Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2014, Article ID 837408, 9 pages
http://dx.doi.org/10.1155/2014/837408

Research Article

Degree of Approximation of Functions $\tilde{f} \in H_\omega$ Class by the $(N_p \cdot E^1)$ Means in the Hölder Metric

Vishnu Narayan Mishra[1,2] **and Kejal Khatri**[1]

[1] *Applied Mathematics and Humanities Department, Sardar Vallabhbhai National Institute of Technology, Ichchhanath Mahadev Dumas Road, Surat, Gujarat 395 007, India*
[2] *L. 1627 Awadh Puri Colony Beniganj, Opp. I.T.I. Ayodhya Main Road, Faizabad, Uttar Pradesh 224 001, India*

Correspondence should be addressed to Vishnu Narayan Mishra; vishnunarayanmishra@gmail.com

Received 28 December 2013; Revised 28 February 2014; Accepted 28 February 2014; Published 30 April 2014

Academic Editor: Ricardo Estrada

A new estimate for the degree of approximation of a function $\tilde{f} \in H_\omega$ class by $(N_p \cdot E^1)$ means of its Fourier series has been determined. Here, we extend the results of Singh and Mahajan (2008) which in turn generalize the result of Lal and Yadav (2001). Some corollaries have also been deduced from our main theorem.

1. Introduction

The degree of approximation of a function f belonging to various classes using different summability method has been determined by several investigators like Khan [1, 2], V. N. Mishra and L. N. Mishra [3], Mishra et al. [4–6], and Mishra [7, 8]. Summability techniques were also applied on some engineering problems; for example, Chen and Jeng [9] implemented the Cesàro sum of order $(C, 1)$ and $(C, 2)$, in order to accelerate the convergence rate to deal with the Gibbs phenomenon, for the dynamic response of a finite elastic body subjected to boundary traction. Chen and Hong [10] used Cesàro sum regularization technique for hyper singularity of dual integral equation. Summability of Fourier series is useful for engineering analysis, for example, [11]. Recently, Mursaleen and Mohiuddine [12] discussed convergence methods for double sequences and their applications in various fields. In sequel, Alexits [13] studied the degree of approximation of the functions in H_α class by the Cesàro means of their Fourier series in the sup-norm. Chandra ([14, 15]), Mohapatra and Chandra ([16, 17]), and Szal [18] have studied the approximation of functions in Hölder metric. Mishra et al. [6] used the technique of approximation of functions in measuring the errors in the input signals and the processed output signals. In 2008, Singh and Mahajan [19] studied error bound of periodic signals in the Hölder metric and generalized the result of Lal and Yadav [20] under much more general assumptions. Analysis of signals or time functions is of great importance, because it conveys information or attributes of some phenomenon. The engineers and scientists use properties of Fourier approximation for designing digital filters. Especially, Psarakis and Moustakides [21] presented a new L_2 based method for designing the finite impulse response (FIR) digital filters and got corresponding optimum approximations having improved performance. We also discuss an example when the Fourier series of the signal has Gibbs phenomenon.

For a 2π-periodic signal $f \in L^p := L^p[0, 2\pi]$, $p \geq 1$, periodic integrable in the sense of Lebesgue. Then the Fourier series of $f(x)$ is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x), \tag{1}$$

with $(n+1)$th partial sum $s_n(f;x)$ called trigonometric polynomial of degree (or order) n and given by

$$s_n(f;x) := \frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx), \quad n \in \mathbb{N} \text{ with } s_0(f;x) = \frac{a_0}{2}. \tag{2}$$

The conjugate series of Fourier series (1) is given by

$$\sum_{n=1}^{\infty}(b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x), \tag{3}$$

with nth partial sum $\tilde{s}_n(f;x)$.

Let $\omega(t)$ and $\omega^*(t)$ denote two given moduli of continuity such that

$$(\omega(t))^{\beta/\alpha} = O(\omega^*(t)) \quad \text{as } t \longrightarrow 0+, \quad \text{for } 0 \le \beta < \alpha \le 1. \tag{4}$$

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic continuous functions defined on $[-\pi,\pi]$ under the sup-norm. The space $L_p[0,2\pi]$ where $p = \infty$ includes the space $C_{2\pi}$. For some positive constant K, the function space H_ω is defined by

$$H_\omega = \{f \in C_{2\pi} : |f(x) - f(y)| \le K\omega(|x-y|)\}, \tag{5}$$

with norm $\|\cdot\|_{\omega^*}$ defined by

$$\|f\|_{\omega^*} = \|f\|_c + \sup_{x,y}\left[\Delta^{\omega^*} f(x,y)\right], \tag{6}$$

where $\omega(t)$ and $\omega^*(t)$ are increasing functions of t,

$$\|f\|_c = \sup_{0\le x\le 2\pi}|f(x)|, \tag{7}$$

$$\Delta^{\omega^*} f(x,y) = \frac{|f(x) - f(y)|}{\omega^*(|x-y|)}, \quad x \ne y, \tag{8}$$

with the understanding that $\Delta^0 f(x,y) = 0$. If there exist positive constants B and K such that $\omega(|x-y|) \le B|x-y|^\alpha$ and $\omega^*(|x-y|) \le K\,|x-y|^\beta$, $0 \le \beta < \alpha \le 1$, then the space

$$H_\alpha = \left\{f \in C_{2\pi} : |f(x) - f(y)| \le K|x-y|^\alpha, 0 < \alpha \le 1\right\} \tag{9}$$

is Banach space [22] and the metric induced by the norm $\|\cdot\|_\alpha$ on H_α is said to be Hölder metric. Clearly H_α is a Banach space which decreases as α increases; that is,

$$H_\alpha \subseteq H_\beta \subseteq C_{2\pi}, \quad \text{for } 0 \le \beta < \alpha \le 1. \tag{10}$$

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with the sequence of nth partial sums $\{s_n\}$. Let $\{p_n\}$ be a nonnegative sequence of constants, real or complex, and let us write

$$P_n = \sum_{k=0}^{n} p_k \ne 0 \quad \forall n \ge 0, \qquad p_{-1} = 0 = P_{-1}, \quad P_n \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{11}$$

The sequence to sequence transformation

$$t_n^N(f;x) = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k} s_k(f;x) \tag{12}$$

defines the sequence $\{t_n^N\}$ of Nörlund means of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series $\sum_{n=0}^{\infty} a_n$ is said to be summable N_p to the sum s if $\lim_{n\to\infty} t_n^N$ exists and is equal to s.

