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Research Article

Some New Riemann-Liouville Fractional Integral Inequalities

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In this paper, some new fractional integral inequalities are established.

1. Introduction

In [1] (see also [2]), the Grüss inequality is defined as the integral inequality that establishes a connection between the integral of the product of two functions and the product of the integrals. The inequality is as follows.

If f and g are two continuous functions on $[a,b]$ satisfying $m \le f(t) \le M$ and $p \le g(t) \le P$ for all $t \in [a,b]$, $m, M, p, P \in \mathbb{R}$, then

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{(b-a)^2}\int_a^b f(t)\,dt\int_a^b g(t)\,dt\right| \le \frac{1}{4}(M-m)(P-p). \tag{1}$$

The literature on Grüss type inequalities is now vast, and many extensions of the classical inequality were intensively studied by many authors. In the past several years, by using the Riemann-Liouville fractional integrals, the fractional integral inequalities and applications have been addressed extensively by several researchers. For example, we refer the reader to [3–9] and the references cited therein. Dahmani et al. [10] gave the following fractional integral inequalities by using the Riemann-Liouville fractional integrals. Let f and g be two integrable functions on $[0,\infty)$ satisfying the following conditions:

$$m \le f(t) \le M, \qquad p \le g(t) \le P$$
$$m, M, p, P \in \mathbb{R}, \quad t \in [0,\infty). \tag{2}$$

For all $t > 0$, $\alpha > 0$, $\beta > 0$, then

$$\begin{aligned}
&\left|\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha(fg)(t) - J^\alpha f(t)\,J^\alpha g(t)\right| \\
&\quad\le \left(\frac{t^\alpha}{\Gamma(\alpha+1)}\right)^2 (M-m)(P-p), \\
&\left(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\beta(fg)(t) + \frac{t^\beta}{\Gamma(\beta+1)}J^\alpha(fg)(t)\right. \\
&\quad\left. - J^\alpha f(t)\,J^\beta g(t) - J^\beta f(t)\,J^\alpha g(t)\right)^2 \\
&\quad\le \left\{\left(M\frac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(t)\right)\left(J^\beta f(t) - m\frac{t^\beta}{\Gamma(\beta+1)}\right)\right. \\
&\qquad + \left(J^\alpha f(t) - m\frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\
&\qquad \times\left.\left(M\frac{t^\beta}{\Gamma(\beta+1)} - J^\beta f(t)\right)\right\} \\
&\quad\times\left\{\left(P\frac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha g(t)\right)\left(J^\beta g(t) - p\frac{t^\beta}{\Gamma(\beta+1)}\right)\right. \\
&\qquad + \left(J^\alpha g(t) - p\frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\
&\qquad \times\left.\left(P\frac{t^\beta}{\Gamma(\beta+1)} - J^\beta g(t)\right)\right\}.
\end{aligned} \tag{3}$$

In this paper, we use the Riemann-Liouville fractional integrals to establish some new fractional integral inequalities of Grüss type. We replace the constants appeared as bounds of the functions f and g, by four integrable functions. From our results, the above inequalities of [10] and the classical Grüss inequalities can be deduced as some special cases.

In Section 2 we briefly review the necessary definitions. Our results are given in Section 3. The proof technique is close to that presented in [10]. But the obtained results are new and also can be applied to unbounded functions as shown in examples.

2. Preliminaries

Definition 1. The Riemann-Liouville fractional integral of order $\alpha \geq 0$ of a function $g \in L^1((0,\infty),\mathbb{R})$ is defined by

$$J^\alpha f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds, \qquad J^0 f(t) = f(t), \tag{4}$$

where Γ is the gamma function.

