Research Article

Certain Bound for q -Starlike and q -Convex Functions with respect to Symmetric Points

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The aim of this paper is to establish the coefficient estimates for the subclasses of q -starlike and q -convex functions with respect to symmetric points involving q -difference operator. Also certain applications based on these results for subclasses of univalent functions defined by convolution are given.

1. Introduction

The q -difference calculus or quantum calculus was initiated at the beginning of 19th century that was initially developed by Jackson [1, 2]. Basic definitions and properties of q -difference calculus can be found in the book mentioned in [3]. The fractional q -difference calculus had its origin in the works by Al-Salam [4] and Agarwal [5]. Recently, the area of q -calculus has attracted the serious attention of researchers. The great interest is due to its application in various branches of mathematics and physics. Mohammed and Darus [6] studied approximation and geometric properties of these q -operators for some subclasses of analytic functions in compact disk.

Let $\mathcal{A}$ denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all univalent functions in $\mathbb{U}$.

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, then we say that the function $f(z)$ is subordinate to $g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$\begin{aligned} w(0) &= 0, \\ |w(z)| &< 1 \quad (z \in \mathbb{U}), \end{aligned} \quad (2)$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}). \quad (3)$$

We denote this subordination by

$$\begin{aligned} f &< g \\ \text{or } f(z) &< g(z) \quad (z \in \mathbb{U}). \end{aligned} \quad (4)$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$ (see [7, 8]).

A q -analogue of the class of starlike functions was first introduced in [9] by means of the q -difference operator $D_q f(z)$ acting on functions $f \in \mathcal{A}$ given by (1) and $0 < q < 1$, and the q -derivative of a function $f(z)$ is defined by (see [1, 2])

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \neq 0), \quad (5)$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q(D_q f(z))$. From (5), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \quad (6)$$

where

$$[k]_q = \frac{1 - q^k}{1 - q}. \quad (7)$$

As $q \rightarrow 1^-$, $[k]_q \rightarrow k$. For a function $h(z) = z^k$, we observe that

$$\begin{aligned} D_q(h(z)) &= D_q(z^k) = \frac{1 - q^k}{1 - q} z^{k-1} \\ &= [k]_q z^{k-1}, \end{aligned} \quad (8)$$

$$\begin{aligned} \lim_{q \rightarrow 1^-} (D_q(h(z))) &= \lim_{q \rightarrow 1^-} ([k]_q z^{k-1}) = k z^{k-1} \\ &= h'(z), \end{aligned}$$

where h' is the ordinary derivative.

As a right inverse, Jackson [2] introduced the q -integral

$$\int_0^z f(t) d_q t = z(1 - q) \sum_{k=0}^{\infty} q^k f(zq^k), \quad (9)$$

provided that the series converges. For a function $h(z) = z^k$, we have

$$\begin{aligned} \int_0^z h(t) d_q t &= \int_0^z t^k d_q t = \frac{z^{k+1}}{[k+1]_q} \quad (k \neq -1) \\ \lim_{q \rightarrow 1^-} \int_0^z h(t) d_q t &= \lim_{q \rightarrow 1^-} \frac{z^{k+1}}{[k+1]_q} = \frac{z^{k+1}}{k+1} \\ &= \int_0^z h(t) dt, \end{aligned} \quad (10)$$

where $\int_0^z h(t) dt$ is the ordinary integral. Note that the q -difference operator plays an important role in the theory of hypergeometric series and quantum physics (see, e.g., [10–14]).

Ma and Minda [15] unified various subclasses of starlike and convex functions for which either quantity $zf'(z)/f(z)$ or quantity $1 + zf''(z)/f'(z)$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function ϕ with positive real part in the unit disc $\mathbb{U}$, with $\phi(0) = 1$, $\phi'(0) > 0$, and ϕ maps $\mathbb{U}$ onto a region starlike, with respect to the real axis.

