Research Article

Proof of Some Conjectures of Melham Using Ramanujan's ${}_1\psi_1$ Formula

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We employ Ramanujan's ${}_1\psi_1$ formula to prove three conjectures of R. S. Melham on representation of an integer n as sums of polygonal numbers.

1. Introduction

Jacobi's classical two-square theorem is as follows.

Theorem 1 (see [1]). *Let $r\{\square + \square\}(n)$ denote the number of representations of n as a sum of two squares, counting order and sign, and let $d_{i,j}(n)$ denote the number of positive divisors of n congruent to i modulo j . Then*

$$r\{\square + \square\}(n) = 4(d_{1,4}(n) - d_{3,4}(n)). \quad (1)$$

The above theorem can also be recasted in terms of Lambert series as

$$\sum_{n=0}^{\infty} r\{\square + \square\}(n) q^n = 1 + 4 \sum_{n=0}^{\infty} \left(\frac{q^{4n+1}}{1 - q^{4n+1}} - \frac{q^{4n+3}}{1 - q^{4n+3}} \right). \quad (2)$$

Similar representation theorems involving squares and triangular numbers were found by Dirichlet [2], Lorenz [3], Legendre [4], and Berndt [5]. For example, the following two theorems are due to Lorenz and Ramanujan, respectively.

Theorem 2 (see [3]). *Let $r\{l\square + m\square\}(n)$ denote the number of representations of n as a sum of l times a square and m times a square. Then*

$$r\{\square + 3\square\}(n) = 2(d_{1,3}(n) - d_{2,3}(n)) + 4(d_{4,12}(n) - d_{8,12}(n)). \quad (3)$$

Theorem 3 (see [5]). *Let $r\{l\Delta + m\Delta\}(n)$ denote the number of representations of n as a sum of l times a triangular number and m times a triangular number. Then*

$$r\{\Delta + 3\Delta\}(n) = d_{1,3}(2n+1) - d_{2,3}(2n+1). \quad (4)$$

Hirschhorn [6, 7] obtained forty-five similar identities (including those obtained by Legendre and Ramanujan) involving squares, triangular numbers, pentagonal numbers, and octagonal numbers employing dissection of the q -series representations of the identities obtained by Jacobi, Dirichlet, and Lorenz. In [8], Baruah and the author obtained twenty-five more such identities involving squares, triangular numbers, pentagonal numbers, heptagonal numbers, octagonal numbers, decagonal numbers, hendecagonal numbers, dodecagonal numbers, and octadecagonal numbers. More works on this topic have been done in [9–11]. In [11], Melham presented 21 conjectured analogues of Jacobi's two-square theorem which are verified using computer algorithms. In [12], Toh offered a uniform approach to prove these conjectures using known formulae for $r\{l\square + m\square\}(n)$. In this paper, we show that some of these conjectures can also be proved by using Ramanujan's famous ${}_1\psi_1$ formula. We prove three conjectures enlisted in the following theorem which have appeared as (6), (7), and (8), respectively, in [11].

Theorem 4. Consider

$$\sum_{n=0}^{\infty} r \{ \Delta + 5\Delta \} (n) q^n = \sum_{n=0}^{\infty} \left(\frac{q^{3n} + q^{7n+1}}{1 - q^{20n+5}} - \frac{q^{13n+9} + q^{17n+12}}{1 - q^{20n+15}} \right), \quad (5)$$

$$\sum_{n=0}^{\infty} r \{ \Delta + 6\Delta \} (n) q^n = \sum_{n=0}^{\infty} \left(\frac{q^{7n}}{1 - q^{24n+3}} + \frac{q^{5n+1}}{1 - q^{24n+9}} - \frac{q^{19n+11}}{1 - q^{24n+15}} - \frac{q^{17n+14}}{1 - q^{24n+21}} \right), \quad (6)$$

$$\sum_{n=0}^{\infty} r \{ 2\Delta + 3\Delta \} (n) q^n = \sum_{n=0}^{\infty} \left(\frac{q^{5n}}{1 - q^{24n+3}} + \frac{q^{7n+2}}{1 - q^{24n+9}} - \frac{q^{17n+10}}{1 - q^{24n+15}} - \frac{q^{19n+16}}{1 - q^{24n+21}} \right). \quad (7)$$

The next section of this paper is devoted to notations, definitions, and preliminary results.