For the convenience of establishing our results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \quad \alpha \geq 0,\ \beta \geq 0, \tag{5}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{6}$$

From Definition 1, if $f(t) = t^\gamma$, then we have

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}, \quad \alpha > 0,\ \gamma > -1,\ t > 0. \tag{7}$$

3. Main Results

Theorem 2. *Let f be an integrable function on $[0,\infty)$. Assume that*

(H_1) *there exist two integrable functions φ_1, φ_2 on $[0,\infty)$ such that*

$$\varphi_1(t) \leq f(t) \leq \varphi_2(t) \quad \forall t \in [0,\infty). \tag{8}$$

Then, for $t > 0$, $\alpha, \beta > 0$, one has

$$J^\beta \varphi_1(t) J^\alpha f(t) + J^\alpha \varphi_2(t) J^\beta f(t) \geq J^\alpha \varphi_2(t) J^\beta \varphi_1(t) + J^\alpha f(t) J^\beta f(t). \tag{9}$$

Proof. From (H_1), for all $\tau \geq 0$, $\rho \geq 0$, we have

$$(\varphi_2(\tau) - f(\tau))(f(\rho) - \varphi_1(\rho)) \geq 0. \tag{10}$$

Therefore

$$\varphi_2(\tau) f(\rho) + \varphi_1(\rho) f(\tau) \geq \varphi_1(\rho)\varphi_2(\tau) + f(\tau) f(\rho). \tag{11}$$

Multiplying both sides of (11) by $(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $\tau \in (0,t)$, we get

$$f(\rho)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(\tau) + \varphi_1(\rho)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau) \geq \varphi_1(\rho)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(\tau) + f(\rho)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau). \tag{12}$$

Integrating both sides of (12) with respect to τ on $(0,t)$, we obtain

$$\begin{aligned} &f(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(\tau)\,d\tau + \varphi_1(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\,d\tau \\ &\geq \varphi_1(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(\tau)\,d\tau + f(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} f(\tau)\,d\tau, \end{aligned} \tag{13}$$

which yields

$$f(\rho) J^\alpha \varphi_2(t) + \varphi_1(\rho) J^\alpha f(t) \geq \varphi_1(\rho) J^\alpha \varphi_2(t) + f(\rho) J^\alpha f(t). \tag{14}$$

Multiplying both sides of (14) by $(t-\rho)^{\beta-1}/\Gamma(\beta)$, $\rho \in (0,t)$, we have

$$J^\alpha \varphi_2(t)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} f(\rho) + J^\alpha f(t)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\varphi_1(\rho) \geq J^\alpha \varphi_2(t)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\varphi_1(\rho) + J^\alpha f(t)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} f(\rho). \tag{15}$$

Integrating both sides of (15) with respect to ρ on $(0,t)$, we get

$$\begin{aligned} &J^\alpha \varphi_2(t)\int_0^t \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} f(\rho)\,d\rho + J^\alpha f(t)\int_0^t \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\varphi_1(\rho)\,d\rho \\ &\geq J^\alpha \varphi_2(t)\int_0^t \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\varphi_1(\rho)\,d\rho + J^\alpha f(t)\int_0^t \frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)} f(\rho)\,d\rho. \end{aligned} \tag{16}$$

Hence, we deduce inequality (9) as requested. This completes the proof. □

As a special case of Theorem 2, we obtain the following result.

Corollary 3. *Let f be an integrable function on $[0,\infty)$ satisfying $m \le f(t) \le M$, for all $t \in [0,\infty)$ and $m, M \in \mathbb{R}$. Then, for $t > 0$ and $\alpha, \beta > 0$, one has*

$$m\frac{t^{\beta}}{\Gamma(\beta+1)}J^{\alpha}f(t) + M\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}f(t) \ge mM\frac{t^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)} + J^{\alpha}f(t)\,J^{\beta}f(t). \tag{17}$$

Example 4. Let f be a function satisfying $t \le f(t) \le t+1$ for $t \in [0,\infty)$. Then, for $t > 0$ and $\alpha > 0$, we have

$$\left(\frac{2t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)J^{\alpha}f(t) \ge \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) + \left(J^{\alpha}f(t)\right)^2. \tag{18}$$

Theorem 5. *Let f and g be two integrable functions on $[0,\infty)$. Suppose that (H_1) holds and moreover one assumes that*