The class of Ma-Minda starlike functions consists of functions $f(z) \in \mathcal{A}$ satisfying the subordination $zf'(z)/f(z) < \phi(z)$. Similarly, the class of Ma-Minda convex functions consists of functions $f \in \mathcal{A}$ satisfying the subordination $1 + zf''(z)/f'(z) < \phi(z)$.

Sakaguchi [16] introduced and studied class S_s^* of starlike functions with respect to symmetric points. Class $S_{q,s}^*(\phi)$ defined below is the generalization of class S_s^* .

Definition 1 (see [17]). Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$ be a univalent starlike function with respect to 1 which maps the unit disk $\mathbb{U}$ onto a region in the right half plane which is

symmetric with respect to the real axis and let $B_1 > 0$. The function $f \in \mathcal{A}$ is in classes $S_s^*(\phi)$ and $C_s(\phi)$:

$$\begin{aligned} \frac{2zf'(z)}{f(z) - f(-z)} &< \phi(z), \\ \frac{2(zf'(z))'}{f'(z) + f'(-z)} &< \phi(z). \end{aligned} \quad (11)$$

Definition 2. A function $f \in \mathcal{A}$ is said to belong to classes $S_{q,s}^*(\phi)$ and $C_{q,s}(\phi)$ if

$$\begin{aligned} \frac{2(zD_q f(z))}{f(z) - f(-z)} &< \phi(z), \\ \frac{2(D_q(zD_q f(z)))}{D_q f(z) + D_q f(-z)} &< \phi(z), \end{aligned} \quad (12)$$

respectively.

As $q \rightarrow 1^-$, we denote subclasses $S_{q,s}^*(\phi)$ and $C_{q,s}(\phi)$ by $S_s^*(\phi)$ and $C_s(\phi)$ which are the class of starlike and convex functions with respect to symmetric points, respectively (see [18]).

The main objective of the present paper is to derive the Fekete-Szegő inequality for functions in classes $S_{q,s}^*(\phi)$ and $C_{q,s}(\phi)$. Also, by using convolution product, we give the applications of our results to the defined functions and in particular we consider classes $S_{q,s}^\lambda(\phi)$ and $C_{q,s}^\lambda(\phi)$ defined by fractional derivatives.

In order to prove the main result, we need the following lemma.

Lemma 3 (see [15]). If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is an analytic function with positive real part in $\mathbb{U}$, then

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2 & \text{if } \nu \leq 0 \\ 2 & \text{if } 0 \leq \nu \leq 1 \\ 4\nu - 2 & \text{if } \nu \geq 1. \end{cases} \quad (13)$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z) = (1+z)/(1-z)$ or one of its rotations. When $0 < \nu < 1$, then the equality holds if and only if $p(z) = (1+z^2)/(1-z^2)$ or one of its rotations. If $\nu = 0$, the equality holds if and only if $p(z) = (1/2 + \lambda/2)((1+z)/(1-z)) + (1/2 - \lambda/2)((1-z)/(1+z))$, ($0 \leq \lambda \leq 1$) or one of its rotations. While $\nu = 1$, equality holds if and only if $p(z)$ is the reciprocal of one of the functions for which the equality holds in the case of $\nu = 0$. Also the above upper bound can be improved as follows: when $0 < \nu < 1$

$$\begin{aligned} |c_2 - \nu c_1^2| + \nu |c_1|^2 &\leq 2 \quad \left(0 < \nu \leq \frac{1}{2}\right), \\ |c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 &\leq 2 \quad \left(\frac{1}{2} < \nu \leq 1\right). \end{aligned} \quad (14)$$

We also need the following results in our investigation.

Lemma 4 (see [19]). *If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$|c_2 - \nu c_1^2| \leq 2 \max \{1, |2\nu - 1|\}. \quad (15)$$

The result is sharp for the function $p(z) = (1 + z^2)/(1 - z^2)$ and $p(z) = (1 + z)/(1 - z)$.