2. Notations and Preliminary Results

Ramanujan's general theta-function $f(a, b)$ is defined by [5, page 34, (18.1)]

$$f(a, b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}, \quad |ab| < 1. \quad (8)$$

Jacobi's famous triple product identity takes the simple form [5, page 35, Entry 19]

$$f(a, b) = (-a; ab)_{\infty} (-b; ab)_{\infty} (ab; ab)_{\infty}, \quad (9)$$

where, here and throughout the paper, for $|q| < 1$,

$$(a; q)_n := \prod_{k=0}^n (1 - aq^k), \quad (a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k). \quad (10)$$

We also use the following notation for the sake of brevity of expressions:

$$\begin{aligned} & [a_1, a_2, a_3, \dots, a_n; q]_{\infty} \\ &= (a_1; q)_{\infty} (a_2; q)_{\infty} (a_3; q)_{\infty} \cdots (a_n; q)_{\infty}. \end{aligned} \quad (11)$$

One important special case of $f(a, b)$ is

$$\psi(q) := f(q, q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2; q^2)_{\infty}}{(q; q^2)_{\infty}}, \quad (12)$$

where the product representations in (12) arise from (9).

From [5, page 46, Entry 30] we find that

$$f(a, b) = f(a^3b, ab^3) + af(a^{-1}b, a^5b^3). \quad (13)$$

Also if $ab = cd$, then [5, page 45, Entry 29]

$$\begin{aligned} f(a, b)(c, d) &= f(ac, bd)f(ad, bc) \\ &+ af(bc^{-1}, ac^2d)f(bd^{-1}, acd^2). \end{aligned} \quad (14)$$

We recall Ramanujan's ${}_1\psi_1$ formula in the form [5, page 34, (17.6)]

$$\sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(az; q)_{\infty} (qa^{-1}z^{-1}; q)_{\infty} (q; q)_{\infty} (ba^{-1}; q)_{\infty}}{(z; q)_{\infty} (qa^{-1}z^{-1}; q)_{\infty} (b; q)_{\infty} (qa^{-1}; q)_{\infty}}, \quad (15)$$

where

$$|ba^{-1}| < |z| < 1. \quad (16)$$

Finally, we note that

$$\sum_{n=0}^{\infty} r \{ a\Delta + b\Delta \} (n) q^n = \psi(q^a) \psi(q^b). \quad (17)$$

3. Proof of the Conjectures (5)–(7)

Proof of Conjecture (5). In view of (17), we rewrite (5) as

$$\psi(q) \psi(q^5) = \sum_{n=0}^{\infty} \left(\frac{q^{3n} + q^{7n+1}}{1 - q^{20n+5}} - \frac{q^{13n+9} + q^{17n+12}}{1 - q^{20n+15}} \right). \quad (18)$$

Replacing q by q^{20} and then setting $z = q^3$, $a = q^5$, and $b = q^{25}$ in (15), we have

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \frac{(q^5; q^{20})_n}{(q^{25}; q^{20})_n} q^{3n} \\ &= \frac{(q^8; q^{20})_{\infty} (q^{12}; q^{20})_{\infty} (q^{20}; q^{20})_{\infty}^2}{(q^3; q^{20})_{\infty} (q^{17}; q^{20})_{\infty} (q^{25}; q^{20})_{\infty} (q^{15}; q^{20})_{\infty}}. \end{aligned} \quad (19)$$

Expanding the q -products on the left side of (19) and noting that $(1 - q^5)(q^{25}; q^{20})_{\infty} = (q^5; q^{20})_{\infty}$, we obtain

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1 - q^{20n+5}} \\ &= \frac{(q^8; q^{20})_{\infty} (q^{12}; q^{20})_{\infty} (q^{20}; q^{20})_{\infty}^2}{(q^3; q^{20})_{\infty} (q^{17}; q^{20})_{\infty} (q^5; q^{20})_{\infty} (q^{15}; q^{20})_{\infty}}. \end{aligned} \quad (20)$$

