Research Article

The Hankel Transform of q -Noncentral Bell Numbers

Cristina B. Corcino,^{1,2} Roberto B. Corcino,^{1,2} Jay M. Ontolan,^{1,2} Charrymae M. Perez-Fernandez,³ and Ednelyn R. Cantallopez³

¹Department of Mathematics, Cebu Normal University, 6000 Cebu, Philippines

²National Research Council of the Philippines-DOST, 1631 Taguig, Philippines

³Department of Mathematics, Mindanao State University, 9700 Marawi, Philippines

Correspondence should be addressed to Roberto B. Corcino; rcorcino@yahoo.com

Received 26 July 2015; Revised 19 September 2015; Accepted 30 September 2015

Academic Editor: Pentti Haukkanen

Copyright © 2015 Cristina B. Corcino et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We define two forms of q -analogue of noncentral Stirling numbers of the second kind and obtain some properties parallel to those of noncentral Stirling numbers. Certain combinatorial interpretation is given for the second form of the q -analogue in the context of 0-1 tableaux which, consequently, yields certain additive identity and some convolution-type formulas. Finally, a q -analogue of noncentral Bell numbers is defined and its Hankel transform is established.

1. Introduction

The *Hankel matrix* H_n of order n of a sequence $A = \{a_0, a_1, \dots, a_n\}$ is given by $H_n = (a_{i+j})_{0 \leq i, j \leq n}$. The *Hankel determinant* h_n of order n of A is the determinant of the corresponding Hankel matrix of order n . That is, $h_n = \det(H_n)$. The *Hankel transform* of the sequence A , denoted by $H(A)$, is the sequence $\{h_n\}$ of Hankel determinants of A . For instance, the Hankel transform of the sequence of Catalan numbers, $C = \{1/(n+1) \binom{2n}{n}\}_{n=1}^{\infty}$, is given by

$$H(C) = \{1, 1, 1, \dots\} \quad (1)$$

and the sequence of the sum of two consecutive Catalan numbers, $a_n = c_n + c_{n+1}$, with c_n being the n th Catalan numbers, has the Hankel transform

$$H(a_n) = \{F_{2n+1}\}_{n=0}^{\infty}, \quad (2)$$

where F_n is the n th Fibonacci number [1].

One remarkable property of Hankel transform is established by Layman [1], which states that the Hankel transform of an integer sequence is invariant under binomial and

inverse binomial transforms. We recall that a sequence $B = (b_n)$ is a binomial transform of $A = (a_n)$, denoted by $B(A)$, if

$$b_n = \sum_{k=0}^n \binom{n}{k} a_k \quad (3)$$

and $C = (c_n)$ is an inverse binomial transform of $A = (a_n)$, denoted by $C(A)$, if

$$c_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k. \quad (4)$$

In fact, Riordan [2, 3] established the so-called *binomial inversion formula* which is given by

$$f_n = \sum_{k=0}^n \binom{n}{k} g_k \iff g_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} f_k. \quad (5)$$

Hence, if A is an integer sequence, B is binomial transform of A , and C is the inverse binomial transform of A , then

$$\begin{aligned} H(B(A)) &= H(A), \\ H(C(A)) &= H(A). \end{aligned} \quad (6)$$

This property played an important role in proving that the Hankel transform of the sequence of ordinary Bell number $\{B_n\}$ [4] and that of r -Bell numbers $\{B_r(n)\}$ [5] are equal. Recently, in the paper by R. B. Corcino and C. B. Corcino [6], this property has also been used in proving that the Hankel transform of the sequence of generalized Bell numbers $\{G_{n,r,\beta}\}$ is given by

$$H(G_{n,r,\beta}) = \prod_{j=0}^n \beta^j j!, \quad (7)$$

where $G_{n,r,\beta}$ is the sum of (r, β) -Stirling numbers $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r,\beta}$

$$G_{n,r,\beta} = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}_{r,\beta} \quad (8)$$

(see [7, 8]).

In that same paper [6], an open problem is disclosed which is to determine the Hankel transform of a q -analogue of generalized Bell numbers. This problem motivates the present authors to define q -analogues of noncentral Stirling and noncentral Bell numbers and, eventually, establish the Hankel transform of a q -analogue of noncentral Bell numbers. Results in this paper may be useful in proving the said open problem.