2. Main Results

Theorem 5. *Let $0 \leq \mu \leq 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \geq 0$.*

If $f(z)$ given by (1) belongs to $S_{q,s}^(\phi)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2} \left[B_2 - \frac{\mu B_1^2}{[2]_q^2} ([3]_q - 1) \right] & \text{if } \mu \leq \sigma_1, \\ \frac{B_1}{2} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{2} \left[B_2 - \frac{\mu B_1^2}{[2]_q^2} ([3]_q - 1) \right] & \text{if } \mu \geq \sigma_2, \end{cases} \quad (16)$$

where

$$\sigma_1 := \frac{[2]_q^2}{([3]_q - 1)} \left(\frac{B_2 - B_1}{B_1^2} \right), \quad (17)$$

$$\sigma_2 := \frac{[2]_q^2}{([3]_q - 1)} \left(\frac{B_1 + B_2}{B_1^2} \right).$$

The result is sharp.

Proof. For $f \in S_{q,s}^*(\phi)$, let

$$p(z) = \frac{2z D_q f(z)}{f(z) - f(-z)} = 1 + b_1 z + b_2 z^2 + \dots \quad (18)$$

From (18), we obtain

$$\begin{aligned} [2]_q a_2 &= b_1, \\ ([3]_q - 1) a_3 &= b_2. \end{aligned} \quad (19)$$

Since $\phi(z)$ is univalent and $p \prec \phi$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \dots \quad (20)$$

is analytic and has positive real part in $\mathbb{U}$. Thus we have

$$p(z) = \phi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right) \quad (21)$$

and from (18) and (21),

$$\begin{aligned} a_2 &= \frac{1}{2[2]_q} B_1 c_1, \\ a_3 &= \frac{1}{[3]_q - 1} \left[\frac{B_1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{4} \right] \\ a_3 - \mu a_2^2 &= \frac{1}{2([3]_q - 1)} \left(c_2 - \nu c_1^2 \right), \end{aligned} \quad (22)$$

where

$$\nu = \frac{1}{2} - \frac{B_2}{2B_1} + \frac{\mu B_1}{2[2]_q^2} ([3]_q - 1). \quad (23)$$

Our result now follows by an application of Lemma 3. To show that these bounds are sharp, we define the functions $K_{\phi n}$ ($n = 2, 3, \dots$) by

$$\frac{2z K'_{\phi n}(z)}{K_{\phi n}(z) - K_{\phi n}(-z)} = \phi(z^{n-1}), \quad (24)$$

$$K_{\phi n}(0) = 0 = [K_{\phi n}]'(0) - 1$$

and the functions F^λ and G_λ ($0 \leq \lambda \leq 1$) by

$$\begin{aligned} \frac{2z F'_\lambda(z)}{F_\lambda(z) - F_\lambda(-z)} &= \phi \left(\frac{z(z + \lambda)}{1 + \lambda z} \right), \\ F_\lambda(0) = 0 &= [F_\lambda]'(0) - 1, \\ \frac{2z G'_\lambda(z)}{G_\lambda(z) - G_\lambda(-z)} &= \phi \left(-\frac{z(z + \lambda)}{1 + \lambda z} \right), \\ G_\lambda(0) = 0 &= [G_\lambda]'(0) - 1. \end{aligned} \quad (25)$$

Clearly the functions $K_{\phi n}, F_\lambda, G_\lambda \in S_{q,s}^*(\phi)$. Also we write $K_\phi := K_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, the equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_λ or one of its rotations.