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)}; \quad (\alpha > 0), \tag{13}$$

then Nörlund summability N_p reduces to the familiar (C,α) summability.

An infinite series $\sum_{n=0}^{\infty} a_n$ is said to be $(C,1)$ summable to s if

$$(C,1) = \frac{1}{(n+1)}\sum_{k=0}^{n} s_k \longrightarrow s \quad \text{as } n \longrightarrow \infty. \tag{14}$$

The E^1 transform is defined as the nth partial sum of E^1 summability and we denote it by $\tilde{E}_n^1$.

If

$$\tilde{E}_n^1(f) = \tilde{E}_n^1(f;x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\tilde{s}_k \longrightarrow s, \quad \text{as } n \longrightarrow \infty, \tag{15}$$

then the infinite series $\sum_{n=0}^{\infty} a_n$ is said to be $(E,1)$ summable to s.

The $(C,1)$ transform of the $(E,1)$ transform E_n^1 defines the $(C,1)(E,1)$ transform of the partial sums s_n of the series $\sum_{n=0}^{\infty} a_n$; that is, the product summability $(C,1)(E,1)$ is obtained by superimposing $(C,1)$ summability on $(E,1)$ summability. Thus, if

$$\begin{aligned}(CE)_n^1 &= \frac{1}{(n+1)}\sum_{k=0}^{n} E_k^1 \\ &= \frac{1}{(n+1)}\sum_{k=0}^{n}\frac{1}{2^k}\sum_{r=0}^{k}\binom{k}{r} s_r \longrightarrow s, \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{16}$$

where E_n^1 denotes the $(E,1)$ transform of s_n, then the series $\sum_{n=0}^{\infty} a_n$ with the partial sums s_n is said to be summable $(C,1)(E,1)$ to the definite number s and we can write

$$(CE)_n^1 \longrightarrow s\,[(C,1)(E,1)], \quad \text{as } n \longrightarrow \infty. \tag{17}$$

The N_p transform of the E^1 transform defines $(N_p \cdot E^1)$ product transform and denote it by $\tilde{t}_n^{NE}(f)$. If

$$\begin{aligned}\tilde{t}_n^{NE}(f) &= \tilde{t}_n^{NE}(f;x) \\ &= \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}\,\tilde{E}_k^1(f) \longrightarrow s, \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{18}$$

then the infinite series $\sum_{n=0}^{\infty} a_n$ is said to be $(N_p \cdot E^1)$ summable to s.

We note that E_n^1, $(CE)_n^1$, and $\tilde{t}_n^{NE}$ are also trigonometric polynomials of degree (or order) n.

The relation between Cesàro mean and Fejér mean is given by

$$\sigma_n(x) = \frac{1}{2\pi} \int_0^{2\pi} [f(x+t) + f(x-t)] F_n(t)\, dt, \tag{19}$$

where the Fejér mean is

$$F_0(t) = 1,$$
$$F_n(t) = \begin{cases} \frac{1}{n+1} \left(\frac{\sin(n+1)(t/2)}{\sin(t/2)} \right)^2, & t \notin 2\pi Z,\ n \geq 1 \\ (n+1), & t \in 2\pi Z, \end{cases} \tag{20}$$

and Cesàro mean is

$$\sigma_n = \frac{s_0(x) + s_1(x) + \cdots + s_{n-1}(x)}{n} = \frac{1}{n} \sum_{k=0}^{n-1} s_k(x). \tag{21}$$

Some important particular cases are as follows:

(1) $(H, 1/(n+1)) \cdot (E_1)$ means, when $a_{n,k} = 1/(n-k+1)\log n$.

(2) $(N, p_n, q_n) \cdot E_1$ means, $a_{n,k} = p_{n-k} q_k / R_n$, where $R_n = \sum_{k=0}^{n} p_k q_{n-k} \neq 0$.

Remark 1. If we take $\omega(|x-y|) = |x-y|^{\alpha}$, then H_{ω} reduces to H_{α} class.

The conjugate function $\tilde{f}(x)$ is defined for almost every x by

$$\begin{aligned} \tilde{f}(x) &= -\frac{1}{2\pi} \int_0^{\pi} \psi(t) \cot\frac{t}{2} dt \\ &= \lim_{h \to 0} \left(-\frac{1}{2\pi} \int_h^{\pi} \psi(t) \cot\frac{t}{2} dt \right); \end{aligned} \tag{22}$$

see [23, page 131].

We write throughout the paper

$$\begin{aligned} \psi_x(t) &= \psi(t) = f(x+t) - f(x-t), \\ \phi_x(t) &= f(x+t) - 2f(x) + f(x-t), \\ (N_p \cdot E^1)_n(t) &= \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \frac{\cos^k(t/2)\cos(k+1)(t/2)}{\sin t/2}. \end{aligned} \tag{23}$$

We note that the series, conjugate to a Fourier series, is not necessarily a Fourier series [23, 24]. Hence, a separate study of conjugate series is desirable and attracted the attention of researchers.

2. Known Results

In 2001, Lal and Yadav [20] established the following theorem to estimate the error between the input signal $f(x)$ and the signal obtained after passing through the $(C,1)(E,1)$-transform.

Theorem 2 (see [20]). *If a function $f : R \to R$ is 2π-periodic function and belongs to class Lip α, $0 < \alpha \leq 1$, then degree of approximation by $(C,1)(E,1)$ means of its Fourier series is given by*

$$\left\| t_n^{C^1 \cdot E^1}(f) - f(x) \right\|_{\infty} = \begin{cases} O(n^{-\alpha}), & 0 < \alpha < 1 \\ O\left(\frac{\log n}{n} \right), & \alpha = 1. \end{cases} \tag{24}$$

Recently, Singh and Mahajan [19] generalized the above result under more general assumptions. They proved the following.