(H_2) there exist ψ_1 and ψ_2 integrable functions on $[0,\infty)$ such that

$$\psi_1(t) \le g(t) \le \psi_2(t), \quad \forall t \in [0,\infty). \tag{19}$$

Then, for $t > 0$, $\alpha, \beta > 0$, the following inequalities hold:

$$\begin{aligned}
&(a)\ J^{\beta}\psi_1(t)\,J^{\alpha}f(t) + J^{\alpha}\varphi_2(t)\,J^{\beta}g(t) \ge J^{\beta}\psi_1(t)\,J^{\alpha}\varphi_2(t) + J^{\alpha}f(t)\,J^{\beta}g(t),\\
&(b)\ J^{\beta}\varphi_1(t)\,J^{\alpha}g(t) + J^{\alpha}\psi_2(t)\,J^{\beta}f(t) \ge J^{\beta}\varphi_1(t)\,J^{\alpha}\psi_2(t) + J^{\beta}f(t)\,J^{\alpha}g(t),\\
&(c)\ J^{\alpha}\varphi_2(t)\,J^{\beta}\psi_2(t) + J^{\alpha}f(t)\,J^{\beta}g(t) \ge J^{\alpha}\varphi_2(t)\,J^{\beta}g(t) + J^{\beta}\psi_2(t)\,J^{\alpha}f(t),\\
&(d)\ J^{\alpha}\varphi_1(t)\,J^{\beta}\psi_1(t) + J^{\alpha}f(t)\,J^{\beta}g(t) \ge J^{\alpha}\varphi_1(t)\,J^{\beta}g(t) + J^{\beta}\psi_1(t)\,J^{\alpha}f(t).
\end{aligned} \tag{20}$$

Proof. To prove (a), from (H_1) and (H_2), we have for $t \in [0,\infty)$ that

$$\left(\varphi_2(\tau) - f(\tau)\right)\left(g(\rho) - \psi_1(\rho)\right) \ge 0. \tag{21}$$

Therefore

$$\varphi_2(\tau)\,g(\rho) + \psi_1(\rho)\,f(\tau) \ge \psi_1(\rho)\,\varphi_2(\tau) + f(\tau)\,g(\rho). \tag{22}$$

Multiplying both sides of (22) by $(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $\tau \in (0,t)$, we get

$$g(\rho)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(\tau) + \psi_1(\rho)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau) \ge \psi_1(\rho)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(\tau) + g(\rho)\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau). \tag{23}$$

Integrating both sides of (23) with respect to τ on $(0,t)$, we obtain

$$\begin{aligned}
&g(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(\tau)\,d\tau + \psi_1(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)\,d\tau\\
&\ge \psi_1(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}\varphi_2(\tau)\,d\tau + g(\rho)\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}f(\tau)\,d\tau.
\end{aligned} \tag{24}$$

Then we have

$$g(\rho)\,J^{\alpha}\varphi_2(t) + \psi_1(\rho)\,J^{\alpha}f(t) \ge \psi_1(\rho)\,J^{\alpha}\varphi_2(t) + g(\rho)\,J^{\alpha}f(t). \tag{25}$$

Multiplying both sides of (25) by $(t-\rho)^{\beta-1}/\Gamma(\beta)$, $\rho \in (0,t)$, we have

$$J^{\alpha}\varphi_2(t)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}g(\rho) + J^{\alpha}f(t)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\psi_1(\rho) \ge J^{\alpha}\varphi_2(t)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}\psi_1(\rho) + J^{\alpha}f(t)\frac{(t-\rho)^{\beta-1}}{\Gamma(\beta)}g(\rho). \tag{26}$$

Integrating both sides of (26) with respect to ρ on $(0,t)$, we get the desired inequality (a).

To prove (b)–(d), we use the following inequalities:

$$\begin{aligned}
&(b)\ \left(\psi_2(\tau) - g(\tau)\right)\left(f(\rho) - \varphi_1(\rho)\right) \ge 0,\\
&(c)\ \left(\varphi_2(\tau) - f(\tau)\right)\left(g(\rho) - \psi_2(\rho)\right) \le 0,\\
&(d)\ \left(\varphi_1(\tau) - f(\tau)\right)\left(g(\rho) - \psi_1(\rho)\right) \le 0.
\end{aligned} \tag{27}$$

□

As a special case of Theorem 5, we have the following corollary.