If $\sigma_1 \leq \mu \leq \sigma_2$, then, in view of Lemma 3, Theorem 5 can be improved. $\square$

Theorem 6. *Let $0 \leq \mu \leq 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \geq 0$.*

Let $f(z)$ given by (1) belongs to $S_{q,s}^(\phi)$. Let σ_3 be given by $\sigma_3 := [2]_q^2 B_2 / ([3]_q - 1) B_1^2$.*

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\begin{aligned} |a_3 - \mu a_2^2| &+ \frac{1}{B_1^2([3]_q - 1)} \left[[2]_q^2 (B_1 - B_2) + \mu B_1^2 ([3]_q - 1) \right] \\ &\cdot |a_2|^2 \leq \frac{B_1}{2}. \end{aligned} \quad (26)$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$|a_3 - \mu a_2^2| + \frac{1}{B_1^2 ([3]_q - 1)} [2]_q^2 (B_1 + B_2) - \mu B_1^2 ([3]_q - 1) \quad (27)$$

$$\cdot |a_2|^2 \leq \frac{B_1}{2}.$$

Corollary 7 (see [18]). Let $0 \leq \mu \leq 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \geq 0$.

If $f(z)$ given by (1) belongs to $S_{q,s}^*(\phi)$ and $q \rightarrow 1^-$ (i.e., $S_{q,s}^*(\phi)$ reduced to $S_s^*(\phi)$) then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{2} \left[B_2 - \frac{\mu B_1^2}{2} \right] & \text{if } \mu \leq \sigma_1, \\ \frac{B_1}{2} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{2} \left[B_2 - \frac{\mu B_1^2}{2} \right] & \text{if } \mu \geq \sigma_2, \end{cases} \quad (28)$$

where

$$\sigma_1 := \frac{2(B_2 - B_1)}{B_1^2}, \quad (29)$$

$$\sigma_2 := \frac{2(B_2 + B_1)}{B_1^2}.$$

The result is sharp.

Theorem 8. Let $0 \leq \mu \leq 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \geq 0$.

If $f(z)$ given by (1) belongs to $C_{q,s}(\phi)$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{6} \left[B_2 - \frac{[3]_q}{2 [2]_q^2} \mu B_1^2 \right] & \text{if } \mu \leq \sigma_1, \\ \frac{B_1}{6} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{6} \left[B_2 - \frac{[3]_q}{2 [2]_q^2} \mu B_1^2 \right] & \text{if } \mu \geq \sigma_2, \end{cases} \quad (30)$$

where

$$\sigma_1 := \frac{2 [2]_q^2}{[3]_q} \left(\frac{B_2 - B_1}{B_1^2} \right), \quad (31)$$

$$\sigma_2 := \frac{2 [2]_q^2}{[3]_q} \left(\frac{B_1 + B_2}{B_1^2} \right).$$

The result is sharp.

Proof. Let

$$p(z) = \frac{2(D_q(zD_q f(z)))}{D_q f(z) + D_q f(-z)} = 1 + b_1 z + b_2 z^2 + \dots \quad (32)$$

Since $\phi(z)$ is univalent and $p < \phi$, the function

$$p_1(z) = \frac{1 + \phi^{-1}(p(z))}{1 - \phi^{-1}(p(z))} = 1 + c_1 z + c_2 z^2 + \dots \quad (33)$$

is analytic and has positive real part in $\mathbb{U}$. Thus we have

$$p(z) = \phi \left(\frac{p_1(z) - 1}{p_1(z) + 1} \right) \quad (34)$$

and from (34),

$$a_2 = \frac{1}{2} \frac{B_1 c_1}{[2]_q},$$

$$a_3 = \frac{B_1}{2 [2]_q [3]_q} \left(\left(c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{2 B_1} \right), \quad (35)$$

$$a_3 - \mu a_2^2 = \frac{1}{2} \frac{B_1}{[2]_q [3]_q} (c_2 - \nu c_1^2),$$

where

$$\nu = \frac{1}{2} - \frac{B_2}{2 B_1} + \frac{\mu B_1 [3]_q}{2 [2]_q}. \quad (36)$$