2. A q -Analogue of $S_a(n, k)$: First Form

The noncentral Stirling numbers of the first and second kind [9], denoted by $s_a(n, k)$ and $S_a(n, k)$, respectively, are defined by means of the following inverse relations:

$$(t)_n = \sum_{k=0}^n \frac{1}{k!} \left[\frac{d^k}{dt^k} (t)_v \right]_{t=a} (t-a)^k, \quad (9)$$

$$(t-a)^n = \sum_{k=0}^n \frac{1}{k!} \left[\Delta^k (t-a)^n \right]_{t=0} (t)_k,$$

where a, t are any real numbers, n is a nonnegative integer, and

$$s_a(n, k) = \frac{1}{k!} \left[\frac{d^k}{dt^k} (t)_v \right]_{t=a} \quad (10)$$

$$S_a(n, k) = \frac{1}{k!} \left[\Delta^k (t-a)^n \right]_{t=0}. \quad (11)$$

These numbers are a certain generalization of the classical Stirling numbers of the first and second kind, respectively, which satisfy the following recurrence relations:

$$\begin{aligned} s_a(n+1, k) &= s_a(n, k-1) + (a-n) s_a(n, k), \\ S_a(n+1, k) &= S_a(n, k-1) + (k-a) S_a(n, k), \end{aligned} \quad (12)$$

with initial conditions

$$\begin{aligned} s_a(0, 0) &= 1, \\ s_a(n, 0) &= (a)_n, \\ s_a(0, k) &= 0, \\ n, k &\neq 0, \\ S_a(0, 0) &= 1, \\ S_a(n, 0) &= (-a)^n, \\ S_a(0, k) &= 0, \\ n, k &\neq 0. \end{aligned} \quad (13)$$

For a more detailed discussion of noncentral Stirling numbers, one may see [9]. Parallel to the ordinary Bell numbers, the noncentral Bell numbers can also be introduced as the sum of noncentral Stirling numbers of the second kind as follows:

$$B_{n,a} = \sum_{k=0}^n S_a(n, k). \quad (14)$$

These numbers are equivalent to r -Bell numbers with the following relation:

$$B_{n,a} = B_{-a}(n). \quad (15)$$

Several mathematicians developed a way of obtaining a generalization of some special numbers. One generalization is a q -analogue of these special numbers. A q -analogue is a mathematical expression parameterized by a quantity q that generalizes a known expression and reduces to the known expression in the limit, as $q \rightarrow 1$. For instance, a polynomial $a_k(q)$ is a q -analogue of the polynomial a_k if $\lim_{q \rightarrow 1} a_k(q) = a_k$. The q -analogues of n , $n!$, $(n)_k$, and $\binom{n}{k}$ are, respectively, given by

$$\begin{aligned} [n]_q &= 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}, \\ [n]_q! &= [n]_q [n-1]_q \cdots [2]_q [1]_q, \\ [n]_{k,q} &= [n]_q [n-1]_q \cdots [n-k+1]_q, \end{aligned} \quad (16)$$

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{[n]_{k,q}}{k!}.$$

The polynomials $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q$ are usually called the q -binomial coefficients. These polynomials possess several properties including the q -binomial inversion formula

$$\begin{aligned} f_n &= \sum_{k=0}^n \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q g_k, \\ \iff g_n &= \sum_{k=0}^n (-1)^{n-k} q^{\binom{n-k}{2}} \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]_q f_k, \end{aligned} \quad (17)$$

and certain generating function

$$\sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (1+x)(1+xq)(1+xq^2) \cdots (1+xq^{n-1}). \quad (18)$$

The q -binomial inversion formula is a q -analogue of the binomial inversion formula in (5).

A q -analogue of both kinds of Stirling numbers was first defined by Carlitz in [10], the second kind of which, known as q -Stirling numbers of the second kind, is defined in terms of the following recurrence relation:

$$S_q[n, k] = S_q[n-1, k-1] + [k]_q S_q[n-1, k] \quad (19)$$

in connection with a problem in Abelian groups, such that when $q \rightarrow 1$, this gives the triangular recurrence relation for the classical Stirling numbers of the second kind $S(n, k)$:

$$S(n, k) = S(n-1, k-1) + kS(n-1, k). \quad (20)$$

A different way of defining a q -analogue of Stirling numbers of the second kind has been adapted in [11] which is given as follows:

$$S_q[n, k] = q^{k-1} S_q[n-1, k-1] + [k]_q S_q[n-1, k]. \quad (21)$$

This type of q -analogue gives the Hankel transform of q -exponential polynomials and numbers which are a certain q -analogue of Bell polynomials and numbers. In the desire to establish the Hankel transform of q -analogue of generalized Bell numbers, the present authors are motivated to define a q -analogue of $S_a(n, k)$ parallel to that of (21) as follows.

Definition 1. For nonnegative integers n and k and real number a , a q -analogue $S_a[n, k]_q$ of $S_a(n, k)$ is defined by

$$S_a[n, k]_q = q^{(k-1)-a} S_a[n-1, k-1]_q + [k-a]_q S_a[n-1, k]_q, \quad (22)$$

where $S_a[0, 0]_q = 1$, $S_a[n, k]_q = 0$ for $n < k$ or $n, k < 0$ and $[k-a]_q = (1/q^a)([k]_q - [a]_q)$.