Also

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1 - q^{20n+5}} &= \sum_{n=0}^{\infty} \frac{q^{3n}}{1 - q^{20n+5}} + \sum_{n=-\infty}^{-1} \frac{q^{3n}}{1 - q^{20n+5}} \\ &= \sum_{n=0}^{\infty} \frac{q^{3n}}{1 - q^{20n+5}} + \sum_{n=1}^{\infty} \frac{q^{-3n}}{1 - q^{-20n+5}} \\ &= \sum_{n=0}^{\infty} \frac{q^{3n}}{1 - q^{20n+5}} - \sum_{n=1}^{\infty} \frac{q^{17n-5}}{1 - q^{20n-5}} \\ &= \sum_{n=0}^{\infty} \frac{q^{3n}}{1 - q^{20n+5}} - \sum_{n=0}^{\infty} \frac{q^{17n+12}}{1 - q^{20n+15}}. \end{aligned} \quad (21)$$

Thus, from (20) and (21), we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{3n}}{1-q^{20n+5}} - \sum_{n=0}^{\infty} \frac{q^{17n+12}}{1-q^{20n+15}} \\ = \frac{(q^8; q^{20})_{\infty} (q^{12}; q^{20})_{\infty} (q^{20}; q^{20})_{\infty}^2}{(q^3; q^{20})_{\infty} (q^{17}; q^{20})_{\infty} (q^5; q^{20})_{\infty} (q^{15}; q^{20})_{\infty}}. \end{aligned} \quad (22)$$

Replacing q by q^{20} and then setting $z = q^3$, $a = q^5$, and $b = q^{25}$ in (15), and proceeding as in case of (22), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{7n}}{1-q^{20n+5}} - \sum_{n=0}^{\infty} \frac{q^{13n+8}}{1-q^{20n+15}} \\ = \frac{(q^{12}; q^{20})_{\infty} (q^8; q^{20})_{\infty} (q^{20}; q^{20})_{\infty}^2}{(q^7; q^{20})_{\infty} (q^{13}; q^{20})_{\infty} (q^5; q^{20})_{\infty} (q^{15}; q^{20})_{\infty}}. \end{aligned} \quad (23)$$

Multiplying (23) by q and adding to (22), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{q^{3n} + q^{7n+1}}{1-q^{20n+5}} - \frac{q^{13n+9} + q^{17n+12}}{1-q^{20n+15}} \right) \\ = \frac{(q^8; q^{20})_{\infty} (q^{12}; q^{20})_{\infty} (q^{20}; q^{20})_{\infty}^2}{(q^3; q^{20})_{\infty} (q^{17}; q^{20})_{\infty} (q^5; q^{20})_{\infty} (q^{15}; q^{20})_{\infty}} \\ + q \frac{(q^{12}; q^{20})_{\infty} (q^8; q^{20})_{\infty} (q^{20}; q^{20})_{\infty}^2}{(q^7; q^{20})_{\infty} (q^{13}; q^{20})_{\infty} (q^5; q^{20})_{\infty} (q^{15}; q^{20})_{\infty}} \\ = \frac{[q^8, q^{12}, q^{20}; q^{20}]_{\infty}}{[q^3, q^5, q^7, q^{13}, q^{15}, q^{17}; q^{20}]_{\infty}} \\ \times ((q^7; q^{20})_{\infty} (q^{13}; q^{20})_{\infty} (q^{20}; q^{20})_{\infty} \\ + q(q^3; q^{20})_{\infty} (q^{17}; q^{20})_{\infty} (q^{20}; q^{20})_{\infty}) \\ = \frac{[q^8, q^{12}, q^{20}; q^{20}]_{\infty}}{[q^3, q^5, q^7, q^{13}, q^{15}, q^{17}; q^{20}]_{\infty}} \\ \times (f(-q^7, -q^{13}) + qf(-q^3, -q^{17})). \end{aligned} \quad (24)$$

Setting $a = q$ and $b = -q^4$ in (13), we find that

$$f(-q^7, -q^{13}) + qf(-q^3, -q^{17}) = f(q, -q^4). \quad (25)$$

Thus, (24) reduces to

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{q^{3n} + q^{7n+1}}{1-q^{20n+5}} - \frac{q^{13n+9} + q^{17n+12}}{1-q^{20n+15}} \right) \\ = \frac{[q^8, q^{12}, q^{20}; q^{20}]_{\infty}}{[q^3, q^5, q^7, q^{13}, q^{15}, q^{17}; q^{20}]_{\infty}} f(q, -q^4). \end{aligned} \quad (26)$$