Theorem 3 (see [19]). *Let $\omega(t)$ defined in (5) be such that*

$$\begin{aligned} &\int_t^{\pi} \frac{\omega(u)}{u^2} du = O(H(t)), \quad H(t) \geq 0, \\ &\int_0^t H(u)\, du = O(tH(t)), \quad \text{as } t \longrightarrow 0^+; \end{aligned} \tag{25}$$

then, for $0 < \beta \leq \alpha \leq 1$ and $f \in H_{\omega}$, we have

$$\left\| t_n^{C^1 \cdot E^1}(f) - f(x) \right\|_{\omega^*} = O\left(\left((n+1)^{-1} H\left(\frac{\pi}{n+1} \right) \right)^{1-\beta/\alpha} \right). \tag{26}$$

Theorem 4 (see [19]). *Consider $\omega(t)$ defined in (5) and for $0 < \beta \leq \alpha \leq 1$ and $f \in H_{\omega}$, we have*

$$\begin{aligned} &\left\| t_n^{C^1 \cdot E^1}(f) - f(x) \right\|_{\omega^*} \\ &= O\left(\left(\omega\left(\frac{\pi}{n} + 1 \right) \right)^{1-\beta/\alpha} + \left((n+1)^{-1} \sum_{k=1}^{n+1} \omega\left(\frac{1}{k+1} \right) \right)^{1-\beta/\alpha} \right). \end{aligned} \tag{27}$$

3. Main Theorem

In this paper, we prove a theorem on the degree of approximation of a function $\tilde{f}(x)$, conjugate to a 2π-periodic function f belonging to $\tilde{f} \in H_{\omega}$ class by $(N_p \cdot E^1)$ means of conjugate series of its Fourier series. This work generalizes the results of Singh and Mahajan [19] on $(N_p \cdot E^1)$ summability of conjugate Fourier series. We will measure the error between the input signal $\tilde{f}(x)$ and the processed output signal $\tilde{t}_n^{NE}(f; x) = (1/P_n) \sum_{k=0}^{n} p_{n-k} \tilde{E}_k^1(f)$, by establishing the following theorems.

Theorem 5. *The functions ω satisfy the following conditions:*

$$\int_t^{\pi} \frac{\omega(u)}{u^2} du = O(H(t)), \quad H(t) \geq 0, \tag{28}$$

$$\int_0^t H(u)\, du = O(tH(t)), \quad \text{as } t \longrightarrow 0^+, \tag{29}$$

where $\omega(t)$ and $\omega^*(t)$ are increasing functions of t. Let N_p be the Nörlund summability matrix generated by the nonnegative $\{p_n\}$ such that

$$(n+1)\,p_n = O(P_n), \quad \forall n \geq 0. \tag{30}$$

Then, for $\tilde{f} \in H_\omega, 0 < \beta \leq \alpha \leq 1$, we have

$$\begin{aligned}\left\|\tilde{t}_n^{NE}(f) - \tilde{f}(x)\right\|_{\omega^*} &= O\left\{\frac{\omega(|x-y|)^{\beta/\alpha}}{\omega^*(|x-y|)}(\log(n+1))^{\beta/\alpha}\right.\\ &\quad \left.\times\left[(n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right\},\end{aligned} \tag{31}$$

and if $\omega(t)$ satisfies (28)*, then for $\tilde{f} \in H_\omega, 0 < \beta \leq \alpha \leq 1$, we have*

$$\begin{aligned}\left\|\tilde{t}_n^{NE}(f) - \tilde{f}(x)\right\|_{\omega^*} &= O\left\{\frac{\omega(|x-y|)^{\beta/\alpha}}{\omega^*(|x-y|)}\left(\log(n+1)\left[\omega\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right.\right.\\ &\quad \left.\left.+\left[\left(\frac{1}{n+1}\right)\sum_{k=0}^{n}\omega\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right)\right\}.\end{aligned} \tag{32}$$

Remark 6. The product transform $(N_p \cdot E^1)$ plays an important role in signal theory as a double digital filter and theory of Machines in Mechanical Engineering [4, 5].

4. Lemmas

In order to prove our main Theorem 5, we require the following lemmas.

Lemma 7. *If $\psi_x(t) = \psi(t) = f(x+t) - f(x-t)$, then for $\tilde{f} \in H_\omega$, we have*

$$\left|\psi_x(t) - \psi_y(t)\right| \leq 2M\omega(|x-y|), \tag{33}$$

$$\left|\psi_x(t) - \psi_y(t)\right| \leq 2M\omega(|t|). \tag{34}$$

It is easy to verify the following.

Lemma 8. *Let $\{p_n\}$ be a nonnegative and nonincreasing sequence satisfies* (30)*; then for $0 < t \leq \pi/(n+1)$, we have*

$$\left(N_p \cdot E^1\right)_n(t) = O\left(\frac{1}{t}\right). \tag{35}$$

Proof. Consider

$$\left(N_p \cdot E^1\right)_n(t) = \frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\frac{\cos^k(t/2)\cos(k+1)(t/2)}{\sin t/2}. \tag{36}$$

Using condition (30), $\sin nt \leq nt, \sin(t/2) \geq (t/\pi)$, for $0 < t \leq \pi$ and $|\cos^k(t/2)| \leq 1$, we have

$$\begin{aligned}\left(N_p \cdot E^1\right)_n(t) &\leq \frac{p_n}{P_n}\sum_{k=0}^{n}\frac{\cos^k(t/2)\cos(k+1)(t/2)}{\sin t/2}\\ &= O\left(\frac{1}{t(n+1)}\right)\left[\sum_{k=0}^{n}\cos^k\left(\frac{t}{2}\right)\cos(k+1)\left(\frac{t}{2}\right)\right]\\ &= \left(\frac{1}{t(n+1)}\right)\sum_{k=0}^{n}\operatorname{Re}\left\{\cos^k\left(\frac{t}{2}\right)e^{i(k+1)t/2}\right\}\\ &= \left(\frac{1}{t(n+1)}\right)\operatorname{Re}\left\{e^{it/2}\left(\frac{1-\cos^{n+1}(t/2)\,e^{\,i(n+1)t/2}}{1-\cos(t/2)\,e^{it/2}}\right)\right\}\\ &= \left(\frac{1}{t^2(n+1)}\right)\operatorname{Re}\left\{i - i\cos(n+1)(t/2)\cos^{n+1}(t/2)\right.\\ &\qquad \left.+\sin(n+1)(t/2)\cos^{n+1}(t/2)\right\}\\ &= O\left(\frac{1}{t^2(n+1)}\right)\sin(n+1)(t/2)\cos^{n+1}(t/2).\\ &= O\left(\frac{1}{t^2(n+1)}(n+1)t\right)\\ &= O\left(\frac{1}{t}\right).\end{aligned} \tag{37}$$