Corollary 6. *Let f and g be two integrable functions on $[0,\infty)$. Assume that*

(H_3) there exist real constants m, M, n, N such that

$$m \le f(t) \le M, \quad n \le g(t) \le N \quad \forall t \in [0,\infty). \tag{28}$$

Then, for $t > 0$, $\alpha, \beta > 0$, we have

$$\begin{aligned}
&(a_1)\ \frac{nt^{\beta}}{\Gamma(\beta+1)}J^{\alpha}f(t)+\frac{Mt^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}g(t)\\
&\qquad\geq \frac{nMt^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+J^{\alpha}f(t)J^{\beta}g(t),\\
&(b_1)\ \frac{mt^{\beta}}{\Gamma(\beta+1)}J^{\alpha}g(t)+\frac{Nt^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}f(t)\\
&\qquad\geq \frac{mNt^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+J^{\beta}f(t)J^{\alpha}g(t),\\
&(c_1)\ \frac{MNt^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+J^{\alpha}f(t)J^{\beta}g(t)\\
&\qquad\geq \frac{Mt^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}g(t)+\frac{Nt^{\beta}}{\Gamma(\beta+1)}J^{\alpha}f(t),\\
&(d_1)\ \frac{mnt^{\alpha+\beta}}{\Gamma(\alpha+1)\Gamma(\beta+1)}+J^{\alpha}f(t)J^{\beta}g(t)\\
&\qquad\geq \frac{mt^{\alpha}}{\Gamma(\alpha+1)}J^{\beta}g(t)+\frac{nt^{\beta}}{\Gamma(\beta+1)}J^{\alpha}f(t).
\end{aligned}\tag{29}$$

Lemma 7. *Let f be an integrable function on $[0,\infty)$ and let φ_1, φ_2 be two integrable functions on $[0,\infty)$. Assume that the condition (H_1) holds. Then, for $t > 0$, $\alpha > 0$, we have*

$$\begin{aligned}
&\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}f^2(t)-\left(J^{\alpha}f(t)\right)^2\\
&\quad=\left(J^{\alpha}\varphi_2(t)-J^{\alpha}f(t)\right)\left(J^{\alpha}f(t)-J^{\alpha}\varphi_1(t)\right)\\
&\qquad-\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\left(\left(\varphi_2(t)-f(t)\right)\left(f(t)-\varphi_1(t)\right)\right)\\
&\qquad+\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\varphi_1 f(t)-J^{\alpha}\varphi_1(t)J^{\alpha}f(t)\\
&\qquad+\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\varphi_2 f(t)-J^{\alpha}\varphi_2(t)J^{\alpha}f(t)\\
&\qquad+J^{\alpha}\varphi_1(t)J^{\alpha}\varphi_2(t)-\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\varphi_1\varphi_2(t).
\end{aligned}\tag{30}$$

Proof. For any $\tau > 0$ and $\rho > 0$, we have

$$\begin{aligned}
&\left(\varphi_2(\rho)-f(\rho)\right)\left(f(\tau)-\varphi_1(\tau)\right)\\
&\qquad+\left(\varphi_2(\tau)-f(\tau)\right)\left(f(\rho)-\varphi_1(\rho)\right)\\
&\qquad-\left(\varphi_2(\tau)-f(\tau)\right)\left(f(\tau)-\varphi_1(\tau)\right)\\
&\qquad-\left(\varphi_2(\rho)-f(\rho)\right)\left(f(\rho)-\varphi_1(\rho)\right)\\
&\quad=f^2(\tau)+f^2(\rho)-2f(\tau)f(\rho)\\
&\qquad+\varphi_2(\rho)f(\tau)+\varphi_1(\tau)f(\rho)-\varphi_1(\tau)\varphi_2(\rho)\\
&\qquad+\varphi_2(\tau)f(\rho)+\varphi_1(\rho)f(\tau)-\varphi_1(\rho)\varphi_2(\tau)\\
&\qquad-\varphi_2(\tau)f(\tau)+\varphi_1(\tau)\varphi_2(\tau)-\varphi_1(\tau)f(\tau)\\
&\qquad-\varphi_2(\rho)f(\rho)+\varphi_1(\rho)\varphi_2(\rho)-\varphi_1(\rho)f(\rho).
\end{aligned}\tag{31}$$

