Research Article

Lower Bounds on Solutions of Quadratic Polynomials Defined over Finite Rings

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Received 17 January 2015; Revised 16 April 2015; Accepted 20 April 2015

Academic Editor: Harvinder S. Sidhu

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Let m be a positive integer and let R_m denote the ring $\mathbb{Z}/(m)$, and let R_m^n denote the Cartesian product of n copies of $\mathbb{Z}/m$. Let $f(\mathbf{x})$ be a quadratic polynomial in $\mathbb{Z}[x_1, \dots, x_n]$. In this paper, we are interested in giving lower bounds on the number of solutions of the quadratic polynomial f over the ring R_m^n .

1. Introduction

Let $R_m = \mathbb{Z}/(m)$ be a finite ring with m positive integer. Let $f(\mathbf{x})$ be a quadratic polynomial in $\mathbb{Z}[x_1, \dots, x_n]$. Write

$$f(\mathbf{x}) = Q(\mathbf{x}) + \mathbf{a} \cdot \mathbf{x} + c, \quad (1)$$

where $\mathbf{a} \in \mathbb{Z}^n$, $c \in \mathbb{Z}$, and $Q(\mathbf{x})$ is a quadratic form given by

$$Q(\mathbf{x}) = \frac{1}{2} \mathbf{x} A \mathbf{x}^T, \quad (2)$$

where A is a symmetric $n \times n$ matrix with integer entries. Throughout this paper (with the exception of Lemma 4), we will assume that $\gcd(\det A, m) = 1$. Let V be the algebraic subset of R_m^n defined by

$$f(\mathbf{x}) \equiv 0 \pmod{m}. \quad (3)$$

If $d \mid m$ and $\mathbf{x}, \mathbf{y} \in R_m^n$, we will say that $\mathbf{x} \equiv \mathbf{y} \pmod{d}$ if $\mathbf{x}$ is congruent to $\mathbf{y}$ modulo the ideal dR_m . For any subset S of R_m^n and divisor d of m , let

$$\eta(S, d) = |\{(s_1, s_2) \in S \times S : s_1 \equiv s_2 \pmod{d}\}|. \quad (4)$$

Let ϕ denote the Euler phi-function, let $\tau(m)$ denote the number of distinct positive divisors of m , and for positive integers m and n set

$$\Phi_n(m) = \sum_{\substack{d>1 \\ d|m}} \phi(d) d^{-n/2}. \quad (5)$$

Considerable attention has been given to the problem of finding zeros of any polynomial over finite fields; see for example [1–8]. A special case of particular interest when the polynomial is quadratic over finite rings is studied in [9] by the author who obtained the following.

Theorem 1. For any subsets S and T of R_m^n , with $|S| \leq |T|$, we have

$$\begin{aligned} & |(S+T) \cap V| \\ & \geq m^{-1} |T| \\ & - m^{-1} |S|^{-1} \sum_{\substack{d>m \\ d|m}} \phi(d) d^{n/2} \eta(S, d)^{1/2} \eta(T, d)^{1/2}. \end{aligned} \quad (6)$$

In this paper, we will make the above result more precise when $S+T$ is a box of points in R_m^n , that is, the image $\overline{\mathcal{B}}$ of a box $\mathcal{B}$ in $\mathbb{Z}^n$ under the canonical mapping of $\mathbb{Z}^n$ onto R_m^n , where

$$\overline{\mathcal{B}} = \{\mathbf{x} \in \mathbb{Z}^n : a_i \leq x_i < a_i + m_i\}, \quad (7)$$

for some $a_i, m_i \in \mathbb{Z}$ with $0 < m_i \leq m$, $1 \leq i \leq n$. In this case we obtain the following.