□

Lemma 9. *Let $\{p_n\}$ be a nonnegative and nonincreasing sequence satisfies* (30)*; then for $\pi/(n+1) \leq t \leq \pi$, we get*

$$\left(N_p \cdot E^1\right)_n(t) = O\left(\frac{1}{t^2(n+1)}\right). \tag{38}$$

Proof. Consider

$$\left(N_p \cdot E^1\right)_n(t) = \frac{1}{P_n}\sum_{k=0}^{n}p_{n-k}\frac{\cos^k(t/2)\cos(k+1)(t/2)}{\sin t/2}. \tag{39}$$

Using condition (30), $\sin(t/2) \geq (t/\pi)$, for $0 < t \leq \pi$, $|\sin t| \leq 1$ for all t and $|\cos^k(t/2)| \leq 1$, we have

$$\begin{aligned}\left(N_p \cdot E^1\right)_n(t) &\leq \frac{p_n}{P_n}\sum_{k=0}^{n}\frac{\cos^k(t/2)\cos(k+1)(t/2)}{\sin t/2}.\\ &= \left(\frac{1}{t(n+1)}\right)\sum_{k=0}^{n}\operatorname{Re}\left\{\cos^k\left(\frac{t}{2}\right)e^{i(k+1)t/2}\right\}\\ &= \left(\frac{1}{t(n+1)}\right)\operatorname{Re}\left\{e^{it/2}\left(\frac{1-\cos^{n+1}(t/2)\,e^{i(n+1)t/2}}{1-\cos(t/2)\,e^{it/2}}\right)\right\}\end{aligned}$$

$$= O\left(\frac{1}{t^2(n+1)}\right)\sin(n+1)(t/2)\cos^{n+1}(t/2).$$
$$= O\left(\frac{1}{t^2(n+1)}\right). \tag{40}$$

□

Lemma 10 (see [19]). *If $\omega(t)$ satisfies conditions* (28) *and* (29), *then*

$$\int_0^u t^{-1}\omega(t)\,dt = O(uH(u)), \quad \text{as } u \longrightarrow 0^+. \tag{41}$$

5. Proof of Theorem 5

The integral representationof $\tilde{s}_n(f;x)$ is given by

$$\begin{aligned}\tilde{s}_n(f;x) &= \sum_{n=1}^{\infty}(b_n\cos nx - a_n\sin nx)\\ &= -\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\left(\frac{\cot(t/2)\cos(n+1/2)t}{2\sin(t/2)}\right)dt\end{aligned} \tag{42}$$

and, therefore,

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi_x(t)\frac{\cos(n+1/2)t}{\sin t/2}dt. \tag{43}$$

Denoting E^1 means of $\tilde{s}_n(f;x)$ by $\tilde{E}_n^1(x)$, we have

$$\begin{aligned}&\frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\{\tilde{s}_n(f;x)-\tilde{f}(x)\}\\ &\quad= \frac{1}{2^{(n+1)}\pi}\int_0^{\pi}\frac{\psi_x(t)}{\sin t/2}\sum_{k=0}^{n}\binom{n}{k}\cos(k+1/2)t\,dt,\\ &\tilde{E}_n^1(x)-\tilde{f}(x)\\ &\quad= \frac{1}{2^{(n+1)}\pi}\int_0^{\pi}\frac{\psi_x(t)}{\sin t/2}\sum_{k=0}^{n}\binom{n}{k}\cos(k+1/2)t\,dt\\ &\quad= \frac{1}{2^{(n+1)}\pi}\int_0^{\pi}\frac{\psi_x(t)}{\sin t/2}\operatorname{Re}\left\{\sum_{k=0}^{n}\binom{n}{k}e^{i(k+1/2)t}\right\}dt\\ &\quad= \frac{1}{2^{(n+1)}\pi}\int_0^{\pi}\frac{\psi_x(t)}{\sin t/2}\operatorname{Re}\left\{e^{it/2}\left(1+e^{it}\right)^n\right\}dt\\ &\quad= \frac{1}{2^{(n+1)}\pi}\int_0^{\pi}\frac{\psi_x(t)}{\sin t/2}\operatorname{Re}\left\{2^n\cos^n\left(\frac{t}{2}\right)e^{i(n+1)t/2}\right\}dt\\ &\quad= \frac{1}{2\pi}\int_0^{\pi}\psi_x(t)\frac{\cos^n(t/2)\cos(n+1)(t/2)}{\sin t/2}\,dt.\end{aligned} \tag{44}$$

Now, using condition (30), then N_p transform of E^1 transform is given by

$$\begin{aligned}&\left|\tilde{t}_n^{NE}(f)-\tilde{f}(x)\right|\\ &\quad= \frac{1}{2\pi}\int_0^{\pi}\frac{|\psi_x(t)|}{t}\\ &\qquad\times\left|\sum_{k=0}^{n}\frac{p_{n-k}}{P_n}\cos^k\left(\frac{t}{2}\right)\cos(k+1)\left(\frac{t}{2}\right)\right|dt\\ &\quad= O\left(\frac{1}{n+1}\right)\int_0^{\pi}\frac{|\psi_x(t)|}{t}\\ &\qquad\times\left|\sum_{k=0}^{n}\cos^k\left(\frac{t}{2}\right)\cos(k+1)\left(\frac{t}{2}\right)\right|dt.\end{aligned} \tag{45}$$

Set

$$\begin{aligned}E_n(x) &= \left|\tilde{t}_n^{NE}(f)-\tilde{f}(x)\right|\\ &= O\left(\frac{1}{n+1}\right)\\ &\quad\times\int_0^{\pi}\frac{|\psi_x(t)|}{t}\left|\sum_{k=0}^{n}\cos^k\left(\frac{t}{2}\right)\cos(k+1)\left(\frac{t}{2}\right)\right|dt,\\ E_n(x,y)&\\ &= |E_n(x)-E_n(y)|\\ &= O\left(\frac{1}{n+1}\right)\int_0^{\pi}\frac{|\psi_x(t)-\psi_y(t)|}{t}\\ &\qquad\times\left|\sum_{k=0}^{n}\cos^k\left(\frac{t}{2}\right)\cos(k+1)\left(\frac{t}{2}\right)\right|dt\\ &= O\left(\frac{1}{n+1}\right)\left[\int_0^{\pi/(n+1)}+\int_{\pi/(n+1)}^{\pi}\right]\\ &= I_1+I_2, \quad (\text{say}).\end{aligned} \tag{46}$$