Our result now follows by an application of Lemma 3. To show that these bounds are sharp, we define the functions $K_{\phi n}$ ($n = 2, 3, \dots$) by

$$\frac{2(K'_{\phi n}(zK'_{\phi n}(z)))}{K'_{\phi n}(z) + K'_{\phi n}(-z)} = \phi(z^{n-1}), \quad (37)$$

$$K_{\phi n}(0) = 0 = [K_{\phi n}]'(0) - 1$$

and the functions F_λ and G_λ ($0 \leq \lambda \leq 1$) by

$$\frac{2(F'_\lambda(zF'_\lambda(z)))}{F'_\lambda(z) + F'_\lambda(-z)} = \phi \left(\frac{z(z+\lambda)}{1+\lambda z} \right),$$

$$F_\lambda(0) = 0 = [F_\lambda]'(0) - 1, \quad (38)$$

$$\frac{2(G'_\lambda(zG'_\lambda(z)))}{G'_\lambda(z) + G'_\lambda(-z)} = \phi \left(-\frac{z(z+\lambda)}{1+\lambda z} \right),$$

$$G_\lambda(0) = 0 = [G_\lambda]'(0) - 1.$$

Clearly the functions $K_{\phi n}, F_\lambda, G_\lambda \in C_{q,s}(\phi)$. Also we write $K_\phi := K_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, the equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_λ or one of its rotations.

If $\sigma_1 \leq \mu \leq \sigma_2$, then, in view of Lemma 3, Theorem 8 can be improved. $\square$

Corollary 9. Let $0 \leq \mu \leq 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \geq 0$.

Let $f(z)$ given by (1) belong to $C_{q,s}(\phi)$. Let σ_3 be given by $\sigma_3 := (2[2]_q^2/[3]_q)(B_2/B_1^2)$.

If $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\begin{aligned} |a_3 - \mu a_2^2| + \frac{2[2]_q^2}{[3]_q} \frac{1}{B_1^2} \left[(B_1 - B_2) + \frac{[3]_q}{2[2]_q^2} \mu B_1^2 \right] \\ \leq \frac{B_1}{6}. \end{aligned} \quad (39)$$

If $\sigma_3 \leq \mu \leq \sigma_2$, then

$$\begin{aligned} |a_3 - \mu a_2^2| + \frac{2[2]_q^2}{[3]_q} \frac{1}{B_1^2} \left[(B_1 + B_2) - \frac{[3]_q}{2[2]_q^2} \mu B_1^2 \right] \\ \leq \frac{B_1}{6}. \end{aligned} \quad (40)$$

Corollary 10 (see [18]). Let $0 \leq \mu \leq 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$, where the coefficients B_n are real with $B_1 > 0$ and $B_2 \geq 0$.

If $f(z)$ given by (1) belongs to $C_{q,s}(\phi)$ and $q \rightarrow 1^-$ (i.e., $C_{q,s}(\phi)$ is reduced to $C_s(\phi)$) then

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1}{6} \left[B_2 - \frac{3}{8} \mu B_1^2 \right] & \text{if } \mu \leq \sigma_1, \\ \frac{B_1}{6} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{6} \left[B_2 - \frac{3}{8} \mu B_1^2 \right] & \text{if } \mu \geq \sigma_2, \end{cases} \quad (41)$$

where

$$\begin{aligned} \sigma_1 &:= \frac{2(B_2 - B_1)}{B_1^2}, \\ \sigma_2 &:= \frac{2(B_2 + B_1)}{B_1^2}. \end{aligned} \quad (42)$$

The result is sharp.

3. Applications to Functions Defined by Fractional Derivative

In order to introduce classes $S_{q,s}^\lambda(\phi)$ and $C_{q,s}^\lambda(\phi)$ we need the following.