Theorem 2. Suppose that $n \geq 4$. Let $\overline{\mathcal{B}}$ be a box in R_m^n whose sides are all of the same length $M < m$; that is, let $\overline{\mathcal{B}}$ be the

image of a box $\mathcal{B}$ as given in (7), where $m_i = M$, $1 \leq i \leq n$. Put $c = [(M+1)/2]$. Then,

$$\begin{aligned} |\overline{\mathcal{B}} \cap V| &\geq \frac{c^n}{m} \left(1 - \Phi_n(m) - 2^n \frac{\tau(m)}{c} \right. \\ &\quad \left. - \frac{m^{(m/2)+1}}{c^n} \left(2^n \frac{c}{m} + 1 \right) \right). \end{aligned} \quad (8)$$

In particular, if $n \geq 6$ and $M \gg m^{1/2+1/n}$, then $\overline{\mathcal{B}} \cap V$ is nonempty.

The second part of Theorem 2 follows immediately from Lemma 3 of the next section. The first part of the theorem will be proven in Section 3.

2. Auxiliary Lemmas

Lemma 3. If $n \geq 4$, then for any integer $m > 1$,

$$\Phi_n(m) \leq \prod_{p|m} (1 + p^{1-(n/2)}) - 1, \quad (9)$$

where the product is over all primes p dividing m . In particular, if $n \geq 6$, then for all $m > 1$, $\Phi_n(m) \leq 2^{2-(n/2)}$.

Proof. Let $s = n/2$ and set $g(m) = \sum_{d|m} \phi(d) d^{-s}$, so that $\Phi_n(m) = g(m) - 1$. Since $\phi(d)$ and d^{-s} are both multiplicative, $g(m)$ is multiplicative. If p is a prime and $e \geq 1$, then

$$\begin{aligned} g(p^e) &= 1 + \sum_{i=1}^e (p^i - p^{i-1}) p^{-si} \\ &= 1 + (1 - p^{-1}) \sum_{i=1}^e p^{(1-s)i} \\ &< 1 + (1 - p^{-1}) p^{1-s} \sum_{i=0}^{\infty} p^{(1-s)i} \\ &= 1 + p^{1-s} \frac{1 - p^{-1}}{1 - p^{1-s}} \leq 1 + p^{1-s}. \end{aligned} \quad (10)$$

The last inequality follows since $s \geq 2$. The first part of the lemma now follows from the multiplicative property of $g(m)$.

Now suppose that $n \geq 6$. Again, letting $s = n/2$ we can say that

$$\Phi_n(m) \leq \sum_{d=2}^{\infty} \phi(d) d^{-s} \leq \frac{1}{2^s} + \frac{2}{3^s} + \frac{2}{4^s} + \sum_{d=5}^{\infty} d^{1-s}, \quad (11)$$

since $\phi(d) < d$. But

$$\sum_{d=5}^{\infty} d^{1-s} < \int_4^{\infty} x^{1-s} dx = (s-2)^{-1} 4^{2-s}. \quad (12)$$

Thus,

$$\begin{aligned} \Phi_n(m) &< \frac{1}{2^s} \left(1 + 2 \left(\frac{2}{3} \right)^s + \frac{1}{2^{s-1}} + \frac{1}{2^{s-4}(s-2)} \right) \\ &\leq \frac{1}{2^s} \cdot 4, \quad \text{for } s \geq 3. \end{aligned} \quad (13)$$

□

To prove Theorem 2, we make use of exponential sums. Let $e_m(x) = e^{(2\pi i/m)x}$. We will abbreviate complete sums $\sum_{x \in R_m^n}()$ by simply $\sum_x()$. Also we will need to use the following fundamental identity: for any $\mathbf{y} \in R_m^n$,

$$\sum_x e_m(\mathbf{x} \cdot \mathbf{y}) = \begin{cases} m^n & \text{if } \mathbf{y} = \mathbf{0}, \\ 0 & \text{if } \mathbf{y} \neq \mathbf{0}. \end{cases} \quad (14)$$