Now, using (34) and Lemma 10, assume that $\omega(t)$ satisfies (28) and (29); we get

$$\begin{aligned}I_1 &= O(1)\int_0^{\pi/(n+1)}t^{-1}\omega(t)\,dt\\ &= O\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right).\end{aligned} \tag{47}$$

Now, using (34) and Lemma 10, assume that $\omega(t)$ satisfies (28) and (29); we get

$$\begin{aligned}I_2 &= O\left(\frac{1}{n+1}\right)\int_{\pi/(n+1)}^{\pi}t^{-2}\omega(t)\,dt\\ &= O\left((n+1)^{-1}H\left(\frac{\pi}{n+1}\right)\right).\end{aligned} \tag{48}$$

Now, using (33), Lemma 8, we get

$$\begin{aligned} I_1 &= O\left(\frac{1}{n+1}\right) \int_0^{\pi/(n+1)} \frac{\left|\psi_x(t)-\psi_y(t)\right|}{t} dt \\ &= O\left(\frac{1}{n+1}\right) \int_0^{\pi/(n+1)} \frac{\omega(|x-y|)}{t} dt \\ &= O(\omega(|x-y|)) \left\{\int_0^{\pi/(n+1)} \frac{1}{t} dt\right\}, \\ &\qquad \text{where } 0 < t \le \frac{\pi}{(n+1)} \\ &= O(\log(n+1)\,\omega(|x-y|)). \end{aligned} \tag{49}$$

Now, using (33), Lemma 9, we get

$$\begin{aligned} I_2 &= O\left(\frac{1}{n+1}\right) \int_{\pi/(n+1)}^{\pi} t^{-2} \omega(|x-y|)\, dt \\ &= O\left(\frac{1}{n+1} \omega(|x-y|)\right) \int_{\pi/(n+1)}^{\pi} t^{-2} dt \\ &= O(\omega(|x-y|)). \end{aligned} \tag{50}$$

Using the fact that we may write $I_k = I_k^{1-\beta/\alpha} I_k^{\beta/\alpha}$, $k = 1, 2$, and combining (47) and (49), we get

$$\begin{aligned} I_1 = O\Bigg(&\left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha} \\ &\times \left[\log(n+1)\,\omega(|x-y|)\right]^{\beta/\alpha}\Bigg). \end{aligned} \tag{51}$$

Combining (48) and (50), we get

$$I_2 = O\left(\left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha} \left[\omega(|x-y|)\right]^{\beta/\alpha}\right). \tag{52}$$

Therefore, from (8), (51), and (52), we have

$$\begin{aligned} &\sup_{x,y} \left|\Delta^{\omega^*} E(x,y)\right| \\ &= \sup_{x,y} \frac{|E_n(x) - E_n(y)|}{\omega^*(|x-y|)} \\ &= O\Bigg\{\frac{\omega(|x-y|)^{\beta/\alpha}}{\omega^*(|x-y|)} (\log(n+1))^{\beta/\alpha} \\ &\qquad \times \left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\Bigg\}. \end{aligned} \tag{53}$$

Since

$$\|E_n(x)\|_c = \sup_{0 \le x \le 2\pi} \left|\tilde{t}_n^{T\cdot E^1}(f) - \tilde{f}(x)\right|, \tag{54}$$

it follows from (47) and (48) that

$$\|E_n(x)\|_c = O\left((n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right). \tag{55}$$

Combining (53) and (55), we get

$$\begin{aligned} &\left\|\tilde{t}_n^{NE}(f) - \tilde{f}(x)\right\|_{\omega^*} \\ &= O\Bigg\{\frac{\omega(|x-y|)^{\beta/\alpha}}{\omega^*(|x-y|)} (\log(n+1))^{\beta/\alpha} \\ &\qquad \times \left[(n+1)^{-1} H\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\Bigg\}. \end{aligned} \tag{56}$$

To prove (32), if $\omega(t)$ satisfies only (28), then, using (34) and second mean value theorem for integrals, we have

$$\begin{aligned} I_1 &= O(\omega(\pi/(n+1))) \left\{\int_0^{\pi/(n+1)} \frac{1}{t} dt\right\}, \\ &\qquad \text{where } 0 < t \le \frac{\pi}{(n+1)} \\ &= O\left(\log(n+1)\,\omega\left(\frac{\pi}{n+1}\right)\right). \end{aligned} \tag{57}$$

Combining (49) and (57), we get

$$\begin{aligned} I_1 = O\Bigg(&\left[\log(n+1)\,\omega\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha} \\ &\times \left[\log(n+1)\,\omega(|x-y|)\right]^{\beta/\alpha}\Bigg). \end{aligned} \tag{58}$$

Again, if $\omega(t)$ satisfies only (28), then using (34) and second mean value theorem for integrals, we have

$$\begin{aligned} I_2 &= O\left(\frac{1}{n+1}\right) \int_{\pi/(n+1)}^{\pi} \frac{\omega(t)}{t^2} dt \\ &= O\left(\frac{1}{n+1}\right) \int_1^{n+1} \omega\left(\frac{\pi}{t}\right) dt \\ &= O\left(\frac{1}{n+1}\right) \sum_{k=0}^{n} \omega\left(\frac{\pi}{n+1}\right). \end{aligned} \tag{59}$$

Combining (50) and (59), we get

$$I_2 = O\left(\left[\left(\frac{1}{n+1}\right) \sum_{k=0}^{n} \omega\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha} \left[\omega(|x-y|)\right]^{\beta/\alpha}\right). \tag{60}$$

Therefore,

$$\sup_{x,y}\left|\Delta^{\omega^*}E(x,y)\right|$$
$$= O\left\{\frac{\omega(|x-y|)^{\beta/\alpha}}{\omega^*(|x-y|)}\right.$$
$$\times\left(\log(n+1)\left[\omega\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right. \quad (61)$$
$$\left.\left.+\left[\left(\frac{1}{n+1}\right)\sum_{k=0}^{n}\omega\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right)\right\},$$

$$\|E_n(x)\|_c = O\left(\log(n+1)\,\omega\left(\frac{\pi}{n+1}\right)\right)$$
$$+ O\left(\left(\frac{1}{n+1}\right)\sum_{k=0}^{n}\omega\left(\frac{\pi}{n+1}\right)\right). \quad (62)$$

