Research Article

The Behavior of Midsets When Repeatedly Taking the Midset of Two Lines in ℓ_1 Geometry

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Received 22 September 2013; Accepted 15 April 2014; Published 8 May 2014

Academic Editor: Hernando Quevedo

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We study the outcome of taking midsets of two lines in ℓ_1 geometry. We establish the algorithm for repeatedly finding these midsets and characterize the limiting midsets. We discuss the issue of angle measurement in Minkowski geometries, especially with respect to the limiting midsets.

1. Introduction

An important class of planar Minkowski geometries are those that are determined by the unit circles $|x|^p + |y|^p = 1$ for $p \geq 1$ [1, 2]. These are designated ℓ_p^2 or simply ℓ_p geometries. The most important example is Euclidean geometry with $p = 2$. Our concern is with ℓ_1 geometry, which is sometimes called taxicab, Manhattan, or city block geometry [3]. Our main interest is in the midsets of two lines in this geometry.

The ℓ_1 midsets of two points or of a point and a line are less nuanced than the midsets of two lines. The ℓ_1 midset of two points, which is all points equidistant from the two points, can have different shapes, but generally it is a piecewise linear set of three segments [3, pages 8-9]. The ℓ_1 midset of a point and a line is an ℓ_1 parabola, which has the appearance of an approximation of an Euclidean parabola by line segments [3, page 39].

The midset of two lines is all points equidistant from the two lines. In Euclidean geometry, repeatedly taking midsets settles at the first or second step into a simple oscillation between pairs of orthogonal lines. The midset of the two intersecting and distinct lines $y = (\tan \varphi)x$ and $y = (\tan(\varphi + \theta))x$ is the perpendicular lines $y = (\tan(\varphi + \theta/2))x$ and $y = (\tan(90^\circ + \varphi + \theta/2))x$. Repeatedly finding midsets oscillates between these two orthogonal lines and a pair of lines rotated 45° from them. If the original lines are orthogonal, that is, $\theta = 90^\circ$, the oscillatory behavior begins at the first step. As seen in Section 2, in ℓ_1 geometry, repeatedly

finding midsets of two lines generally does not produce this oscillatory behavior, unless the original lines are orthogonal. Subsequently, “midset” indicates “ ℓ_1 midset.”

In Minkowski geometries, normality to a line is not reflexive; that is, the line perpendicular to the line that is perpendicular to the original line is not parallel to the original line, except in the case that the space’s unit circle is a Radon curve [2, pages 233-234], [4, pages 137-138], and [5, page 145]. Analogously, we show that in general taking midsets is not reflexive in ℓ_1 geometry; that is, the midset of the midset of two lines is not the original lines.

Without loss of generality, place the origin at the intersection of the original lines. In ℓ_1 geometry the distance from a point to a line is measured either parallel to the x -axis or parallel to the y -axis. For $y = \alpha x$, if $|\alpha| > 1$, then the distance is measured parallel to the x -axis; if $|\alpha| < 1$, then the distance is measured parallel to the y -axis; and if $|\alpha| = 1$, then either direction gives the same distance.

There are ten configurations for the original lines. They are labeled according to the location of their portion in the closed upper half plane. For our purposes, octants are open sets that do not include their boundaries, which are the axes and the lines $y = \pm x$. The configurations are as follows.

- (1) The lines are in octants separated by another octant.
- (2) The lines are $y = x$ and $y = -x$, or the lines are the axes.

since Configuration 2 is relatively simple, only Configuration 1 is discussed subsequently.

3. Midsets for Configuration 1

The algorithm for the sequence of slopes of the midsets of initial lines in Configuration 1 follows.

Algorithm 6. Beginning with the lines

$$y = a_0x, \quad y = b_0x \quad (5)$$

with $0 < a_0 < 1$ and $b_0 < -1$, the slopes of the first midset are

$$a_1 = \frac{b_0(a_0 - 1)}{b_0 - 1}, \quad b_1 = \frac{b_0(a_0 + 1)}{b_0 + 1} \quad (6)$$

with $-1 < a_1 < 0$ and $b_1 > 1$. The slopes of the second midset, which is the midset of the first midset (6), are

$$a_2 = \frac{b_1(a_1 + 1)}{b_1 + 1}, \quad b_2 = \frac{b_1(a_1 - 1)}{b_1 - 1} \quad (7)$$

with $0 < a_2 < 1$ and $b_2 < -1$. The slopes of the third midset, which is the midset of the second midset, are