Let $f(\mathbf{x})$ and $Q(\mathbf{x})$ be as defined by (1) and (2). By viewing R_m^n as a $\mathbb{Z}$ -module, the Gauss sums

$$\begin{aligned} G_m(Q, \mathbf{y}) &= \sum_{x \in R_m^n} e_m(Q(\mathbf{x}) + \mathbf{y} \cdot \mathbf{x}), \\ G_m(f, \mathbf{y}) &= \sum_{x \in R_m^n} e_m(f(\mathbf{x}) + \mathbf{y} \cdot \mathbf{x}) \end{aligned} \quad (15)$$

are well defined whether we take $\mathbf{y} \in \mathbb{Z}^n$ or $\mathbf{y} \in R_m^n$.

For any $n \times n$ matrix A with integer entries, we define $\ker_m(A)$ by

$$\ker_m(A) = \{ \mathbf{x} \in R_m^n : A\mathbf{x}^T \equiv \mathbf{0}^T \pmod{m} \}. \quad (16)$$

We need the following lemmas.

Lemma 4 (see [10, Theorem 4]). Let $Q(\mathbf{x})$ be a quadratic form given by (2), where now we allow A to be any symmetric integral matrix with even diagonal entries. Then, given $\mathbf{y} \in R_m^n$, the Gauss sum $G_m(Q, \mathbf{y})$ is zero unless $Q(\mathbf{x}) + \mathbf{y} \cdot \mathbf{x} \equiv 0 \pmod{m}$ for all $\mathbf{x} \in \ker_m(A)$, in which case $|G_m(Q, \mathbf{y})|^2 = m^n |\ker_m A|$.

Lemma 5. Let $f(\mathbf{x})$ be a quadratic polynomial as given by (1) with $\gcd(\det A, m) = 1$. Let $\mathbf{y} \in \mathbb{Z}^n$, $\lambda \in \mathbb{Z}$, $\lambda \not\equiv 0 \pmod{m}$, and set $d = (\lambda, m)$. Then,

$$|G_m(\lambda f, \mathbf{y})| = \begin{cases} (md)^{n/2} & \text{if } \mathbf{y} \equiv \mathbf{0} \pmod{d}, \\ 0 & \text{if } \mathbf{y} \not\equiv \mathbf{0} \pmod{d}. \end{cases} \quad (17)$$

Proof. By (1), we have

$$\begin{aligned} |G_m(\lambda f, \mathbf{y})| &= \left| \sum_{\mathbf{x}} e_m(\lambda f(\mathbf{x}) + \mathbf{y} \cdot \mathbf{x}) \right| \\ &= \left| \sum_{\mathbf{x}} e_m(\lambda Q(\mathbf{x}) + (\lambda \mathbf{a} + \mathbf{y}) \cdot \mathbf{x} + \lambda c) \right| \\ &= \left| \sum_{\mathbf{x}} e_m(\lambda Q(\mathbf{x}) + (\lambda \mathbf{a} + \mathbf{y}) \cdot \mathbf{x}) \right|. \end{aligned} \quad (18)$$

Now $\lambda Q(\mathbf{x}) = (1/2)\mathbf{x}(\lambda A)\mathbf{x}^T$, so that by Lemma 4, $G_m(\lambda f, \mathbf{y}) = 0$ unless $\mathbf{y}$ satisfies the following condition:

$$\forall \mathbf{x} \in \ker_m(\lambda A), \quad (19)$$

$$(\lambda \mathbf{a} + \mathbf{y}) \cdot \mathbf{x} + \lambda Q(\mathbf{x}) \equiv 0 \pmod{m}.$$

Now

$$\begin{aligned} \ker_m(\lambda A) &= \{ \mathbf{x} \in R_m^n : \lambda A\mathbf{x}^T \equiv \mathbf{0} \pmod{m} \} \\ &= \left\{ \mathbf{x} \in R_m^n : A\mathbf{x}^T \equiv \mathbf{0} \pmod{\frac{m}{d}} \right\}, \end{aligned} \quad (20)$$

but as $\gcd(\det A, m) = 1$ we conclude that

$$\ker_m(\lambda A) = \left\{ \mathbf{x} \in R_m^n : \mathbf{x} \equiv \mathbf{0} \pmod{\frac{m}{d}} \right\}. \quad (21)$$