Remark 2. When $q \rightarrow 1$, the above definition will reduce to the recurrence relation of the noncentral Stirling numbers of the second kind established by Koutras [9] which is given by

$$S_a(n, k) = S_a(n-1, k-1) + (k-a) S_a(n-1, k). \quad (23)$$

Remark 3. It can easily be verified that

$$S_a[n, 0]_q = [-a]_q^n = -q^{-na} [a]_q^n. \quad (24)$$

By proper application of (22), we can easily obtain two other forms of recurrence relations and certain generating function.

Theorem 4. For nonnegative integers n and k and real number a , the q -analogue $S_a[n, k]_q$ satisfies the following vertical and horizontal recurrence relations:

$$S_a[n+1, k+1]_q = q^{k-a} \sum_{j=k}^n [k+1-a]_q^{n-j} S_a[j, k]_q, \quad (25)$$

$$S_a[n, k]_q = \sum_{j=0}^{n-k} (-1)^j q^{a-k-j} \frac{\{a\}_{k+j+1, q}}{\{a\}_{k+1, q}} S_a[n+1, k+j+1]_q, \quad (26)$$

respectively, where

$$\{a\}_{i, q} = \prod_{h=1}^{i-1} \frac{1}{q^{h-1}} ([h]_q - [a]_q) = \prod_{h=1}^{j-1} q^{a-h+1} [h-a]_q \quad (27)$$

with initial value $S_a[0, 0]_q = 1$. Moreover, a horizontal generating function for $S_a[n, k]_q$ is given by

$$\sum_{k=0}^n S_a[n, k]_q [t+a]_{k, q} = [t]_q^n. \quad (28)$$

Explicit formulas and generating functions of a given sequence of numbers or polynomials are useful tools in giving combinatorial interpretation of the numbers or polynomials. In the subsequent theorems, we establish the exponential and rational generating functions and two explicit formulas for $S_a[n, k]_q$. One of these explicit formulas is in symmetric function form which will be used to give combinatorial interpretation of $S_a[n, k]_q$ in the context of some A -tableaux.

A q -analogue of the difference operator, known as q -difference operator, was defined and thoroughly discussed in [12, 13]. More precisely, the q -difference operator of degree n , denoted by $\Delta_{q, h}^n$, is defined to be a mapping that assigns to every function f the function $\Delta_{q, h}^n f$ defined by the rule

$$\Delta_{q, h}^n f(x) = \left[\prod_{j=0}^{n-1} (E_h - q^j) \right] f(x), \quad n \geq 1, \quad (29)$$

where E_h is the shift operator defined by $E_h f(x) = f(x+h)$. When $h=1$, we use the notation

$$\Delta_{q, h}^n = \Delta_q^n. \quad (30)$$

As convention, define $\Delta_{q, h}^0 = 1$ (the identity map). The following is the explicit formula for the q -difference operator:

$$\Delta_{q, h}^k f(x) = \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q f(x+jh). \quad (31)$$

The new q -analogue of Newton's Interpolation Formula in [13] states that, for

$$f_q(x) = a_0 + a_1 [x-x_0]_q + \cdots + a_m [x-x_0]_q [x-x_1]_q \cdots [x-x_{m-1}]_q, \quad (32)$$

we have

$$\begin{aligned} f_q(x) &= f_q(x_0) + \frac{\Delta_{q^h, h} f_q(x_0) [x - x_0]_q}{[1]_{q^h}! [h]_q} \\ &+ \frac{\Delta_{q^h, h}^2 f_q(x_0) [x - x_0]_q [x - x_1]_q}{[2]_{q^h}! [h]_q^2} + \dots \\ &+ \frac{\Delta_{q^h, h}^m f_q(x_0) [x - x_0]_q [x - x_1]_q \cdots [x - x_{m-1}]_q}{[m]_{q^h}! [h]_q^m}, \end{aligned} \quad (33)$$

where $x_k = x_0 + kh$, $k = 1, 2, \dots$, such that when $x_0 = 0$ and $h = 1$, this can be simplified as

$$\begin{aligned} f_q(x) &= f_q(0) + \frac{\Delta_q f_q(0) [x]_q}{[1]_q!} \\ &+ \frac{\Delta_q^2 f_q(0) [x]_q [x - 1]_q}{[2]_q!} + \dots \\ &+ \frac{\Delta_q^m f_q(0) [x]_q [x - 1]_q \cdots [x - (m - 1)]_q}{[m]_q!}. \end{aligned} \quad (34)$$