Now

$$f(q, -q^4) = (-q; -q^5)_{\infty} (q^4; -q^5)_{\infty} (-q^5; -q^5)_{\infty}. \quad (27)$$

We have

$$\begin{aligned} (-q; -q^5)_{\infty} &= (-q; q^{10})_{\infty} (q^6; q^{10})_{\infty} \\ &= \frac{(q^2; q^{20})_{\infty} (q^6; q^{10})_{\infty}}{(q; q^{10})_{\infty}} \\ &= \frac{[q^2, q^6, q^{16}; q^{20}]_{\infty}}{[q, q^{11}; q^{20}]_{\infty}}. \end{aligned} \quad (28)$$

Similarly

$$\begin{aligned} (q^4; -q^5)_{\infty} &= \frac{[q^4, q^{14}, q^{18}; q^{20}]_{\infty}}{[q^9, q^{19}; q^{20}]_{\infty}}, \\ (-q^5; -q^5)_{\infty} &= \frac{[q^{10}, q^{10}, q^{20}; q^{20}]_{\infty}}{[q^5, q^{15}; q^{20}]_{\infty}}. \end{aligned} \quad (29)$$

Hence from (27), (28), and (29) we have

$$f(q, -q^4) = \frac{[q^2, q^4, q^6, q^{10}, q^{10}, q^{14}, q^{16}, q^{18}, q^{20}; q^{20}]_{\infty}}{[q, q^5, q^9, q^{11}, q^{15}, q^{19}; q^{20}]_{\infty}}. \quad (30)$$

Using (30) in (26), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{q^{3n} + q^{7n+1}}{1-q^{20n+5}} - \frac{q^{13n+9} + q^{17n+12}}{1-q^{20n+15}} \right) \\ = \frac{[q^2, q^4, q^6, q^8, q^{10}, q^{10}, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{20}; q^{20}]_{\infty}}{[q, q^3, q^5, q^5, q^7, q^9, q^{11}, q^{13}, q^{15}, q^{15}, q^{17}, q^{19}; q^{20}]_{\infty}} \\ = ([q^2, q^4, q^6, q^8, q^{10}, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}; q^{20}]_{\infty} \\ \times [q^{10}, q^{20}; q^{20}]_{\infty}) \\ \times ([q, q^3, q^5, q^7, q^9, q^{11}, q^{13}, q^{15}, q^{17}, q^{19}; q^{20}]_{\infty} \\ \times [q^5, q^{15}; q^{20}]_{\infty})^{-1} \\ = \frac{(q^2; q^2)_{\infty} (q^{10}; q^{10})_{\infty}}{(q; q^2)_{\infty} (q^5; q^{10})_{\infty}} \\ = \psi(q) \psi(q^5). \end{aligned} \quad (31)$$

This completes the proof of (18). $\square$

Proof of Conjecture (6). In view of (17), we rewrite (6) as

$$\begin{aligned} \psi(q) \psi(q^6) &= \sum_{n=0}^{\infty} \left(\frac{q^{7n}}{1-q^{24n+3}} + \frac{q^{5n+1}}{1-q^{24n+9}} \right. \\ &\quad \left. - \frac{q^{19n+11}}{1-q^{24n+15}} - \frac{q^{17n+14}}{1-q^{24n+21}} \right). \end{aligned} \quad (32)$$

Replacing q by q^{24} and then setting $z = q^7$, $a = q^3$, and $b = q^{27}$ in (15), and proceeding as in case of (22), we find that

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{7n}}{1 - q^{24n+3}} - \frac{q^{17n+14}}{1 - q^{24n+21}} \right) \\ &= \frac{(q^{10}; q^{24})_{\infty} (q^{14}; q^{24})_{\infty} (q^{24}; q^{24})_{\infty}^2}{(q^7; q^{24})_{\infty} (q^{17}; q^{24})_{\infty} (q^3; q^{24})_{\infty} (q^{21}; q^{24})_{\infty}}. \end{aligned} \quad (33)$$