Thus, setting $\mathbf{x} \equiv (m/d)\mathbf{t} \pmod{m}$, we see that (19) is equivalent to saying that, for all $\mathbf{t} \in \mathbb{Z}^n$,

$$\begin{aligned} \lambda \left(\frac{m}{d} \right) \mathbf{a} \cdot \mathbf{t} + \lambda \left(\frac{m}{d} \right) \mathbf{y} \cdot \mathbf{t} + \lambda \left(\frac{m}{d} \right)^2 Q(\mathbf{t}) \\ \equiv 0 \pmod{m}. \end{aligned} \quad (22)$$

But $\lambda(m/d) = m(\lambda/d) \equiv 0 \pmod{m}$ and similarly $\lambda(m/d)^2 = m(\lambda/d)(m/d) \equiv 0 \pmod{m}$ so that (22) simplifies to the congruence $(m/d)\mathbf{y} \cdot \mathbf{t} \equiv 0 \pmod{m}$; that is, $\mathbf{y} \cdot \mathbf{t} \equiv 0 \pmod{d}$. Hence, $\mathbf{y}$ satisfies (19) if and only if, for all $\mathbf{t} \in \mathbb{Z}^n$, $\mathbf{y} \cdot \mathbf{t} \equiv 0 \pmod{d}$, that is $\mathbf{y} \equiv \mathbf{0} \pmod{d}$. If $\mathbf{y} \equiv \mathbf{0} \pmod{d}$ then by Lemma 4, and (21), we have

$$|G_m(\lambda f, \mathbf{y})|^2 = m^2 |\ker_m(\lambda A)| = m^n d^n. \quad (23)$$

□

3. Proof of Theorem 2

Let S and T be subsets of R_m^n , and let V be the set of points in R_m^n satisfying $f(\mathbf{x}) \equiv 0 \pmod{m}$. Let N be the number of triples $(\mathbf{s}, \mathbf{t}, \mathbf{v}) \in S \times T \times V$ such that $\mathbf{s} + \mathbf{t} \equiv \mathbf{v} \pmod{m}$.

Lemma 6. Let N be the number of triples $(\mathbf{s}, \mathbf{t}, \mathbf{v}) \in S \times T \times V$ such that $\mathbf{s} + \mathbf{t} \equiv \mathbf{v} \pmod{m}$. Then, for any subsets S and T of R_m^n , with $|S| \leq |T|$ we have

$$\begin{aligned} |N - m^{-1} |S| |T|| \\ \leq m^{-1} \sum_{\substack{d>1 \\ d|m}} \phi(d) d^{-n/2} d^n \eta(S, d)^{1/2} \eta(T, d)^{1/2}. \end{aligned} \quad (24)$$

Proof. By the fundamental identity (14),

$$\begin{aligned} N &= m^{-n} \sum_{\mathbf{x} \in V} \sum_{\mathbf{s} \in S} \sum_{\mathbf{t} \in T} e_m(\mathbf{y} \cdot (\mathbf{s} + \mathbf{t} - \mathbf{x})) \\ &= m^{-n-1} \sum_{\mathbf{x} \in R_m^n} \left[\sum_{\lambda \in R_m} e_m(\lambda f(\mathbf{x})) \right] \\ &\quad \cdot \sum_{\mathbf{s} \in S} \sum_{\mathbf{t} \in T} e_m(\mathbf{y} \cdot (\mathbf{s} + \mathbf{t} - \mathbf{x})) \\ &= m^{-n-1} \sum_{\lambda} \sum_{\mathbf{y}} \Psi(\mathbf{y}) \sum_{\mathbf{x}} e_m(\lambda f(\mathbf{x}) - \mathbf{y} \cdot \mathbf{x}), \end{aligned} \quad (25)$$

where

$$\Psi(\mathbf{y}) = \sum_{\mathbf{s} \in S} \sum_{\mathbf{t} \in T} e_m(\mathbf{y} \cdot (\mathbf{s} + \mathbf{t})). \quad (26)$$

