta(K) = 1$ and $\nu(\lambda)$-$\lim x_{k_n} = \xi$. Then there exists a positive integer $N \in \mathbb{N}$ such that for $n \geq N$

$$\nu_{t_n(x)-\xi}(t) > 1 - \epsilon. \tag{19}$$

Put $K_\epsilon(\lambda) = \{n \in \mathbb{N} : \nu_{t_n(x)-\xi}(t) \leq 1 - \epsilon\}$ and $K' = \{k_{N+1}, k_{N+2}, \ldots\}$. Then $\delta(K') = 1$ and $K_\epsilon(\lambda) \subseteq \mathbb{N} - K'$ which implies that $\delta(K_\epsilon(\lambda)) = 0$. Hence $x = (x_k)$ is statistically λ-summable to ξ in PN-space.

Conversely, let sequence $x = (x_k)$ is statistically λ-summable to ξ. For $q = 1, 2, 3, \ldots$ and $t > 0$, write

$$\begin{aligned} K_q(\lambda) &= \left\{ j \in \mathbb{N} : \nu_{t_{k_j}(x)-\xi}(t) \geq 1 - \frac{1}{q} \right\}, \\ M_q(\lambda) &= \left\{ j \in \mathbb{N} : \nu_{t_{k_j}(x)-\xi}(t) < 1 - \frac{1}{q} \right\}. \end{aligned} \tag{20}$$

Then $\delta(K_q(\lambda)) = 0$ and

$$M_1(\lambda) \supset M_2(\lambda) \supset \cdots M_i(\lambda) \supset M_{i+1}(\lambda) \supset \cdots, \tag{21}$$

$$\delta\left(M_q(\lambda)\right) = 1, \quad q = 1, 2, \ldots. \tag{22}$$

Now we have to show that for $j \in M_q(\lambda)$, $x = (x_{k_j})$ is $(\lambda)_\nu$-summable to ξ. Suppose that $x = (x_{k_j})$ is not $(\lambda)_\nu$-summable to ξ. Therefore there is $\epsilon > 0$ such that $\nu_{t_{k_j}-\xi}(t) \geq \epsilon$ for infinitely many terms. Let

$$M_\epsilon(\lambda) = \left\{ j \in \mathbb{N} : \nu_{t_{k_j}-\xi}(t) < \epsilon \right\}, \tag{23}$$

and $\epsilon > 1/q$ with $q = 1, 2, 3 \ldots$. Then

$$\delta\left(M_\epsilon(\lambda)\right) = 0, \tag{24}$$

and, by (21), $M_q(\lambda) \subset M_\epsilon(\lambda)$. Hence $\delta(M_q(\lambda)) = 0$, which contradicts (22) and therefore $x = (x_{k_j})$ is $(\lambda)_\nu$-summable to ξ. □

Similarly we can prove the following dual statement.

Theorem 9. *A sequence $x = (x_k)$ is λ-statistically summable in PN-space to ξ if and only if there exists a subset $K = \{k_1 < k_2 < \cdots < k_n < \cdots\} \subseteq \mathbb{N}$ such that $\delta_\lambda(K) = 1$ and ν-$\lim x_{k_n} = \xi$.*

3. Statistically Complete through de la Vallée-Poussin Mean

In this section, we define the notions of statistically λ-Cauchy and statistically λ-complete with respect to probabilistic normed space and prove related results.

Definition 10. A sequence $x = (x_k)$ is said to be statistically λ-Cauchy in (X, ν, τ) or simply $S(\lambda)_\nu$-Cauchy if, for every $\epsilon > 0$ and $\theta \in (0, 1)$, there exists a number $N = N(\epsilon)$ such that, for all $j, h \geq N$, the set $S_\epsilon(\lambda) = \{j \in \mathbb{N} : \nu_{t_j(x)-t_h(x)}(\epsilon) \leq 1 - \theta\}$ has natural density zero (briefly, $\delta(S_\epsilon(\lambda)) = 0$); that is,

$$\lim_n \frac{1}{n} \left| \left\{ j \leq n : \nu_{t_j(x)-t_h(x)}(\epsilon) \leq 1 - \theta \right\} \right| = 0. \tag{25}$$

Theorem 11. *A sequence $x = (x_k)$ is statistically λ-summable in PN-space; then it is statistically λ-Cauchy.*