Multiplying (31) by $(t-\tau)^{\alpha-1}/\Gamma(\alpha)$, $\tau \in (0,t)$, $t > 0$ and integrating the resulting identity with respect to τ, from 0 to t, we get

$$\begin{aligned}
&\left(\varphi_2(\rho)-f(\rho)\right)\left(J^{\alpha}f(t)-J^{\alpha}\varphi_1(t)\right)\\
&\qquad+\left(J^{\alpha}\varphi_2(t)-J^{\alpha}f(t)\right)\left(f(\rho)-\varphi_1(\rho)\right)\\
&\qquad-J^{\alpha}\left(\left(\varphi_2(t)-f(t)\right)\left(f(t)-\varphi_1(t)\right)\right)\\
&\qquad-\left(\varphi_2(\rho)-f(\rho)\right)\left(f(\rho)-\varphi_1(\rho)\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&\quad=J^{\alpha}f^2(t)+f^2(\rho)\frac{t^{\alpha}}{\Gamma(\alpha+1)}-2f(\rho)J^{\alpha}f(t)\\
&\qquad+\varphi_2(\rho)J^{\alpha}f(t)+f(\rho)J^{\alpha}\varphi_1(t)\\
&\qquad-\varphi_2(\rho)J^{\alpha}\varphi_1(t)+f(\rho)J^{\alpha}\varphi_2(t)\\
&\qquad+\varphi_1(\rho)J^{\alpha}f(t)-\varphi_1(\rho)J^{\alpha}\varphi_2(t)\\
&\qquad-J^{\alpha}\varphi_2 f(t)+J^{\alpha}\varphi_1\varphi_2(t)-J^{\alpha}\varphi_1 f(t)\\
&\qquad-\varphi_2(\rho)f(\rho)\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\varphi_1(\rho)\varphi_2(\rho)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&\qquad-\varphi_1(\rho)f(\rho)\frac{t^{\alpha}}{\Gamma(\alpha+1)}.
\end{aligned}\tag{32}$$

Multiplying (32) by $(t-\rho)^{\alpha-1}/\Gamma(\alpha)$, $\rho \in (0,t)$, $t > 0$ and integrating the resulting identity with respect to ρ, from 0 to t, we have

$$\begin{aligned}
&\left(J^{\alpha}\varphi_2(t)-J^{\alpha}f(t)\right)\left(J^{\alpha}f(t)-J^{\alpha}\varphi_1(t)\right)\\
&\qquad+\left(J^{\alpha}\varphi_2(t)-J^{\alpha}f(t)\right)\left(J^{\alpha}f(t)-J^{\alpha}\varphi_1(t)\right)\\
&\qquad-J^{\alpha}\left(\left(\varphi_2(t)-f(t)\right)\left(f(t)-\varphi_1(t)\right)\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&\qquad-J^{\alpha}\left(\left(\varphi_2(t)-f(t)\right)\left(f(t)-\varphi_1(t)\right)\right)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&\quad=\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}f^2(t)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}f^2(t)\\
&\qquad-2J^{\alpha}f(t)J^{\alpha}f(t)+J^{\alpha}\varphi_2(t)J^{\alpha}f(t)\\
&\qquad+J^{\alpha}\varphi_1(t)J^{\alpha}f(t)-J^{\alpha}\varphi_1(t)J^{\alpha}\varphi_2(t)\\
&\qquad+J^{\alpha}\varphi_2(t)J^{\alpha}f(t)+J^{\alpha}\varphi_1(t)J^{\alpha}f(t)\\
&\qquad-J^{\alpha}\varphi_1(t)J^{\alpha}\varphi_2(t)-\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\varphi_2 f(t)\\
&\qquad+\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\varphi_1\varphi_2(t)-\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\varphi_1 f(t)\\
&\qquad-\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\varphi_2 f(t)+\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\varphi_1\varphi_2(t)\\
&\qquad-\frac{t^{\alpha}}{\Gamma(\alpha+1)}J^{\alpha}\varphi_1 f(t),
\end{aligned}\tag{33}$$

which implies (30). □

If $\varphi_1(t) \equiv m$ and $\varphi_2(t) \equiv M$, $m, M \in \mathbb{R}$, for all $t \in [0,\infty)$, then inequality (30) reduces to the following corollary [10, Lemma 3.2].