$$a_3 = \frac{b_2(a_2 - 1)}{b_2 - 1}, \quad b_3 = \frac{b_2(a_2 + 1)}{b_2 + 1} \quad (8)$$

with $-1 < a_3 < 0$ and $b_3 > 1$. In general, for odd $n \geq 1$,

$$a_n = \frac{b_{n-1}(a_{n-1} - 1)}{b_{n-1} - 1}, \quad b_n = \frac{b_{n-1}(a_{n-1} + 1)}{b_{n-1} + 1} \quad (9)$$

with $-1 < a_n < 0$ and $b_n > 1$. For even $n \geq 2$,

$$a_n = \frac{b_{n-1}(a_{n-1} + 1)}{b_{n-1} + 1}, \quad b_n = \frac{b_{n-1}(a_{n-1} - 1)}{b_{n-1} - 1} \quad (10)$$

with $0 < a_n < 1$ and $b_n < -1$.

Table 1 contains the positive angle of the limiting even-numbered midsets for initial lines at five evenly spaced angles θ_0 from 4.5° to 40.5° incremented by 9.0° , giving slopes $a_0 = \tan \theta_0$, and at five evenly spaced angles φ_0 from 94.5° to 130.5° also incremented by 9.0° , giving slopes $b_0 = \tan \varphi_0$. The angle of the second line in the limiting midset is obtained by adding 90° . The angles and slopes of the limiting odd-numbered midsets can be obtained by a 45° rotation of the limiting lines of the even-numbered midsets. Limiting angles are rounded to two decimal places. No more than five pairs of taking midsets are required to obtain each tabulated value. For the entries on the main diagonal, the initial lines are orthogonal; that is, $\varphi_0 - \theta_0 = 90^\circ$. Euclidean angle measure is used only for reference. Indeed, most formulas involve only the slopes of the lines. See Section 4 about angle measurement.

Theorem 7. *If Algorithm 6 converges, it approaches a cycle of period two, alternating between pairs of orthogonal lines.*

TABLE 1: Limiting angles in degrees are given for even-numbered midsets for selected lines $y = a_0x = (\tan \theta_0)x$ and $y = b_0x = (\tan \varphi_0)x$ in Configuration 1.

$\theta_0 \setminus \varphi_0$	94.5	103.5	112.5	121.5	130.5
4.5	4.50	9.10	14.11	19.91	27.10
13.5	9.10	13.50	18.20	23.53	30.01
22.5	14.11	18.20	22.50	27.31	33.07
31.5	19.91	23.53	27.31	31.50	36.46
40.5	27.10	30.01	33.07	36.46	40.50

Proof. Consider n even. The proof for n odd is almost identical. From (10)

$$a_{n+2} = \frac{b_{n+1}(a_{n+1} + 1)}{b_{n+1} + 1}, \quad b_{n+2} = \frac{b_{n+1}(a_{n+1} - 1)}{b_{n+1} - 1} \quad (11)$$

and from (9)

$$a_{n+1} = \frac{b_n(a_n - 1)}{b_n - 1}, \quad b_{n+1} = \frac{b_n(a_n + 1)}{b_n + 1}. \quad (12)$$

Substituting (12) into (11) gives

$$a_{n+2} = \frac{b_n(a_n + 1)(a_nb_n - 1)}{(b_n - 1)(a_nb_n + 1 + 2b_n)}, \quad (13)$$

$$b_{n+2} = \frac{b_n(a_n - 1)(a_nb_n + 1 - 2b_n)}{(b_n - 1)(a_nb_n - 1)}. \quad (14)$$

In (13) set $a_{n+2} = a_n$, and in (14) set $b_{n+2} = b_n$. Each of these gives $b_n = -1/a_n$. $\square$

The slopes of the limiting two lines are designated by a and b . It may be somewhat surprising that in most cases the limiting midsets are not the geometry's natural lines of symmetry, which are the axes and $y = \pm x$.

An interesting symmetry in the outcomes of the iterative process is illustrated by the symmetry about the main diagonal in Table 1. If the initial lines are in Configuration 1, then $-1 < a_0b_0 < 0$, $a_0b_0 < -1$, or $a_0b_0 = -1$. For the last condition, Theorem 2 says that the iterative process of finding midsets alternates between the initial lines and a pair of lines obtained by a 45° rotation. In Table 1, instances are on the main diagonal.