Using (28) with $t = x$, we get

$$\sum_{k=0}^n S_a[n, k] [x + a]_{k, q} = [x]_q^n, \quad (35)$$

which can be expressed further as

$$\sum_{k=0}^n S_a[n, k] [x]_q [x - 1]_q \cdots [x - (k - 1)]_q = [x - a]_q^n. \quad (36)$$

Applying the above Newton Interpolation Formula and the identity in (31) with $h = 1$,

$$\Delta_q^k f(x) = \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q f(x + j), \quad (37)$$

we get

$$\begin{aligned} S_a[n, k] &= \frac{\Delta_q^k f_q(0)}{[k]_q!} \\ &= \frac{1}{[k]_q!} \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j - a]_q^n. \end{aligned} \quad (38)$$

Thus, we obtain the following explicit formula.

Theorem 5. The explicit formula for $S_a[n, k]_q$ is given by

$$S_a[n, k]_q = \frac{1}{[k]_q!} \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j - a]_q^n. \quad (39)$$

Remark 6. The above theorem reduces to

$$S_a(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (j - a)^n \quad (40)$$

when $q \rightarrow 1$ which is the explicit formula of the noncentral Stirling numbers of the second kind.

Remark 7. For brevity, (39) can be expressed as

$$S_a[n, k]_q = \frac{1}{[k]_q!} \left[\Delta_q^k [x - a]_q^n \right]_{x=0} \quad (41)$$

which reduces to (11) as $q \rightarrow 1$.

Theorem 8. For nonnegative integers n and k and real number a , the q -analogue $S_a[n, k]_q$ has a generating function

$$\sum_{n \geq 0} S_a[n, k]_q \frac{[t]_q^n}{[n]_q!} = \frac{1}{[k]_q!} \left[\Delta_q^k e_q([x - a]_q [t]_q) \right]_{x=0}. \quad (42)$$

Proof. Using the explicit formula in Theorem 5, we obtain

$$\begin{aligned} \sum_{n \geq 0} S_a[n, k]_q \frac{[t]_q^n}{[n]_q!} &= \sum_{n \geq 0} \frac{1}{[k]_q!} \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q [j - a]_q^n \frac{[t]_q^n}{[n]_q!} \\ &= \frac{1}{[k]_q!} \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q \sum_{n \geq 0} \frac{([j - a]_q [t]_q)^n}{[n]_q!} \\ &= \frac{1}{[k]_q!} \sum_{j=0}^k (-1)^{k-j} q^{\binom{k-j}{2}} \begin{bmatrix} k \\ j \end{bmatrix}_q e_q([j - a]_q [t]_q). \end{aligned} \quad (43)$$

Using (31), we prove the theorem. $\square$

Remark 9. When $q \rightarrow 1$, the above theorem becomes

$$\begin{aligned} \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} e^{(j-a)t} \\ = e^{-at} \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (e^t)^j = e^{-at} \frac{1}{k!} (e^t - 1)^k \end{aligned} \quad (44)$$

which is the exponential generating function of the noncentral Stirling numbers of the second kind.

The following theorem contains the rational generating function for $S_a[n, k]_q$.

Theorem 10. For nonnegative integers n and k and complex number a , the q -analogue $S_a[n, k]_q$ satisfies the rational generating function

$$\Psi_k(t) = \sum_{n \geq k} S_a[n, k]_q [t]_q^n = \frac{q^{\binom{k}{2} - ka} [t]_q^k}{\prod_{j=0}^{k-1} (1 - [j - a]_q [t]_q)}. \quad (45)$$

Proof. For $k = 0$, we have

$$\begin{aligned} \Psi_0(t) &= \sum_{n \geq 0} S_a[n, 0]_q [t]_q^n = \sum_{n \geq 0} [-a]_q^n [t]_q^n \\ &= \frac{1}{(1 - [-a]_q [t]_q)}. \end{aligned} \quad (46)$$

With $k > 0$ and using Definition 1, we obtain

$$\begin{aligned}\Psi_k(t) &= \sum_{n \geq k} S_a[n, k]_q [t]_q^n = q^{(k-1)-a} [t]_q \\ &\cdot \sum_{n \geq k} S_a[n-1, k-1]_q [t]_q^{n-1} + [k-a]_q [t]_q \\ &\cdot \sum_{n \geq k} S_a[n-1, k]_q [t]_q^{n-1} = q^{(k-1)-a} [t]_q \Psi_{k-1}(t) \\ &+ [k-a]_q [t]_q \Psi_k(t).\end{aligned}\quad (47)$$