Corollary 8. *Let f be an integrable function on $[0,\infty)$ satisfying $m \le f(t) \le M$, for all $t \in [0,\infty)$. Then, for all $t > 0$, $\alpha > 0$, one has*

$$\begin{aligned}\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(t) - \left(J^\alpha f(t)\right)^2 &= \left(M\frac{t^\alpha}{\Gamma(\alpha+1)} - J^\alpha f(t)\right)\left(J^\alpha f(t) - m\frac{t^\alpha}{\Gamma(\alpha+1)}\right) \\ &\quad - \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha\left((M - f(t))(f(t) - m)\right).\end{aligned} \tag{34}$$

Theorem 9. *Let f and g be two integrable functions on $[0,\infty)$ and let φ_1, φ_2, ψ_1, and ψ_2 be four integrable functions on $[0,\infty)$ satisfying the conditions (H_1) and (H_2) on $[0,\infty)$. Then, for all $t > 0$, $\alpha > 0$, one has*

$$\begin{aligned}&\left|\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t) - J^\alpha f(t)\,J^\alpha g(t)\right| \\ &\qquad \le \sqrt{T(f,\varphi_1,\varphi_2)\,T(g,\psi_1,\psi_2)},\end{aligned} \tag{35}$$

where $T(u,v,w)$ is defined by

$$\begin{aligned}T(u,v,w) &= \left(J^\alpha w(t) - J^\alpha u(t)\right)\left(J^\alpha u(t) - J^\alpha v(t)\right) \\ &\quad + \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha vu(t) - J^\alpha v(t)\,J^\alpha u(t) \\ &\quad + \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha wu(t) - J^\alpha w(t)\,J^\alpha u(t) \\ &\quad + J^\alpha v(t)\,J^\alpha w(t) - \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha vw(t).\end{aligned} \tag{36}$$

Proof. Let f and g be two integrable functions defined on $[0,\infty)$ satisfying (H_1) and (H_2). Define

$$\begin{aligned}H(\tau,\rho) &:= \left(f(\tau) - f(\rho)\right)\left(g(\tau) - g(\rho)\right), \\ &\qquad \tau,\rho \in (0,t),\ t > 0.\end{aligned} \tag{37}$$

Multiplying both sides of (37) by $(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}/\Gamma^2(\alpha)$, $\tau,\rho \in (0,t)$ and integrating the resulting identity with respect to τ and ρ, from 0 to t, we can state that

$$\begin{aligned}&\frac{1}{2\Gamma^2(\alpha)}\iint_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}H(\tau,\rho)\,d\tau\,d\rho \\ &\qquad = \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t) - J^\alpha f(t)\,J^\alpha g(t).\end{aligned} \tag{38}$$

Applying the Cauchy-Schwarz inequality to (38), we have

$$\begin{aligned}&\left(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t) - J^\alpha f(t)\,J^\alpha g(t)\right)^2 \\ &\qquad \le \left(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(t) - \left(J^\alpha f(t)\right)^2\right) \\ &\qquad\quad \times \left(\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha g^2(t) - \left(J^\alpha g(t)\right)^2\right).\end{aligned} \tag{39}$$

Since $(\varphi_2(t) - f(t))(f(t) - \varphi_1(t)) \ge 0$ and $(\psi_2(t) - g(t))(g(t) - \psi_1(t)) \ge 0$, for $t \in [0,\infty)$, we have

$$\begin{aligned}&\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha\left((\varphi_2(t) - f(t))(f(t) - \varphi_1(t))\right) \ge 0, \\ &\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha\left((\psi_2(t) - g(t))(g(t) - \psi_1(t))\right) \ge 0.\end{aligned} \tag{40}$$