Replacing q by q^{24} and then setting $z = q^5$, $a = q^9$, and $b = q^{33}$ in (15), and proceeding in a way similar to obtaining (22), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{5n}}{1 - q^{24n+9}} - \frac{q^{19n+10}}{1 - q^{24n+15}} \right) \\ &= \frac{(q^{14}; q^{24})_{\infty} (q^{10}; q^{24})_{\infty} (q^{24}; q^{24})_{\infty}^2}{(q^5; q^{24})_{\infty} (q^{19}; q^{24})_{\infty} (q^9; q^{24})_{\infty} (q^{15}; q^{24})_{\infty}}. \end{aligned} \quad (34)$$

Multiplying (34) by q and then adding to (33), we find that

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{7n}}{1 - q^{24n+3}} + \frac{q^{5n+1}}{1 - q^{24n+9}} - \frac{q^{19n+11}}{1 - q^{24n+15}} - \frac{q^{17n+14}}{1 - q^{24n+21}} \right) \\ &= \frac{(q^{10}; q^{24})_{\infty} (q^{14}; q^{24})_{\infty} (q^{24}; q^{24})_{\infty}^2}{(q^7; q^{24})_{\infty} (q^{17}; q^{24})_{\infty} (q^3; q^{24})_{\infty} (q^{21}; q^{24})_{\infty}} \\ &+ q \frac{(q^{14}; q^{24})_{\infty} (q^{10}; q^{24})_{\infty} (q^{24}; q^{24})_{\infty}^2}{(q^5; q^{24})_{\infty} (q^{19}; q^{24})_{\infty} (q^9; q^{24})_{\infty} (q^{15}; q^{24})_{\infty}} \\ &= \frac{[q^{10}, q^{14}; q^{24}]_{\infty}}{[q^3, q^5, q^7, q^9, q^{15}, q^{17}, q^{19}, q^{21}, q^{24}]_{\infty}} \\ &\times (f(-q^5, -q^{19}) f(-q^9, -q^{15}) + q f(-q^7, -q^{17}) f(-q^3, -q^{21})). \end{aligned} \quad (35)$$

Setting $a = -q$, $b = -q^{11}$, $c = -q^4$, and $d = -q^8$ in (14), we find that

$$\begin{aligned} & f(-q^5, -q^{19}) f(-q^9, -q^{15}) + q f(-q^7, -q^{17}) f(-q^3, -q^{21}) \\ &= f(q, q^{11}) f(-q^4, -q^8). \end{aligned} \quad (36)$$

Using (36) in (35), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{7n}}{1 - q^{24n+3}} + \frac{q^{5n+1}}{1 - q^{24n+9}} - \frac{q^{19n+11}}{1 - q^{24n+15}} - \frac{q^{17n+14}}{1 - q^{24n+21}} \right) \\ &= \frac{[q^{10}, q^{14}; q^{24}]_{\infty}}{[q^3, q^5, q^7, q^9, q^{15}, q^{17}, q^{19}, q^{21}, q^{24}]_{\infty}} \\ &\times f(q, q^{11}) f(-q^4, -q^8). \end{aligned} \quad (37)$$

Now

$$\begin{aligned} f(q, q^{11}) &= (-q; q^{12})_{\infty} (-q^{11}; q^{12})_{\infty} (q^{12}; q^{12})_{\infty} \\ &= \frac{(q^2; q^{24})_{\infty} (q^{22}; q^{24})_{\infty} (q^{12}; q^{12})_{\infty}}{(q; q^{12})_{\infty} (q^{11}; q^{12})_{\infty}} \\ &= \frac{[q^2, q^{12}, q^{22}, q^{24}; q^{24}]_{\infty}}{[q, q^{11}, q^{13}, q^{23}; q^{24}]_{\infty}}, \end{aligned} \quad (38)$$

$$\begin{aligned} f(-q^4, -q^8) &= (q^4; q^{12})_{\infty} (q^8; q^{12})_{\infty} (q^{12}; q^{12})_{\infty} \\ &= [q^4, q^8, q^{12}, q^{16}, q^{20}, q^{24}; q^{24}]_{\infty}. \end{aligned}$$