Hence,

$$\begin{aligned}\Psi_k(t) &= \frac{q^{(k-1)-a} [t]_q}{1 - [k-a]_q [t]_q} \Psi_{k-1}(t), \\ \Psi_k(t) &= \frac{q^{(k-1)-a} [t]_q}{1 - [k-a]_q [t]_q} \frac{q^{(k-2)-a} [t]_q}{1 - [k-1-a]_q [t]_q} \Psi_{k-2}(t) \\ &\vdots \\ \Psi_k(t) &= \frac{q^{\binom{k}{2}-ka} [t]_q^k}{\prod_{j=0}^{k-1} (1 - [j-a]_q [t]_q)}.\end{aligned}\quad (48)$$

□

The rational generating function in Theorem 10 can then be expressed as

$$\begin{aligned}\sum_{n \geq k} S_a[n, k]_q [t]_q^n &= q^{\binom{k}{2}-ka} [t]_q^k \prod_{j=0}^{k-1} \frac{1}{(1 - [j-a]_q [t]_q)} \\ &= q^{\binom{k}{2}-ka} [t]_q^k \prod_{j=0}^{k-1} \sum_{n \geq 0} [j-a]_q^n [t]_q^n = q^{\binom{k}{2}-ka} [t]_q^k \\ &\cdot \sum_{n \geq k} \sum_{s_1+s_2+\dots+s_k=n-k} \prod_{j=0}^{k-1} [j-a]_q^{s_j} [t]_q^{s_j} = q^{\binom{k}{2}-ka} [t]_q^k \\ &\cdot \sum_{n \geq k} \left\{ \sum_{s_1+s_2+\dots+s_k=n-k} \prod_{j=0}^{k-1} [j-a]_q^{s_j} \right\} [t]_q^{n-k} \\ &= q^{\binom{k}{2}-ka} \sum_{n \geq k} \left\{ \sum_{s_1+s_2+\dots+s_k=n-k} \prod_{j=0}^{k-1} [j-a]_q^{s_j} \right\} [t]_q^n.\end{aligned}\quad (49)$$

Hence,

$$S_a[n, k]_q = q^{\binom{k}{2}-ka} \sum_{s_1+s_2+\dots+s_k=n-k} \prod_{j=0}^{k-1} [j-a]_q^{s_j}. \quad (50)$$

This sum may be written further as follows.

Theorem 11. For nonnegative integers n and k and complex number a , the explicit formula for $S_a[n, k]_q$ in homogeneous symmetric function form is given by

$$S_a[n, k]_q = \sum_{0 \leq j_1 \leq j_2 \leq \dots \leq j_{n-k} \leq k} q^{\binom{k}{2}-ka} \prod_{i=1}^{n-k} [j_i - a]_q. \quad (51)$$

3. A q -Analogue of $S_a(n, k)$: Second Form

We now define another q -analogue of the noncentral Stirling numbers, denoted by $S_a^*[n, k]_q$, as follows:

$$S_a^*[n, k]_q := q^{ka - \binom{k}{2}} S_a[n, k]_q. \quad (52)$$

Hence,

$$S_a^*[n, k]_q = \sum_{0 \leq j_1 \leq j_2 \leq \dots \leq j_{n-k} \leq k} \prod_{i=1}^{n-k} [j_i - a]_q. \quad (53)$$

All other properties parallel to those of $S_a[n, k]_q$ can easily be established by imbedding the factor $q^{ka - \binom{k}{2}}$ in the derivations or multiplying this factor directly by both sides of the equation that yields the formula/identity for $S_a[n, k]_q$.

The following definition contains the concept of A -tableau which is useful in describing $S_a^*[n, k]_q$, combinatorially.

Definition 12 (see [14]). An A -tableau is a list ϕ of columns c of Ferrer's diagram of a partition λ (by decreasing order of length) such that the lengths $|c|$ are part of the sequence $A = (a_i)_{i \geq 0}$, a strictly increasing sequence of nonnegative integers.

Let ω be a function from the set of nonnegative integers N to a ring K . Suppose Φ is an A -tableau with r columns of lengths $|c| \leq h$. We use $T_a^A(h, r)$ to denote the set of such A -tableaux. Then, we set

$$\omega_A(\Phi) = \prod_{c \in \Phi} \omega(|c|). \quad (54)$$

Note that Φ might contain a finite number of columns whose lengths are zero since $0 \in A = \{0, 1, 2, \dots, k\}$ and if $\omega(0) \neq 0$.

From this point onward, whenever an A -tableau is mentioned, it is always associated with the sequence $A = \{0, 1, 2, \dots, k\}$.

We are now ready to mention the following theorem.