The condition $-1 < a_0b_0 < 0$ corresponds to $90^\circ < \varphi_0 - \theta_0 < 135^\circ$, and examples are above the main diagonal in Table 1. For $a_0b_0 < -1$, $45^\circ < \varphi_0 - \theta_0 < 90^\circ$ and examples are below the main diagonal in Table 1. To distinguish initial lines and outcomes of iterations for these two cases, we introduce primes for those with initial lines with $a_0b_0 < -1$. That is, use $\theta'_0, \varphi'_0, a'_0, b'_0, a'_n, b'_n, a',$ and b' .

Theorem 8. *Start with the two initial lines $y = a_0x$ and $y = b_0x$ with $-1 < a_0b_0 < 0$ corresponding to $90^\circ < \varphi_0 - \theta_0 < 135^\circ$, and further two initial lines $y = a'_0x$ and $y = b'_0x$ with $a'_0b'_0 < -1$ corresponding to $45^\circ < \varphi'_0 - \theta'_0 < 90^\circ$. Assume further the identities $\theta'_0 = \varphi_0 - 90^\circ$ and $\varphi'_0 = \theta_0 + 90^\circ$. Then $a'_n = -1/b_n$ and $b'_n = -1/a_n$ for all even $n \geq 0$.*

Proof. We use mathematical induction. For $n = 0$, $a'_0 = \tan \theta'_0 = \tan(\varphi_0 - 90^\circ) = -1/\tan \varphi_0 = -1/b_0$. Similarly, $b'_0 = -1/a_0$. Assume that

$$a'_n = -\frac{1}{b_n}, \quad b'_n = -\frac{1}{a_n} \quad (15)$$

for n even. From (13) and (14)

$$\begin{aligned} a'_{n+2} &= \frac{b'_n(a'_n + 1)(a'_n b'_n - 1)}{(b'_n - 1)(a'_n b'_n + 1 + 2b'_n)}, \\ b'_{n+2} &= \frac{b'_n(a'_n + 1)(a'_n b'_n + 1 - 2b'_n)}{(b'_n - 1)(a'_n b'_n - 1)}. \end{aligned} \quad (16)$$

Substituting (15) into (16) and using (13) and (14) give $a'_{n+2} = -1/b_{n+2}$ and $b'_{n+2} = -1/a_{n+2}$. $\square$

Theorem 9. *If $-1 < a_0 b_0 < 0$, then for n even, $a_{n+2} > a_n$, $b_{n+2} < b_n$, and $-1 < a_{n+2} b_{n+2} < a_n b_n$, and the angle $\varphi_n - \theta_n$ between the midset's lines is monotonically decreasing toward 90° . If $a'_0 b'_0 < -1$, then for n even, $a'_{n+2} < a'_n$, $b'_{n+2} > b'_n$, and $a'_n b'_n < a'_{n+2} b'_{n+2} < -1$, and the angle $\varphi'_n - \theta'_n$ between the midset's lines is monotonically increasing toward 90° . Symmetrically placed initial lines have the same pairs of orthogonal lines in the limit, which always exists.*

Proof. The proof for the case $-1 < a_0 b_0 < 0$ is outlined. The other case's proof is very similar. Using (13) with $n = 0$, set $a_2 > a_0$, which reduces to $a_0 b_0 > -1$. Reversing the algebraic steps gives $a_2 > a_0$. Similarly, using (14) with $n = 0$, set $b_2 < b_0$, which reduces to $a_0 b_0 > -1$. Reversing the steps gives $b_2 < b_0$. To see that $-1 < a_2 b_2$, for $n = 0$ multiply (13) and (14), which becomes

$$a_2 b_2 = \frac{b_0^2 (a_0 + 1)^2 (a_0 b_0 + 1 - 2b_0)}{(b_0 - 1)^2 (a_0 b_0 + 1 + 2b_0)}. \quad (17)$$

This is seen to be greater than -1 by setting it greater than -1 and performing the reversible steps of cross-multiplying by the negative denominator, expanding, cancelling terms, and factoring, giving

$$(a_0 b_0 + 1)(a_0^2 b_0^2 + 1 - 2b_0^2) < 0. \quad (18)$$

The first factor is positive since $-1 < a_0 b_0$, and the second factor is negative for $-1 < a_0 b_0 < 0$ and $b_0 < -1$. Mathematical induction on n gives the results for even n . The slopes $\{a_n\}$ are an increasing sequence, which is bounded above, since $a_n < 1$ for all n . The limiting slope exists, and, using Theorems 7 and 8, the process of taking midsets monotonically approaches a cycle alternating between pairs of orthogonal lines, since $a' = -1/b = -1/(-1/a) = a$. $\square$