Combining (61) and (62), we get

$$\left\|\widetilde{t}_n^{NE}(f)-\widetilde{f}(x)\right\|_{\omega^*}$$
$$= O\left\{\frac{\omega(|x-y|)^{\beta/\alpha}}{\omega^*(|x-y|)}\right.$$
$$\times\left(\log(n+1)\left[\omega\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right. \quad (63)$$
$$\left.\left.+\left[\left(\frac{1}{n+1}\right)\sum_{k=0}^{n}\omega\left(\frac{\pi}{n+1}\right)\right]^{1-\beta/\alpha}\right)\right\}.$$

This completes the proof of Theorem 5.

6. Applications

The theory of approximation is a very extensive field and the study of theory of trigonometric approximation is of great mathematical interest and of great practical importance. As mentioned in [21], the L_p space in general and L_2 and L_∞ in particular play an important role in the theory of signals and filters. From the point of view of the applications, sharper estimates of infinite matrices [25] are useful to get bounds for the lattice norms (which occur in solid state physics) of matrix valued functions and enable to investigate perturbations of matrix valued functions and compare them. The following corollaries may be derived from Theorem 5.

If $\omega(|x-y|) \le B|x-y|^{\alpha}$, $\omega^*(|x-y|) \le K|x-y|^{\beta}$, $0 \le \beta < \alpha \le 1$ and set

$$H(u) = \begin{cases} u^{\alpha-1}, & 0<\alpha<1 \\ \log\left(\dfrac{1}{u}\right), & \alpha = 1, \end{cases} \quad (64)$$

then we get Corollary 11.

Corollary 11. *If $\widetilde{f} \in H_\alpha$, $0<\beta\le\alpha\le 1$, then*

$$\left\|\widetilde{t}_n^{NE}(f)-\widetilde{f}(x)\right\|_{\beta}$$
$$= \begin{cases} O\left((\log(n+1))^{\beta/\alpha}(n+1)^{\beta-\alpha}\right), & 0<\alpha<1 \\ O\left(\log(n+1)(n+1)^{\beta-1}\right), & \alpha = 1. \end{cases} \quad (65)$$

If we put $\beta = 0$ in Corollary 11, then we get Corollary 12.

Corollary 12. *If $\widetilde{f} \in Lip\,\alpha$, $0<\alpha\le 1$, then*

$$\left\|\widetilde{t}_n^{NE}(f)-\widetilde{f}(x)\right\|_c = \begin{cases} O\left((n+1)^{-\alpha}\right), & 0<\alpha<1 \\ O\left(\dfrac{\log(n+1)}{(n+1)}\right), & \alpha = 1. \end{cases} \quad (66)$$

7. Example

In the example, we see how the sequence of averages (i.e., $\sigma_n^1(x)$-means or $(C,1)$ mean) and Nörlund mean N_p of partial sums of a Fourier series is better behaved than the sequence of partial sums $s_n(x)$ itself.

Let

$$f(x) = \begin{cases} -1, & -\pi\le x<0, \\ 1, & 0\le x<\pi, \end{cases} \quad (67)$$

with $f(x+2\pi) = f(x)$ for all real x. Fourier series of $f(x)$ is given by

$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx, \quad -\pi\le x\le\pi. \quad (68)$$

Then nth partial sum $s_n(x)$ of Fourier series (68) and nth Cesàro sum for $\delta = 1$, that is, $\sigma_n^1(x)$ for the series (68), are given by

$$s_n(x) = \frac{4}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\cdots+\frac{1}{n}\sin nx\right),$$
$$\sigma_n^1(x) = \frac{2}{\pi}\sum_{k=1}^{n}\left(1-\frac{k}{n}\right)\left(\frac{1-(-1)^k}{k}\right)\sin kx. \quad (69)$$

From Theorem 20 of Hardy's "Divergent Series," if a Nörlund method N_p has increasing weights $\{p_n\}$, then it is stronger than $(C,1)$.

Now, take N_p to be the Nörlund matrix generated by $p_n = n+1$, then Nörlund means N_p is given by

$$t_n^N(f;x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\,s_k(f;x). \quad (70)$$

In this graph, we observe that $\sigma_n^1(x), t_n^N(f;x)$ converges to $f(x)$ faster than $s_n(x)$ in the interval $[-\pi,\pi]$. We further note that near the points of discontinuities, that is, $-\pi, 0$ and π, the graph of s_5 and s_{10} shows peaks that move closer to the line passing through points of discontinuity as n increases (Gibbs phenomenon), but in the graph of $\sigma_n^1(x), t_n^N(f;x)$,

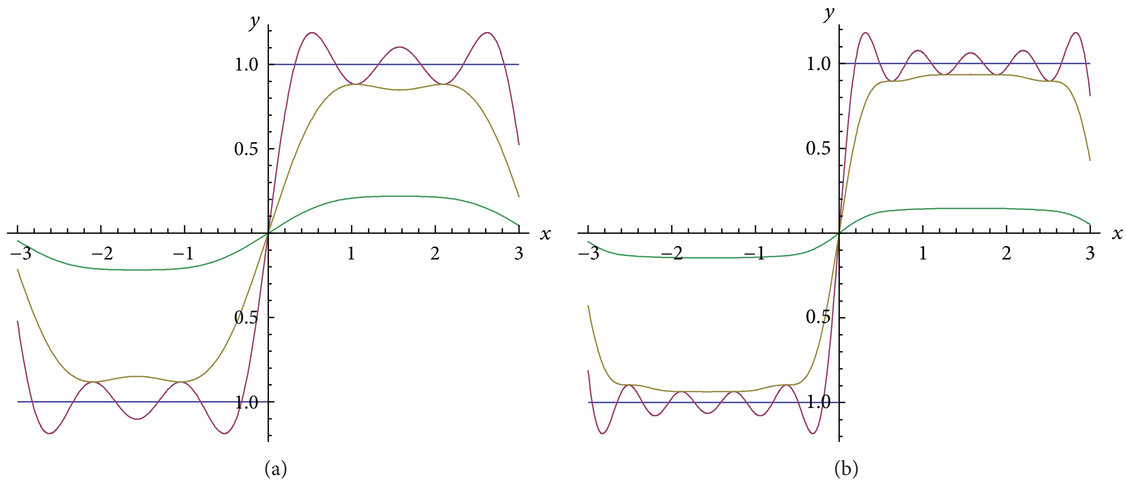


FIGURE 1: Graph of $f(x)$ (blue), $s_n(x)$ (pink), $\sigma_n^1(x)$ (yellow), $t_n^N(f;x)$ (green), $n = 5$ and 10.