Theorem 13. Let $\omega : N \rightarrow K$ denote a function from N to a ring K (column weights according to length) which is defined by $\omega(|c|) = [|c| - a]_q$, where a is a complex number and $|c|$ is the length of column c of an A -tableau in $T_a^A(k, n-k)$. Then,

$$S_a^*[n, k]_q = \sum_{\phi \in T_a^A(k, n-k)} \prod_{c \in \phi} \omega(|c|). \quad (55)$$

Proof. This can easily be proved using Definition 12 and (53). $\square$

3.1. Combinatorics of A -Tableaux. Suppose that, for some numbers a_1 and a_2 , we have $a = a_1 + a_2$. Then, (53) yields

$$S_a^*[n, k]_q = \sum_{0 \leq j_1 \leq j_2 \leq \dots \leq j_{n-k} \leq k} \prod_{i=1}^{n-k} [(j_i - a_1) - a_2]_q. \quad (56)$$

That is, for any $\phi \in T_a^A(k, n-k)$,

$$\omega_A(\phi) = \prod_{c \in \phi} [(j_i - a_1) - a_2]_q, \quad (57)$$

where $|c| \in \{0, 1, 2, \dots, k\}$. Note that the weight of each column of ϕ can be considered as a finite sum with additive constant a_2 ; that is, for each $c \in \phi$, we can write

$$\omega(|c|) = \frac{1}{q^{a_2}} (\omega^*(|c|) - [a_2]_q), \quad (58)$$

where $\omega^*(|c|) = [|c| - a_1]_q$. The following theorem determines how an additive constant affects the recurrence formula for $S_a[n, k]_q$.

From Theorem 13,

$$S_a^*[n, k]_q = \sum_{\phi \in T_a^A(k, n-k)} \omega_A(\phi) = \sum_{\phi \in T_a^A(k, n-k)} \prod_{c \in \phi} \omega(|c|), \quad (59)$$

where

$$\begin{aligned} \omega_A(\phi) &= \prod_{c \in \phi} [|c| - a]_q, \quad \text{where } |c| \in \{0, 1, \dots, k\} \\ &= \prod_{i=1}^{n-k} [j_i - a]_q, \quad \text{where } j_i \in \{0, 1, \dots, k\}. \end{aligned} \quad (60)$$

If $a = a_1 + a_2$ for some a_1 and a_2 , then, by (58),

$$\begin{aligned} \omega_A(\phi) &= \prod_{i=1}^{n-k} \frac{1}{q^{a_2}} (\omega^*(j_i) - [a_2]_q), \\ &\quad \text{where } \omega^*(j_i) = [j_i - a_1]_q \\ &= q^{-(n-k)a_2} (\omega^*(j_1) - [a_2]_q) \\ &\quad \cdot (\omega^*(j_2) - [a_2]_q) \cdots (\omega^*(j_{n-k}) - [a_2]_q) \\ &= q^{-(n-k)a_2} \sum_{l=0}^{n-k} (-[a_2]_q)^{n-k-l} \\ &\quad \cdot \sum_{j_1 \leq q_1 \leq q_2 \leq \dots \leq q_l \leq j_{n-k}} \prod_{i=1}^l \omega^*(q_i). \end{aligned} \quad (61)$$

Suppose B_ϕ is the set of all A -tableaux corresponding to ϕ such that, for each $\psi \in B_\phi$, either

$$\begin{aligned} &\psi \text{ has no column whose weight is } -[a_2]_q, \text{ or} \\ &\psi \text{ has one column whose weight is } -[a_2]_q, \text{ or} \\ &\psi \text{ has two columns whose weights are } -[a_2]_q, \text{ or} \\ &\vdots \\ &\psi \text{ has } (n-k) \text{ columns whose weights are} \\ &\quad -[a_2]_q. \end{aligned} \quad (62)$$

Then, we may write

$$\omega_A(\phi) = \sum_{\psi \in B_\phi} \omega_A(\psi). \quad (63)$$

Now, if r columns in ψ have weights other than $-[a_2]_q$, then

$$\begin{aligned} \omega_A(\psi) &= \prod_{c \in \psi} \omega^*(|c|) \\ &= q^{-(n-k)a_2} (-[a_2]_q)^{n-k-r} \prod_{i=1}^r \omega^*(q_i), \end{aligned} \quad (64)$$

where $q_1, q_2, \dots, q_r \in \{j_1, j_2, \dots, j_{n-k}\}$. Note that, for each r , there correspond

$$\binom{n-k}{r} \quad (65)$$

tableaux with r columns having weights $\omega^*(j_i) = [j_i - a_1]_q$. It can easily be verified that

$$|T_a^A(k, n-k)| = \binom{(n-k) + k}{n-k} = \binom{n}{n-k} = \binom{n}{k}. \quad (66)$$