4. Definitions of ℓ_1 Angle and Angle Bisector

Thompson discusses at length the “ambiguity about the definition of angle” [1, page 273], and Busemann declares

the “great arbitrariness of angle measure” [6, page 275]. Angle measure remains an active area of study in geometry and analysis [7–10]. Trigonometric functions are generally defined in terms of lines, rays, or vectors. For example, calling them vectors, the cosine of a pair is the dot product of the vectors divided by the product of their lengths. Angle measure is virtually unique in Euclidean geometry, but there are many natural choices in Minkowski geometry, depending upon the properties that are desirable. For example, in ℓ_1 geometry, the angle between (x, y) on the unit circle near $(1, 0)$ is $2y$, if the angle is defined by the arc length from the x -axis along the unit circle. Defining the sine function as the length of the opposite side divided by the hypotenuse in ℓ_1 measure, the ratio of sine to the angle is the constant $1/2$. In order to preserve certain properties of derivatives of trigonometric functions with respect to their arguments, Thompson selects a measure of angle that preserves the property that the limit as the angle approaches zero of the sine divided by the angle is 1 [1, pages. 264–266]. Generally, the angle measure between two vectors or rays that share the same Euclidean angle measurement is different, since the size of the angle depends upon the specific directions of the vectors or rays.

Except for special situations, such as the lines being the axes, the limiting line $y = ax$ in Section 3 is not an angle bisector. Various definitions of angle give different angle bisectors. Here, as elsewhere, Euclidean angle measure is used only for easy reference. The slope is the fundamental quantity.

We consider four options of ℓ_1 angle bisector for lines in Configuration 1.

- (1) Use Euclidian (ℓ_2) angle measure and bisectors.
- (2) Use ℓ_1 angles based on the subtended arc length of the unit circle $|x| + |y| = 1$ and their bisectors.
- (3) Use the first step in Algorithm 6; that is, in Configuration 1, use $y = b_1 x$ in (4).
- (4) Use the lines $y = ax$ obtained in the limit in Section 3.

Using Euclidian measure follows Busemann [6, page 277] and [11]. Option 2 is the same as using area measure, since in ℓ_1 geometry the subtended arc length has a constant of proportionality with the Euclidean area.

For an example, take the lines $y = (\tan 4.5^\circ)x = 0.07870x$ and $y = (\tan 130.5^\circ)x = -1.171x$ in Configuration 1, and compare the bisectors obtained in the second octant.

For Option 1, the Euclidean angle bisector has the equation $y = (\tan[(4.5^\circ + 130.5^\circ)/2])x = (\tan 67.5^\circ)x = 2.414x$.

For Option 2, the ℓ_1 unit circle centered at the origin is the square with vertices at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$. Its circumference is 8. The ℓ_1 length of the subtended arc along the portion $y = 1 - x$ of the ℓ_1 unit circle between $y = 0.07870x$ and the positive y -axis is $2 - 2(1 - [1/(1 + 0.07870)]) = 1.854$. The length of the subtended arc between $y = -1.171x$ and the positive y -axis is $2[1/(1 + 1.171)] = 0.9212$. One-half the length of the subtended arc between $y = 0.07870x$ and $y = -1.171x$ is $(1.854 + 0.9212)/2 = 1.388$. Add this length to the arc length between the x -axis and $y = 0.07870x$ to obtain $2(1 - [1/(1 + 0.07870)]) + 1.388 = 1.534$.

The tangent of the angle that this bisector makes with the x -axis is $[1.534/2]/[1 - (1.534/2)] = 3.292$, which corresponds to the angle $\tan^{-1} 3.292 = 73.10^\circ$.

For Option 3, $b_1 = [-1.171/(-0.171)](1.079) = 7.389$, which corresponds to the angle $\tan^{-1} 7.389 = 82.29^\circ$.

For Option 4, using the value in Table 1, the limiting line is $y = [\tan(27.10^\circ + 45^\circ)]x = (\tan 72.10^\circ)x = 3.096x$.

Depending upon which option is selected, the angle that the bisector makes with the x -axis is 67.5° , 73.10° , 82.29° , or 72.10° , in this example.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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