Peeling off the $\lambda = 0$ term yields

$$\begin{aligned} N &= m^{-1} |S| \cdot |T| \\ &\quad + m^{-n-1} \sum_{\lambda \neq 0} \sum_{\mathbf{y}} \Psi(\mathbf{y}) \sum_{\mathbf{x}} e_m(\lambda f(\mathbf{x}) - \mathbf{y} \cdot \mathbf{x}). \end{aligned} \quad (27)$$

Thus (from (27)),

$$\begin{aligned} N - m^{-1} |S| \cdot |T| \\ = m^{-n-1} \sum_{\substack{1 \leq d < m \\ d|m}} \sum_{\substack{1 \leq \lambda < m \\ (\lambda, m) = d}} \sum_{\mathbf{y}} \Psi(\mathbf{y}) \sum_{\mathbf{x}} e_m(\lambda f(\mathbf{x}) - \mathbf{y} \cdot \mathbf{x}), \end{aligned} \quad (28)$$

so that by Lemma 5,

$$\begin{aligned} |N - m^{-1} |S| |T|| \\ \leq m^{-n-1} \sum_{\substack{d|m \\ d < m}} \sum_{\substack{1 \leq \lambda < m \\ (\lambda, m) = d}} m^{n/2} d^{n/2} \sum_{\mathbf{y}}^* |\Psi(\mathbf{y})|, \end{aligned} \quad (29)$$

where the sum $\sum^*$ on $\mathbf{y}$ is over all $\mathbf{y} \equiv \mathbf{0} \pmod{d}$. On replacing d by m/d and $\sum_{\mathbf{y}}^*$ by $\sum_{\mathbf{y}}^{**}$, the sum over all $\mathbf{y} \equiv \mathbf{0} \pmod{(m/d)}$, we obtain

$$\begin{aligned} |N - m^{-1} |S| |T|| \\ \leq m^{-n/2-1} \sum_{\substack{d|m \\ d < m}} \phi\left(\frac{m}{d}\right) d^{n/2} \sum_{\mathbf{y}}^* |\Psi(\mathbf{y})| \\ = m^{-1} \sum_{\substack{d>1 \\ d|m}} \phi(d) d^{-n/2} \sum_{\mathbf{y}}^{**} |\Psi(\mathbf{y})|. \end{aligned} \quad (30)$$

Now,

$$\begin{aligned} \sum_{\mathbf{y}}^{**} |\Psi(\mathbf{y})| |\Psi(\mathbf{y})| &= \sum_{\mathbf{y}}^{**} \left| \sum_{\mathbf{s} \in S} e_m(\mathbf{y} \cdot \mathbf{s}) \right| \left| \sum_{\mathbf{t} \in T} e_m(\mathbf{y} \cdot \mathbf{t}) \right| \\ &\leq \left[\sum_{\mathbf{y}}^{**} \left| \sum_{\mathbf{s} \in S} e_m(\mathbf{y} \cdot \mathbf{s}) \right|^2 \right]^{1/2} \\ &\quad \cdot \left[\sum_{\mathbf{y}}^{**} \left| \sum_{\mathbf{t} \in T} e_m(\mathbf{y} \cdot \mathbf{t}) \right|^2 \right]^{1/2}. \end{aligned} \quad (31)$$