Thus, $\forall \phi \in T_a^A(k, n-k)$, B_ϕ contains a total of

$$\binom{n}{k} \binom{n-k}{r} \quad (67)$$

tableaux with r columns of weights $\omega^*(j_i)$. However, only

$$\binom{r+k}{r} \quad (68)$$

tableaux with r columns in B_ϕ are distinct. Hence, every distinct tableau ψ with r columns of weights other than $-[a_2]_q$ appears

$$\frac{\binom{n}{k} \binom{n-k}{r}}{\binom{r+k}{r}} = \binom{n}{r+k} \quad (69)$$

times in the collection. Thus,

$$\begin{aligned} & \sum_{\phi \in T_a^A(k, n-k)} \omega_A(\phi) \\ &= \sum_{r=0}^{n-k} \binom{n}{r+k} q^{-(n-k)a_2} (-[a_2]_q)^{n-k-r} \\ & \cdot \sum_{\varphi \in \bar{B}_r} \prod_{c \in \varphi} \omega^*(|c|), \end{aligned} \quad (70)$$

where $\bar{B}_r$ denotes the set of all tableaux φ having r columns of weights $\omega^*(j_i) = [j_i - a_1]_q$. Reindexing the double sum, we get

$$\begin{aligned} & \sum_{\phi \in T_a^A(k, n-k)} \omega_A(\phi) \\ &= \sum_{j=k}^n \binom{n}{j} q^{-(n-k)a_2} (-[a_2]_q)^{n-j} \sum_{\varphi \in \bar{B}_{j-k}} \prod_{c \in \varphi} \omega^*(|c|), \end{aligned} \quad (71)$$

where $\bar{B}_{j-k}$ is the set of all tableaux φ with $j-k$ columns of weights $\omega^*(j_i) = [j_i - a_1]_q$ for each $i = 1, 2, \dots, j-k$. Clearly $\bar{B}_{j-k} = T_{a_1}^A(k, j-k)$. Hence,

$$\begin{aligned} & \sum_{\phi \in T_a^A(k, n-k)} \omega_A(\phi) \\ &= \sum_{j=k}^n \binom{n}{j} q^{-(n-k)a_2} (-[a_2]_q)^{n-j} \sum_{\varphi \in T_{a_1}^A(k, j-k)} \omega_A(\varphi). \end{aligned} \quad (72)$$

Applying Theorem 13, we obtain the following theorem.

Theorem 14. The q -analogue $S_a^*[n, k]_q$ satisfies the following identity:

$$\begin{aligned} & S_a^*[n, k]_q \\ &= \sum_{j=k}^n (-1)^{n-j} \binom{n}{j} q^{-(n-k)a_2} [a_2]_q^{n-j} S_{a_1}^*[j, k]_q, \end{aligned} \quad (73)$$

where $a = a_1 + a_2$ for some numbers a_1 and a_2 .

Suppose

ϕ_1 is a tableau with $k-m$ columns whose lengths are in the set $\{0, 1, \dots, m\}$,

ϕ_2 is a tableau with $n-k-j$ columns whose lengths are in the set $\{m+1, m+2, \dots, m+j+1\}$.

Then,

$$\begin{aligned} & \phi_1 \in T^{A_1}(m, k-m), \\ & \phi_2 \in T^{A_2}(j, n-k-j), \end{aligned} \quad (74)$$

where $A_1 = \{0, 1, \dots, m\}$ and $A_2 = \{m+1, m+2, \dots, m+j+1\}$. Notice that, by joining the columns of ϕ_1 and ϕ_2 , we obtain

an A -tableau ϕ with $n-m-j$ columns whose lengths are in the set $A = A_1 \cup A_2 = \{0, 1, \dots, m+j+1\}$. That is, $\phi \in T^A(m+j+1, n-m-j)$. Then,

$$\begin{aligned} & \sum_{\phi \in T^A(m+j+1, n-m-j)} \omega_A(\phi) \\ &= \sum_{k=m}^{n-j} \left\{ \sum_{\phi_1 \in T^{A_1}(m, k-m)} \omega_{A_1}(\phi_1) \right\} \\ & \cdot \left\{ \sum_{\phi_2 \in T^{A_2}(j, n-k-j)} \omega_{A_2}(\phi_2) \right\}. \end{aligned} \quad (75)$$

Note that

$$\begin{aligned} & \sum_{\phi_2 \in T^{A_2}(j, n-k-j)} \omega_{A_2}(\phi_2) = \sum_{\phi_2 \in T^{A_2}(j, n-k-j)} \prod_{c \in \phi_2} [|c| - a]_q \\ &= \sum_{m+1 \leq g_1 \leq \dots \leq g_{n-k-j} \leq m+j+1} \prod_{i=1}^{n-k-j} [g_i - a]_q \\ &= \sum_{0 \leq g_1 \leq \dots \leq g_{n-k-j} \leq j} \prod_{i=1}^{n-k-j} [g_i + (m+1) - a]_q. \end{aligned} \quad (76)$$

Thus,

$$\begin{aligned} & \sum_{0 \leq g_1 \leq \dots \leq g_{n-m-j} \leq m+j+1} \prod_{i=1}^{n-m-j} [g_i - a]_q \\ &= \sum_{k=m}^{n-j} \left\{ \sum_{0 \leq g_1 \leq \dots \leq g_{k-m} \leq m} \prod_{i=1}^{k-m} [g_i - a]_q \right\} \\ & \cdot \left\{ \sum_{0 \leq g_1 \leq \dots \leq g_{n-k-j} \leq j} \prod_{i=1}^{n-k-j} [g_i + m+1 - a]_q \right\}. \end{aligned} \quad (77)$$

By (53), we obtain the following theorem.