$n = 5, 10$ the peaks become flatter (Figure 1). The Gibbs phenomenon is an overshoot a peculiarity of the Fourier series and other eigen function series at a simple discontinuity; that is, the convergence of Fourier series is very slow near the point of discontinuity. Thus, the product summability means of the Fourier series of $f(x)$ overshoot the Gibbs Phenomenon and show the smoothing effect of the method. Thus, $\sigma_n^1(x), t_n^N(f;x)$ is the better approximant than $s_n(x)$ and N_p method is stronger than $(C,1)$ method.

8. Conclusion

Several results concerning the degree of approximation of periodic signals (functions) by product summability means of Fourier series and conjugate Fourier series in generalized Hölder metric and Hölder metric have been reviewed. Using graphical representation $\sigma_n^1(x), t_n^N(f;x)$ is a better approximant to $s_n(x)$, but till now nothing has been done to show this. Some interesting application of the operator $(N_p \cdot E^1)$ used in this paper is pointed out in Remark 6. Also, the result of our theorem is more general rather than the results of any other previous proved theorems, which will enrich the literate of summability theory of infinite series.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to express their deep gratitude to the anonymous learned referee(s) and the editor for their valuable suggestions and constructive comments, which resulted in the subsequent improvement of this paper. Special thanks are due to Professor Ricardo Estrada, Academic Editor of IJMMS, and Professor Mohammed Mahmoud, Editorial office, Hindawi Publishing Corporation, for kind cooperation, smooth behavior during communication, and their efforts to send the reports of the paper timely. The authors are also grateful to all the editorial board members and reviewers of Hindawi prestigious journal, that is, International Journal of Mathematics and Mathematical Sciences. The second author Kejal Khatri acknowledges the Ministry of Human Resource Development, New Delhi, India, for providing the financial support and to carry out her research work under the guidance of Dr. Vishnu Narayan Mishra at S.V.N.I.T., Surat (Gujarat), India. The first author Vishnu Narayan Mishra acknowledges Cumulative Professional Development Allowance (CPDA), SVNIT, Surat (Gujarat), India, for supporting this research paper. Both authors carried out the proof. All the authors conceived of the study and participated in its design and coordination. Both the authors read and approved the final manuscript.

References

[1] H. H. Khan, "On the degree of approximation of a functions belonging to the class Lip(α, p)," *Indian Journal of Pure and Applied Mathematics*, vol. 5, pp. 132–136, 1974.

[2] H. H. Khan, *Approximation of classes of functions [Ph.D. thesis]*, AMU, Aligarh, India, 1974.

[3] V. N. Mishra and L. N. Mishra, "Trigonometric approximation in $L_p(p \geq 1)$ spaces," *International Journal of Contemporary Mathematical Sciences*, vol. 7, pp. 909–918, 2012.

[4] V. N. Mishra, K. Khatri, and L. N. Mishra, "Product $(N, p_n)(C, 1)$ summability of a sequence of Fourier coefficients," *Mathematical Sciences*, vol. 6, p. 38, 2012.

[5] V. N. Mishra, K. Khatri, and L. N. Mishra, "Product summability transform of conjugate series of Fourier series," *International Journal of Mathematics and Mathematical Sciences*, vol. 2012, Article ID 298923, 13 pages, 2012.

[6] V. N. Mishra, H. H. Khan, K. Khatri, and L. N. Mishra, "On approximation of conjugate of signals (Functions) belonging to the generalized weighted $W(L_r, \xi(t))$, $(r \geq 1)$-class by product summability means of conjugate series of fourier

series," *International Journal of Mathematical Analysis*, vol. 6, no. 35, pp. 1703–1715, 2012.

[7] V. N. Mishra, "On the degree of approximation of signals (Functions) belonging to the weighted $W(L_p, \xi(t))$, $(p \geq 1)$-class by almost matrix summability method of its conjugate Fourier series," *International Journal of Applied Mathematics and Mechanics*, vol. 5, no. 7, pp. 16–27, 2009.

[8] V. N. Mishra, "On the degree of approximation of signals (Functions) belonging to Generalized Weighted $W(L_p, \xi(t))$, $(p \geq 1)$-class by product summability method," *Journal of International Academy of Physical Sciences*, vol. 14, no. 4, pp. 413–423, 2010.

[9] J. T. Chen and Y. S. Jeng, "Dual series representation and its applications to a string subjected to support motions," *Advances in Engineering Software*, vol. 27, no. 3, pp. 227–238, 1996.

[10] J. T. Chen and H.-K. Hong, "Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series," *Applied Mechanics Reviews*, vol. 52, no. 1, pp. 17–32, 1999.

[11] J. T. Chen, H.-K. Hong, C. S. Yeh, and S. W. Chyuan, "Integral representations and regularizations for a divergent series solution of a beam subjected to support motions," *Earthquake Engineering and Structural Dynamics*, vol. 25, no. 9, pp. 909–925, 1996.

[12] M. Mursaleen and S. A. Mohiuddine, *Convergence Methods For Double Sequences and Applications*, Springer, 2014.

[13] G. Alexits, *Convergence Problems of Orthogonal Series*, Pergamon Elmsford, New York, NY, USA, 1961.

[14] P. Chandra, "On the degree of approximation of a class of functions by means of Fourier series," *Acta Mathematica Hungarica*, vol. 52, no. 3-4, pp. 199–205, 1988.

[15] P. Chandra, "A note on the degree of approximation of continuous functions," *Acta Mathematica Hungarica*, vol. 62, no. 1-2, pp. 21–23, 1993.

[16] R. N. Mohapatra and P. Chandra, "Hölder continuous functions and their Euler, Borel and Taylor means," *Math. Chronicle*, vol. 11, pp. 81–96, 1982.