Setting $\mathbf{y} \equiv (m/d)\mathbf{u} \pmod{m}$ and letting $\mathbf{u}$ run through a complete set of representatives for R_d^n , we can say

$$\begin{aligned} \sum_{\mathbf{y}}^{**} \left| \sum_{\mathbf{t} \in T} e_m(\mathbf{y} \cdot \mathbf{t}) \right|^2 \\ = \sum_{\mathbf{y}}^{**} \sum_{\mathbf{s}_1 \in S} \sum_{\mathbf{s}_2 \in S} e_m(\mathbf{y} \cdot (\mathbf{s}_1 - \mathbf{s}_2)) \\ = \sum_{\mathbf{s}_1 \in S} \sum_{\mathbf{s}_2 \in S} \sum_{\mathbf{y}}^{**} e_m(\mathbf{y} \cdot (\mathbf{s}_1 - \mathbf{s}_2)) \\ = \sum_{\mathbf{s}_1 \in S} \sum_{\mathbf{s}_2 \in S} \sum_{\mathbf{u} \in R_d^n} e_m\left(\frac{m}{d} \mathbf{u} \cdot (\mathbf{s}_1 - \mathbf{s}_2)\right) \\ = \sum_{\mathbf{s}_1 \in S} \sum_{\mathbf{s}_2 \in S} \sum_{\mathbf{u}} e_d(\mathbf{u} \cdot (\mathbf{s}_1 - \mathbf{s}_2)) = d^n \eta(S, d). \end{aligned} \quad (32)$$

Thus, by (31),

$$\sum_{\mathbf{y}}^{**} |\Psi(\mathbf{y})| \leq d^n \eta(S, d)^{1/2} \eta(T, d)^{1/2}, \quad (33)$$

and therefore, by (30),

$$\begin{aligned} & |N - m^{-1} |S| |T|| \\ & \leq m^{-1} \sum_{\substack{d>1 \\ d|m}} \phi(d) d^{-n/2} d^n \eta(S, d)^{1/2} \eta(T, d)^{1/2}. \end{aligned} \quad (34)$$

□

Proof of Theorem 2. Let

$$\begin{aligned} B &= \{\mathbf{x} \in \mathbb{Z}^n : a_i \leq x_i < a_i + M, 1 \leq i \leq n\}, \\ S &= \{\mathbf{x} \in \mathbb{Z}^n : 0 \leq x_i < c\}, \\ T &= S + (a_1, a_2, \dots, a_n), \end{aligned} \quad (35)$$

for some $a_i \in \mathbb{Z}$, $1 \leq i \leq n$, where $c = [(M+1)/c]$. Let $\bar{B}$, $\bar{S}$, and $\bar{T}$ be the images of B , S , and T in R_m^n under the canonical mapping of $\mathbb{Z}^n$ onto R_m^n . Then, $\bar{S} + \bar{T} \subset \bar{B}$ and $|\bar{S}| = |\bar{T}| = [(M+1)/c]^n \geq 2^{-n} |\bar{B}|$. We claim that for any divisor d of m ,

$$\eta(\bar{S}, d) \leq c^n \left(\left\lfloor \frac{c}{d} \right\rfloor + 1 \right)^n, \quad (36)$$

and that the same inequality holds for $\eta(\bar{T}, d)$. Let $\mathbf{s} = (s_1, \dots, s_n)$ be a fixed point in S . If $\mathbf{u} = (u_1, \dots, u_n)$ is a point in S such that $\mathbf{u} \equiv \mathbf{s} \pmod{d}$, then $u_i = s_i + dk_i$, for some $k_i \in \mathbb{Z}$, $1 \leq i \leq n$. Since $0 \leq u_i < c$, there are at most $[c/d] + 1$ choices for each u_i , and thus at most $([c/d] + 1)^n$ choices for $\mathbf{u}$. Since $|S| = c^n$, we obtain (36). It is clear that $\eta(\bar{S}, d) = \eta(\bar{T}, d)$ so that (36) holds also for $\eta(\bar{T}, d)$.