Using (38) in (37), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{7n}}{1 - q^{24n+3}} + \frac{q^{5n+1}}{1 - q^{24n+9}} - \frac{q^{19n+11}}{1 - q^{24n+15}} - \frac{q^{17n+14}}{1 - q^{24n+21}} \right) \\ &= \frac{[q^2, q^4, q^8, q^{10}, q^{12}, q^{14}, q^{16}, q^{20}, q^{22}, q^{24}; q^{24}]_{\infty}}{[q, q^3, q^5, q^7, q^9, q^{11}, q^{13}, q^{15}, q^{17}, q^{19}, q^{21}, q^{23}; q^{24}]_{\infty}} \\ &= ([q^2, q^4, q^6, q^8, q^{10}, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{22}, q^{24}; q^{24}]_{\infty} \\ &\times [q^{12}, q^{24}; q^{24}]_{\infty}) \\ &\times ([q, q^3, q^5, q^7, q^9, q^{11}, q^{13}, q^{15}, q^{17}, q^{19}, q^{21}, q^{23}; q^{24}]_{\infty} \\ &\times [q^6, q^{12}; q^{24}]_{\infty})^{-1} \\ &= \frac{(q^2; q^2)_{\infty} (q^{12}; q^{12})_{\infty}}{(q, q^2)_{\infty} (q^6; q^{12})_{\infty}} \\ &= \psi(q) \psi(q^6). \end{aligned} \quad (39)$$

This completes the proof of (32). $\square$

Proof of Conjecture (7). In view of (17), we rewrite (7) as

$$\begin{aligned} \psi(q^2) \psi(q^3) &= \sum_{n=0}^{\infty} \left(\frac{q^{5n}}{1 - q^{24n+3}} + \frac{q^{7n+2}}{1 - q^{24n+9}} \right. \\ &\quad \left. - \frac{q^{17n+10}}{1 - q^{24n+15}} - \frac{q^{19n+16}}{1 - q^{24n+21}} \right). \end{aligned} \quad (40)$$

Replacing q by q^{24} and then setting $z = q^5$, $a = q^3$, and $b = q^{27}$ in (15), and proceeding as in case of (22), we find that

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{5n}}{1 - q^{24n+3}} - \frac{q^{19n+16}}{1 - q^{24n+21}} \right) \\ &= \frac{(q^8; q^{24})_{\infty} (q^{16}; q^{24})_{\infty} (q^{24}; q^{24})_{\infty}^2}{(q^5; q^{24})_{\infty} (q^{19}; q^{24})_{\infty} (q^3; q^{24})_{\infty} (q^{21}; q^{24})_{\infty}}. \end{aligned} \quad (41)$$

Also, replacing q by q^{24} and then setting $z = q^7$, $a = q^9$, and $b = q^{33}$ in (15), and proceeding similar to (22), we obtain

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{7n}}{1 - q^{24n+9}} - \frac{q^{17n+8}}{1 - q^{24n+15}} \right) \\ &= \frac{(q^{16}; q^{24})_{\infty} (q^8; q^{24})_{\infty} (q^{24}; q^{24})_{\infty}^2}{(q^7; q^{24})_{\infty} (q^{17}; q^{24})_{\infty} (q^9; q^{24})_{\infty} (q^{15}; q^{24})_{\infty}}. \end{aligned} \quad (42)$$

Multiplying (42) by q^2 and then adding to (41), we find that

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{5n}}{1 - q^{24n+3}} + \frac{q^{7n+2}}{1 - q^{24n+9}} - \frac{q^{17n+10}}{1 - q^{24n+15}} - \frac{q^{19n+16}}{1 - q^{24n+21}} \right) \\ &= \frac{(q^8; q^{24})_{\infty} (q^{16}; q^{24})_{\infty} (q^{24}; q^{24})_{\infty}^2}{(q^5; q^{24})_{\infty} (q^{19}; q^{24})_{\infty} (q^3; q^{24})_{\infty} (q^{21}; q^{24})_{\infty}} \\ &+ q^2 \frac{(q^{16}; q^{24})_{\infty} (q^8; q^{24})_{\infty} (q^{24}; q^{24})_{\infty}^2}{(q^7; q^{24})_{\infty} (q^{17}; q^{24})_{\infty} (q^9; q^{24})_{\infty} (q^{15}; q^{24})_{\infty}} \\ &= \frac{[q^8, q^{16}; q^{24}]_{\infty}}{[q^3, q^5, q^7, q^9, q^{15}, q^{17}, q^{19}, q^{21}; q^{24}]_{\infty}} \\ &\times (f(-q^7, -q^{17}) f(-q^9, -q^{15}) \\ &+ q^2 f(-q^5, -q^{19}) f(-q^3, -q^{21})). \end{aligned} \quad (43)$$