Thus, from Lemma 7, we get

$$\begin{aligned}&\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha f^2(t) - \left(J^\alpha f(t)\right)^2 \\ &\quad \le \left(J^\alpha\varphi_2(t) - J^\alpha f(t)\right)\left(J^\alpha f(t) - J^\alpha\varphi_1(t)\right) \\ &\qquad + \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha\varphi_1 f(t) - J^\alpha\varphi_1(t)\,J^\alpha f(t) \\ &\qquad + \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha\varphi_2 f(t) - J^\alpha\varphi_2(t)\,J^\alpha f(t) \\ &\qquad + J^\alpha\varphi_1(t)\,J^\alpha\varphi_2(t) - \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha\varphi_1\varphi_2(t) \\ &\quad = T(f,\varphi_1,\varphi_2),\end{aligned} \tag{41}$$

$$\begin{aligned}&\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha g^2(t) - \left(J^\alpha g(t)\right)^2 \\ &\quad \le \left(J^\alpha\psi_2(t) - J^\alpha g(t)\right)\left(J^\alpha g(t) - J^\alpha\psi_1(t)\right) \\ &\qquad + \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha\psi_1 g(t) - J^\alpha\psi_1(t)\,J^\alpha g(t) \\ &\qquad + \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha\psi_2 g(t) - J^\alpha\psi_2(t)\,J^\alpha g(t) \\ &\qquad + J^\alpha\psi_1(t)\,J^\alpha\psi_2(t) - \frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha\psi_1\psi_2(t) \\ &\quad = T(g,\psi_1,\psi_2).\end{aligned} \tag{42}$$

From (39), (41), and (42), we obtain (35). □

Remark 10. If $T(f,\varphi_1,\varphi_2) = T(f,m,M)$ and $T(g,\psi_1,\psi_2) = T(g,p,P)$, $m, M, p, P \in \mathbb{R}$, then inequality (35) reduces to

$$\begin{aligned}&\left|\frac{t^\alpha}{\Gamma(\alpha+1)}J^\alpha fg(t) - J^\alpha f(t)\,J^\alpha g(t)\right| \\ &\qquad \le \left(\frac{t^\alpha}{2\Gamma(\alpha+1)}\right)^2(M-m)(P-p).\end{aligned} \tag{43}$$

See [10, Theorem 3.1].

Example 11. Let f and g be two functions satisfying $t \leq f(t) \leq t+1$ and $t-1 \leq g(t) \leq t$ for $t \in [0,\infty)$. Then, for $t > 0$ and $\alpha > 0$, we have

$$\left|\frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\alpha} fg(t) - J^{\alpha} f(t) J^{\alpha} g(t)\right| \leq \sqrt{T(f,t,t+1)\, T(g,t-1,t)}, \tag{44}$$

where

$$\begin{aligned} T(f,t,t+1) &= \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - J^{\alpha} f(t)\right) \\ &\quad \times \left(J^{\alpha} f(t) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) \\ &\quad + \frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\alpha}(tf)(t) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} J^{\alpha} f(t) \\ &\quad + \frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\alpha}((t+1) f)(t) \\ &\quad - \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) J^{\alpha} f(t) \\ &\quad + \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) \\ &\quad - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right), \end{aligned} \tag{45}$$

$$\begin{aligned} T(g,t-1,t) &= \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - J^{\alpha} g(t)\right) \\ &\quad \times \left(J^{\alpha} g(t) - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) \\ &\quad + \frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\alpha}((t-1) g)(t) \\ &\quad - \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right) J^{\alpha} g(t) \\ &\quad + \frac{t^{\alpha}}{\Gamma(\alpha+1)} J^{\alpha}(tg)(t) - \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) J^{\alpha} g(t) \\ &\quad + \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) \\ &\quad - \frac{t^{\alpha}}{\Gamma(\alpha+1)}\left(\frac{2t^{\alpha+2}}{\Gamma(\alpha+3)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right). \end{aligned} \tag{46}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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