Definition 11 (see [20]). Let $f(z)$ be analytic in a simply connected region of the Z -plane containing the origin. The fractional derivative of f of order λ is defined by

$$\begin{aligned} D_z^\lambda f(z) &:= \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^\lambda} d\zeta \\ (0 \leq \lambda < 1), \end{aligned} \quad (43)$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring that $\log(z-\zeta)$ is real for $z-\zeta > 0$. Using the above Definition 11 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [20] introduced the operator $\Omega^\lambda : A \rightarrow A$ defined by

$$(\Omega^\lambda f)(z) = \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z), \quad (\lambda \neq 2, 3, 4, \dots). \quad (44)$$

Classes $S_{q,s}^\lambda(\phi)$ and $C_{q,s}^\lambda(\phi)$ consist of functions $f \in \mathcal{A}$ for which $\Omega^\lambda f \in S_{q,s}^*(\phi)$ and $\Omega^\lambda f \in C_{q,s}(\phi)$, respectively.

For a fixed $g \in \mathcal{A}$, let $S_{q,s}^g(\phi)$ be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in S_{q,s}^*(\phi)$ and let $C_{q,s}^g(\phi)$ be the class of functions $f \in \mathcal{A}$ for which $(f * g) \in C_s(\phi)$.

Classes $S_{q,s}^\lambda(\phi)$ and $C_{q,s}^\lambda(\phi)$ are the special case of classes $S_{q,s}^g(\phi)$ and $C_{q,s}^g(\phi)$, respectively, when

$$g(z) = z + \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} z^n. \quad (45)$$

Let

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n \quad (g_n > 0), \quad (46)$$

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Since $f \in S_{q,s}^g(\phi)$ ($C_{q,s}^g(\phi)$) if and only if $f * g \in S_{q,s}^*(\phi)$ ($C_{q,s}(\phi)$), we obtain the coefficient estimates for functions in classes $S_{q,s}^g(\phi)$ and $C_{q,s}^g(\phi)$, from the corresponding estimates for functions in classes $S_{q,s}^*(\phi)$ and $C_{q,s}(\phi)$.

Applying Theorem 5 for the function $(f * g)(z) = z + g_2 a_2 z^2 + g_3 a_3 z^3 + \dots$, we get the following theorem for an obvious change of μ .

Theorem 12. Let the function $\phi(z)$ be given by $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$ and let $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$, ($g_n > 0$). If $f(z)$ given by (1) belongs to $S_{q,s}^g(\phi)$, then

$$\begin{aligned} |a_3 - \mu a_2^2| \\ \leq \begin{cases} \frac{1}{2g_3} \left[B_2 - \frac{([3]_q - 1)\mu g_3}{[2]_q^2} B_1^2 \right] & \text{if } \mu \leq \sigma_1, \\ \frac{B_1}{2g_3} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{2g_3} \left[B_2 - \frac{([3]_q - 1)\mu g_3}{[2]_q^2} B_1^2 \right] & \text{if } \mu \geq \sigma_2, \end{cases} \end{aligned} \quad (47)$$

where

$$\begin{aligned} \sigma_1 &:= \frac{[2]_q^2 g_2^2}{([3]_q - 1) g_3} \left(\frac{B_2 - B_1}{B_1^2} \right), \\ \sigma_2 &:= \frac{[2]_q^2 g_2^2}{([3]_q - 1) g_3} \left(\frac{B_1 + B_2}{B_1^2} \right). \end{aligned} \quad (48)$$

The result is sharp.

Since

$$(\Omega^\lambda f)(z) = z + \sum_{n=0}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n, \quad (49)$$

we have

$$\begin{aligned} g_2 &:= \frac{\Gamma(3)\Gamma(2-\lambda)}{\Gamma(3-\lambda)} = \frac{2}{2-\lambda}, \\ g_3 &:= \frac{\Gamma(4)\Gamma(2-\lambda)}{\Gamma(4-\lambda)} = \frac{6}{(2-\lambda)(3-\lambda)}. \end{aligned} \quad (50)$$