Setting $a = q^2$, $b = -q^{10}$, $c = -q^5$, and $d = -q^7$ in (14), we find that

$$\begin{aligned} & f(-q^7, -q^{17}) f(-q^9, -q^{15}) + q^2 f(-q^5, -q^{19}) f(-q^3, -q^{21}) \\ &= f(q^2, q^{10}) f(-q^5, -q^7). \end{aligned} \quad (44)$$

Hence, (43) reduces to

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{7n}}{1 - q^{24n+3}} + \frac{q^{5n+1}}{1 - q^{24n+9}} - \frac{q^{19n+11}}{1 - q^{24n+15}} - \frac{q^{17n+14}}{1 - q^{24n+21}} \right) \\ &= \frac{[q^8, q^{16}; q^{24}]_{\infty}}{[q^3, q^5, q^7, q^9, q^{15}, q^{17}, q^{19}, q^{21}; q^{24}]_{\infty}} \\ &\times f(q^2, q^{10}) f(-q^5, -q^7). \end{aligned} \quad (45)$$

Now

$$\begin{aligned} f(q^2, q^{10}) &= (-q^2; q^{12})_{\infty} (-q^{10}; q^{12})_{\infty} (q^{12}; q^{12})_{\infty} \\ &= \frac{(q^4; q^{24})_{\infty} (q^{20}; q^{24})_{\infty} (q^{12}; q^{12})_{\infty}}{(q^2; q^{12})_{\infty} (q^{10}; q^{12})_{\infty}} \\ &= \frac{[q^4, q^{12}, q^{20}, q^{24}; q^{24}]_{\infty}}{[q^2, q^{10}, q^{14}, q^{22}; q^{24}]_{\infty}}, \\ f(-q^5, -q^7) &= (q^5; q^{12})_{\infty} (q^7; q^{12})_{\infty} (q^{12}; q^{12})_{\infty} \\ &= [q^5, q^7, q^{12}, q^{17}, q^{19}, q^{24}; q^{24}]_{\infty}. \end{aligned} \quad (46)$$

Using (46) in (45), we have

$$\begin{aligned} & \sum_{n=0}^{\infty} \left(\frac{q^{7n}}{1 - q^{24n+3}} + \frac{q^{5n+1}}{1 - q^{24n+9}} - \frac{q^{19n+11}}{1 - q^{24n+15}} - \frac{q^{17n+14}}{1 - q^{24n+21}} \right) \\ &= \frac{[q^4, q^5, q^7, q^8, q^{12}, q^{12}, q^{16}, q^{17}, q^{19}, q^{20}, q^{24}, q^{24}; q^{24}]_{\infty}}{[q^2, q^3, q^5, q^7, q^9, q^{10}, q^{14}, q^{15}, q^{17}, q^{19}, q^{21}, q^{22}; q^{24}]_{\infty}} \\ &= \frac{[q^4, q^8, q^{12}, q^{16}, q^{20}, q^{24}; q^{24}]_{\infty} [q^6, q^{12}, q^{18}, q^{24}; q^{24}]_{\infty}}{[q^2, q^6, q^{10}, q^{14}, q^{18}, q^{22}; q^{24}]_{\infty} [q^3, q^9, q^{15}, q^{21}; q^{24}]_{\infty}} \\ &= \frac{(q^4; q^4)_{\infty} (q^6; q^6)_{\infty}}{(q^2; q^4)_{\infty} (q^3; q^6)_{\infty}} \\ &= \psi(q^2) \psi(q^3). \end{aligned} \quad (47)$$

This completes the proof of (40). $\square$

4. Concluding Remarks

All conjectures in [11] can easily be reformulated as theta function identity using Ramanujan's ${}_1\psi_1$ formula. However, these identities might be too complicated to actually have a proof.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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