Proof. Suppose that $\nu(S_\lambda)$-$\lim x_k = \ell$. Let $\epsilon > 0$ be a given number and choose $q > 0$ such that

$$\tau\left((1 - q), (1 - q)\right) > 1 - \epsilon. \tag{26}$$

Then, for $t > 0$, we have $\delta(A_q(\lambda)) = 0$, where $A_q(\lambda) = \{j \in \mathbb{N} : \nu_{t_j(x)-\ell}(t/2) \leq 1-q\}$ which implies that

$$\delta\left(A_q^c(\lambda)\right) = \delta\left(\left\{j \in \mathbb{N} : \nu_{t_j(x)-\ell}\left(\frac{t}{2}\right) > 1-q\right\}\right) = 1. \quad (27)$$

Let $m \in A_q^c(\lambda)$. Then $\nu_{t_m(x)-\ell}(t/2) > 1-q$.

Now, let

$$B_\epsilon(\lambda) = \left\{j \in \mathbb{N} : \nu_{t_j(x)-t_m(x)}(t) \leq 1-\epsilon\right\}. \quad (28)$$

We need to show that $B_\epsilon(\lambda) \subset A_q(\lambda)$. Let $j \in B_\epsilon(\lambda)$. Then $\nu_{t_j(x)-t_m(x)}(t) \leq 1-\epsilon$ and hence $\nu_{t_j(x)-\ell}(t/2) \leq 1-q$; that is, $j \in A_q(\lambda)$. Otherwise, if $\nu_{t_j(x)-\ell}(t/2) > 1-q$, then

$$\begin{aligned} 1-\epsilon \geq \nu_{t_j(x)-t_m(x)}(t) &\geq \tau\left(\nu_{t_j(x)-\ell}\left(\frac{t}{2}\right), \nu_{t_m(x)-\ell}\left(\frac{t}{2}\right)\right) \\ &> \tau\left((1-q),(1-q)\right) > 1-\epsilon, \end{aligned} \quad (29)$$

which is not possible. Therefore $B_\epsilon(\lambda) \subset A_q(\lambda)$ and hence a sequence x is statistically λ-Cauchy in PN-space. □

Definition 12. Let (X, ν, τ) be a PN-space. Then,

(i) PN-space is said to be complete if every Cauchy sequence is convergent in (X, ν, τ);

(ii) PN-space is said to be statistically λ-complete or simply $S(\lambda)_\nu$-complete if every statistically λ-Cauchy sequence in (X, ν, τ) is statistically λ-summable.

Theorem 13. *Every probabilistic normed space (X, ν, τ) is statistically λ-complete but not complete in general.*

Proof. Suppose that $x = (x_k)$ is statistically λ-Cauchy in PN-space but not statistically λ-summable. Then there exists $M \in \mathbb{N}$ such that

$$\begin{aligned} \delta\left(E_\epsilon(\lambda)\right) &= \delta\left(\left\{j \in \mathbb{N} : \nu_{t_j(x)-t_m(x)}(t) \leq 1-\epsilon\right\}\right) = 0, \\ \delta\left(F_\epsilon(\lambda)\right) &= \delta\left(\left\{j \in \mathbb{N} : \nu_{t_j(x)-\ell}\left(\frac{t}{2}\right) > 1-\epsilon\right\}\right) = 0. \end{aligned} \quad (30)$$

This implies that $\delta(F_\epsilon^c(\lambda)) = 1$. Since

$$\nu_{t_j(x)-t_m(x)}(t) \geq 2\nu_{t_j(x)-\ell}\left(\frac{t}{2}\right) > 1-\epsilon, \quad (31)$$

if $\nu_{t_j(x)-\ell}(t/2) > (1-\epsilon)/2$, then $\delta(E_\epsilon^c(\lambda)) = 0$; that is, $\delta(E_\epsilon(\lambda)) = 1$, which leads to a contradiction, since $x = (x_k)$ was statistically λ-Cauchy. Hence $x = (x_k)$ must be statistically λ-summable in PN-space.

A probabilistic normed space is not complete in general; we verify this by the following example.

Example 14. Let $X = (0,1)$ and $\nu_x(t) = t/(t+|x|)$ for $t > 0$. Then (X, ν, τ) is a probabilistic normed space but not complete, since the sequence $(1/(n+1))$ is Cauchy with respect to (X, ν, τ) but not convergent with respect to the present PN-space. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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