[17] R. N. Mohapatra and P. Chandra, "Degree of approximation of functions in the H?lder metric," *Acta Mathematica Hungarica*, vol. 41, no. 1-2, pp. 67–76, 1983.

[18] B. Szal, "On the rate of strong summability by matrix means in the generalized Hölder metric," *Journal of Inequalities in Pure and Applied Mathematics*, vol. 9, no. 1, article 28, 2008.

[19] T. Singh and P. Mahajan, "Error bound of periodic signals in the Hölder metric," *International Journal of Mathematics and Mathematical Science*, vol. 2008, Article ID 495075, 9 pages, 2008.

[20] S. Lal and K. N. S. Yadav, "On degree of approximation of function belonging to the Lipschitz class by $(C, 1)(E, 1)$means of its Fourier series," *Bulletin of Calcutta Mathematical Society*, vol. 93, pp. 191–196, 2001.

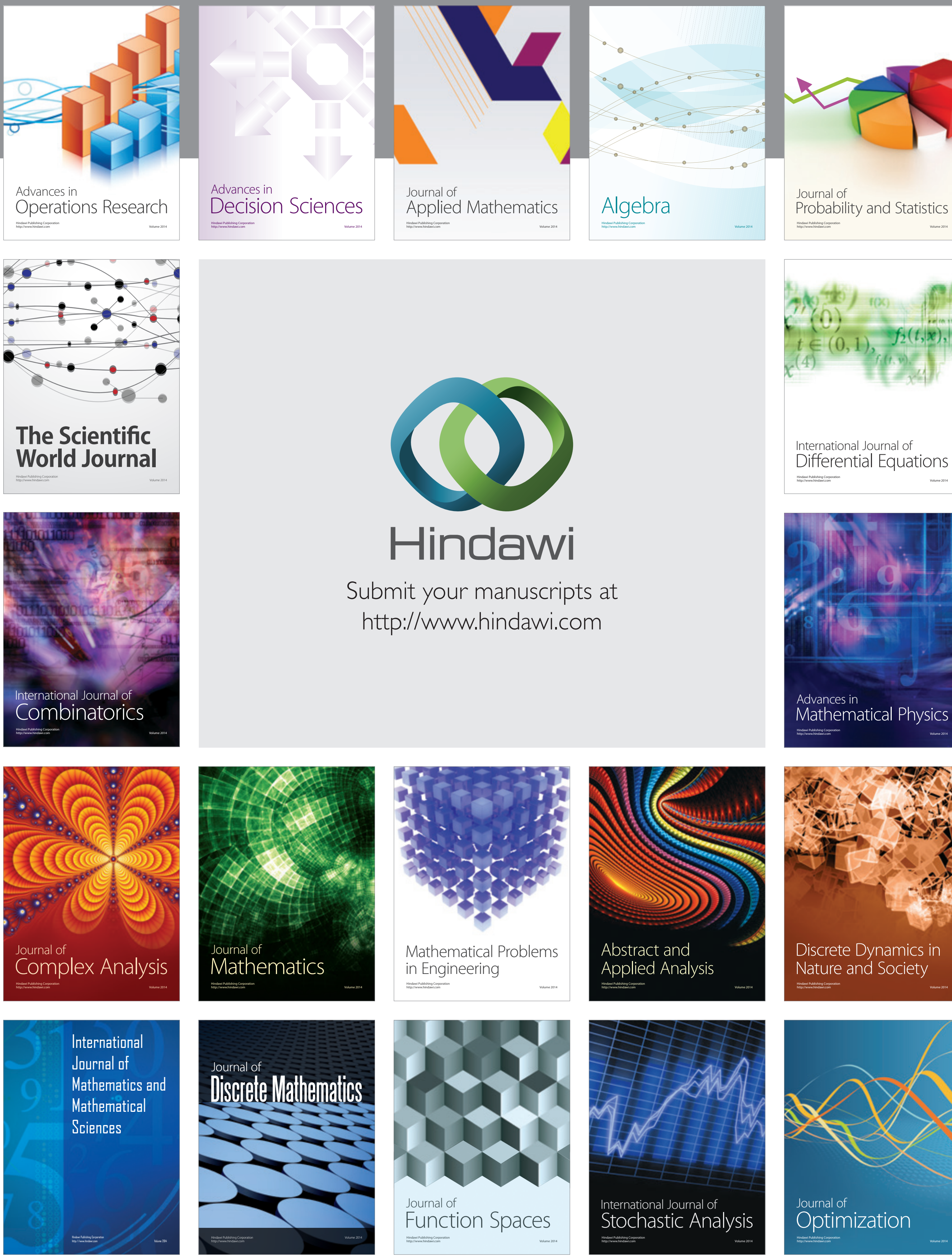
[21] E. Z. Psarakis and G. V. Moustakides, "An L_2-based method for the design of 1-D zero phase FIR digital filters," *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, vol. 44, no. 7, pp. 591–601, 1997.

[22] G. Das, T. Ghosh, and B. K. Ray, "Degree of approximation of functions in the Hölder metric by (e, c) Means," *Proceedings of the Indian Academy of Science*, vol. 105, pp. 315–327, 1995.

[23] A. Zygmund, *Trigonometric Series*, Cambridge University Press, London, UK, 3rd edition, 2002.

[24] G. Bachman, L. Narici, and E. Beckenstien, *Fourier and Wavelet Analysis*, Springer, New York, NY, USA, 2000.

[25] M. I. Gil', "Estimates for entries of matrix valued functions of infinite matrices," *Mathematical Physics Analysis and Geometry*, vol. 11, no. 2, pp. 175–186, 2008.


Advances in
Operations Research
Advances in
Decision Sciences
Journal of
Applied Mathematics
Algebra
Journal of
Probability and Statistics
The Scientific
World Journal
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of
Differential Equations
International Journal of
Combinatorics
Advances in
Mathematical Physics
Journal of
Complex Analysis
Journal of
Mathematics
Mathematical Problems
in Engineering
Abstract and
Applied Analysis
Discrete Dynamics in
Nature and Society
International
Journal of
Mathematics and
Mathematical
Sciences
Journal of
Discrete Mathematics
Journal of
Function Spaces
International Journal of
Stochastic Analysis
Journal of
Optimization