We now apply Theorem 1 with S and T replaced by the sets $\bar{S}$ and $\bar{T}$ just defined. We will abbreviate the sum $\sum_{d>1, d|m}$ by simple $\sum^*$. From Lemma 6 and (36) we have

$$\begin{aligned} |N - m^{-1} |S| |T|| & \leq m^{-1} \sum^* \phi(d) d^{n/2} c^n \left(\left\lfloor \frac{c}{d} \right\rfloor + 1 \right)^n \\ & \leq m^{-1} \sum^* \phi(d) d^{n/2} c^n d^{-n} (c+d)^n \\ & = m^{-1} c^n \sum^* \phi(d) d^{-n/2} c^n (c+d)^n. \end{aligned} \quad (37)$$

Now, since $(c+d)^n \leq c^n + 2^n(c^{n-1}d + cd^{n-1}) + d^n$, we have

$$\begin{aligned} |N - m^{-1} |S| |T|| & \leq m^{-1} c^{2n} \sum^* \phi(d) d^{-n/2} \\ & \quad + m^{-1} 2^n c^{2n-1} \sum^* \phi(d) d^{1-n/2} \\ & \quad + m^{-1} 2^n c^{n+1} \sum^* \phi(d) d^{(n/2)-1} \\ & \quad + m^{-1} c^n \sum^* \phi(d) d^{n/2}. \end{aligned} \quad (38)$$

The first sum on the right-hand side of (38) is just $\Phi_n(m)$. We make crude estimates for the remaining sums. For $n \geq 4$ we have

$$\begin{aligned} \sum^* \phi(d) d^{1-(n/2)} & \leq \sum^* \phi(d) d^{-1} \leq \sum^* 1 < \tau(m), \\ \sum^* \phi(d) d^{(n/2)-1} & \leq m^{n/2-1} \sum^* \phi(d) \leq m^{n/2-1} \cdot m \\ & = m^{n/2}, \\ \sum^* \phi(d) d^{(n/2)} & \leq m^{n/2} \sum^* \phi(d) \leq m^{n/2+1}. \end{aligned} \quad (39)$$

Thus, by (38), we see that

$$\begin{aligned} N & \geq m^{-1} c^{2n} - m^{-1} c^{2n} \Phi_n(m) - m^{-1} 2^n c^{2n-1} \tau(m) \\ & \quad - m^{-1} 2^n c^{n+1} m^{n/2} - m^{-1} c^n m^{(n/2)+1} = m^{-1} c^{2n} (1 \\ & \quad - \Phi_n(m) - 2^n c^{-1} \tau(m) \\ & \quad - c^{-n} m^{(n/2)+1} (2^n c m^{-1} + 1)). \end{aligned} \quad (40)$$

Theorem 2 now follows from the observation that

$$|\bar{B} \cap V| \geq |(\bar{S} \cap \bar{T}) \cap V| \geq N |S|^{-1} = N c^{-n}. \quad (41)$$

□

4. Remarks

- (1) It is clear from (39) and Lemma 3 that if $n \geq 8$ then in the statement of Theorem 2 we can replace $\tau(m)$ by $2^{3-(n/2)}$.
- (2) Let $f(\mathbf{x})$ be given by (1), let V be the set of zeros of $f(\mathbf{x})$ in R_m^n , and again suppose that $\gcd(\det A, m) = 1$. Lemma 5 provides us with an easy means of estimating $|V|$. For $n \geq 4$, we obtain

$$|V| = m^{n-1} \prod_{p|m} (1 + \theta_p p^{1-(n/2)}), \quad (42)$$

where the product is over all primes p dividing m , and for each such p , θ_p is a real number of absolute value ≤ 1 . Equation (42) follows from the observation that

$$\begin{aligned} |V| &= m^{-1} \sum_{\mathbf{x} \in R_m^n} \sum_{\lambda=0}^{m-1} e(\lambda f(\mathbf{x})) \\ &= m^{n-1} + m^{-1} \sum_{\lambda=1}^{m-1} \sum_{\mathbf{x}} e(\lambda f(\mathbf{x})). \end{aligned} \quad (43)$$