Theorem 13. Let the function $g(z) = z + \sum_{n=2}^{\infty} g_n z^n$, ($g_n > 0$). If $f(z)$ given by (1) belongs to $C_{q,s}^g(\phi)$, then

$$\begin{aligned} &|a_3 - \mu a_2^2| \\ &\leq \begin{cases} \frac{1}{6g_3} \left[B_2 - \frac{[3]_q}{2[2]_q^2} \frac{\mu g_3}{g_2^2} B_1^2 \right] & \text{if } \mu \leq \sigma_1, \\ \frac{B_1}{6g_3} & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{1}{6g_3} \left[B_2 - \frac{[3]_q}{2[2]_q^2} \frac{\mu g_3}{g_2^2} B_1^2 \right] & \text{if } \mu \geq \sigma_2, \end{cases} \end{aligned} \quad (51)$$

where

$$\begin{aligned} \sigma_1 &:= \frac{2[2]_q^2 g_2^2}{[3]_q g_3} \left(\frac{B_2 - B_1}{B_1^2} \right), \\ \sigma_2 &:= \frac{2[2]_q^2 g_2^2}{[3]_q g_3} \left(\frac{B_1 + B_2}{B_1^2} \right). \end{aligned} \quad (52)$$

The result is sharp.

For g_2 and g_3 given by (50), Theorems 12 and 13 reduce to the following.

Corollary 14. Let $\lambda < 2$. If $f(z)$ given by (1) belongs to $S_{q,s}^\lambda(\phi)$, then

$$\begin{aligned} &|a_3 - \mu a_2^2| \\ &\leq \begin{cases} \frac{(2-\lambda)(3-\lambda)}{12} [B_2 - \xi \mu B_1^2] & \text{if } \mu \leq \sigma_1, \\ \frac{(2-\lambda)(3-\lambda)}{12} B_1 & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{(2-\lambda)(3-\lambda)}{12} [B_2 - \xi \mu B_1^2] & \text{if } \mu \geq \sigma_2, \end{cases} \end{aligned} \quad (53)$$

where

$$\begin{aligned} \xi &= \frac{3(2-\lambda)([3]_q - 1)}{2(3-\lambda)[2]_q^2} \\ \sigma_1 &:= \frac{2(3-\lambda)[2]_q^2}{3(2-\lambda)([3]_q - 1)} \left(\frac{B_2 - B_1}{B_1^2} \right), \\ \sigma_2 &:= \frac{2(3-\lambda)[2]_q^2}{3(2-\lambda)([3]_q - 1)} \left(\frac{B_1 + B_2}{B_1^2} \right). \end{aligned} \quad (54)$$

Corollary 15. Let $\lambda < 2$. If $f(z)$ given by (1) belongs to $C_{q,s}^\lambda(\phi)$, then

$$\begin{aligned} &|a_3 - \mu a_2^2| \leq \begin{cases} \frac{(2-\lambda)(3-\lambda)}{36} \left[B_2 - \frac{3[3]_q(2-\lambda)}{4[2]_q^2(3-\lambda)} \mu B_1^2 \right] & \text{if } \mu \leq \sigma_1, \\ \frac{(2-\lambda)(3-\lambda)}{36} B_1 & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\ -\frac{(2-\lambda)(3-\lambda)}{36} \left[B_2 - \frac{3[3]_q(2-\lambda)}{4[2]_q^2(3-\lambda)} \mu B_1^2 \right] & \text{if } \mu \geq \sigma_2, \end{cases} \end{aligned} \quad (55)$$

where

$$\begin{aligned} \sigma_1 &:= \frac{2[2]_q(3-\lambda)}{3[3]_q(2-\lambda)} \left(\frac{B_2 - B_1}{B_1^2} \right), \\ \sigma_2 &:= \frac{2[2]_q(3-\lambda)}{3[3]_q(2-\lambda)} \left(\frac{B_1 + B_2}{B_1^2} \right). \end{aligned} \quad (56)$$

Conflict of Interests

The authors declare that there is no conflict of interest regarding the publication of this paper.

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