By Lemma 5, we then have

$$|V| = m^{n-1} + \theta_m m^{n/2-1} \sum_{\substack{1 \leq d < m \\ d|m}} \phi\left(\frac{m}{d}\right) d^{n/2}, \quad (44)$$

for some $\theta_m \in \mathbb{R}$, with $|\theta_m| \leq 1$. Since

$$\Phi_n(m) = \sum_{\substack{d>1 \\ d|m}} \phi(d) d^{-n/2} = \sum_{\substack{1 \leq d < m \\ d|m}} \phi\left(\frac{m}{d}\right) \left(\frac{d}{m}\right)^{n/2}, \quad (45)$$

we obtain

$$|V| = m^{n-1} [1 + \theta_m \Phi_n(m)]. \quad (46)$$

To obtain (42), we apply (46) in turn to each prime power dividing m and use the Chinese remainder theorem to compute $|V|$. That is, for each divisor d of m , we let $v(d)$ be the number of points in R_d^n satisfying the congruence $f(\mathbf{x}) \equiv 0 \pmod{d}$. Consequently, if $m = \prod_{i=1}^s p_i^{e_i}$ then $v(m) = \prod_{i=1}^s v(p_i^{e_i})$. Thus, by (46), we have

$$|V| = v(m) = \prod_{i=1}^s p_i^{e_i(n-1)} (1 + \theta_i \Phi_n(p_i^{e_i})), \quad (47)$$

for some $\theta_i \in \mathbb{R}$ with $|\theta_i| \leq 1$, $1 \leq i \leq s$, and by Lemma 3 we have

$$\Phi_n(p^e) \leq p^{1-(n/2)}, \quad (48)$$

for any prime power p^e , when $n \geq 4$. Equation (42) is now immediate.

Equation (42) indicates that we obtain roughly the expected quota of zeros for $f(\mathbf{x})$, namely, m^{n-1} , when $\gcd(\det A, m) = 1$. When $\gcd(\det A, m) \neq 1$, this is no longer the case. For example, suppose that $m = pq$, where p and q are distinct primes. Let α be quadratic nonresidue $\pmod{p}$, let β be a quadratic nonresidue $\pmod{q}$, and let $f(\mathbf{x}) = f(x_1, x_2, x_3, x_4)$ be defined by

$$f(\mathbf{x}) = p(x_1^2 - \beta x_2^2) + q(x_3^2 - \alpha x_4^2). \quad (49)$$

If $\mathbf{x}$ is an integral solution of the congruence $f(\mathbf{x}) \equiv 0 \pmod{m}$ then $x_1^2 - \beta x_2^2 \equiv 0 \pmod{q}$ and $x_3^2 - \alpha x_4^2 \equiv 0 \pmod{p}$, so that $x_1 \equiv x_2 \equiv 0 \pmod{q}$ and $x_3 \equiv x_4 \equiv 0 \pmod{p}$. Thus, if V is the set of points in R_m^4 satisfying $f(\mathbf{x}) \equiv 0 \pmod{m}$, then $|V| = p^2 q^2 = m^2$, rather than expected quota of m^3 . This example indicates that Theorem 2 does not hold when $\gcd(\det A, m) \neq 1$.

We have not been able to obtain an analogue of Theorem 2 when $\gcd(\det A, m) \neq 1$. The main difficulty is that $\ker_m(\lambda A)$ no longer leads to such a simple description as in the case when $\gcd(\det A, m) = 1$; see (21). To overcome this difficulty, one may be able to use the description of $\ker_m(\lambda A)$ given in section 4 of [11] which involved the invariant factors of A .

Conflict of Interests

The author has no competing interests.

Acknowledgments

The author is very grateful to the referees and the editors for their valuable comments and some helpful suggestions that improved the clarity and readability of the paper. The author is thankful to Jazan University (JU), Jazan, Saudi Arabia, for providing excellent research facilities to carry out this research.

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