Theorem 15. The q -analogue $S_a^*[n, k]_q$ satisfies the following convolution-type identity:

$$\begin{aligned} & S_a^*[n+1, m+j+1]_q \\ &= \sum_{k=0}^n S_a^*[k, m]_q S_{a-m-1}^*[n-k, j]_q. \end{aligned} \quad (78)$$

The next theorem provides another form of convolution-type identity.

Theorem 16. The q -analogue $S_a^*[n, k]_q$ satisfies the following second form of convolution formula:

$$S_a^*[m+j, n]_q = \sum_{k=m}^{n-j} S_a^*[m, k]_q S_{a-k}^*[j, n-k]_q. \quad (79)$$

Proof. Let

ϕ_1 be a tableau with $m - k$ columns whose lengths are in $A_1 = \{0, 1, \dots, k\}$,

ϕ_2 be a tableau with $j - n + k$ columns whose lengths are in $A_2 = \{k, k + 1, \dots, n\}$.

Then, $\phi_1 \in T^{A_1}(k, m - k)$ and $\phi_2 \in T^{A_2}(n - k, j - n + k)$. Using the same argument above, we can easily obtain the convolution formula. $\square$

4. q -Noncentral Bell Number and Its Hankel Transform

In this section, we define a q -analogue of the noncentral Bell numbers and obtain some combinatorial properties that will be used to establish its Hankel transform.

Definition 17. A q -analogue of the noncentral Bell numbers, denoted by $\tilde{B}_{n,a}^q$, is defined by

$$\tilde{B}_{n,a}^q = \sum_{k=0}^n \tilde{S}_a[n, k], \quad (80)$$

where $\tilde{S}_a[n, k] = q^{-ka} S_a^*[n, k]_q = q^{-\binom{k}{2}} S_a[n, k]_q$. For brevity, we use the term q -noncentral Bell numbers for $\tilde{B}_{n,a}^q$.

We observe that, multiplying both of (22) by $q^{ka - \binom{k}{2}}$, we obtain

$$\begin{aligned} q^{ka - \binom{k}{2}} S_a[n, k]_q &= q^{(k-1)a - \binom{k-1}{2}} S_a[n-1, k-1]_q \\ &\quad + [k-a]_q q^{ka - \binom{k}{2}} S_a[n-1, k]_q, \\ S_a^*[n, k]_q &= S_a^*[n-1, k-1]_q \\ &\quad + [k-a]_q S_a^*[n-1, k]_q, \\ \tilde{S}_a[n, k]_q &= q^{-a} \tilde{S}_a[n-1, k-1]_q \\ &\quad + [k-a]_q \tilde{S}_a[n-1, k]_q. \end{aligned} \quad (81)$$

Note that when $a = 0$, we get

$$\tilde{S}_0[n, k]_q = \tilde{S}_0[n-1, k-1]_q + [k]_q \tilde{S}_0[n-1, k]_q, \quad (82)$$

which coincides with the recurrence relation of the modified q -Stirling numbers of the second kind $\tilde{S}[n, k]$ in [11] and that of the q -Stirling numbers of the second kind in [15]. In fact,

$$\begin{aligned} \tilde{S}_0[n, k]_q &= q^{-\binom{k}{2}} S_0[n, k]_q = q^{-\binom{k}{2}} S[n, k]_q \\ &= \tilde{S}[n, k]. \end{aligned} \quad (83)$$

That is, $\tilde{S}_a[n, k]_q$ is a proper q -analogue of $S_a(n, k)$ that generalizes $\tilde{S}[n, k]$.

Note that our objective is to define a q -analogue of $B_{n,a}$ such that when $a = 0$, it reduces to the q -exponential number

$$\tilde{e}_n = \tilde{e}_n[1] = \sum_{k=0}^n \tilde{S}[n, k] = \varphi_n(1), \quad (84)$$

in [11, 15], whose Hankel transform is known to be

$$H(\tilde{e}_n) = H(\varphi_n(1)) = q^{\binom{n+1}{3}} \prod_{j=0}^n [j]! \quad (85)$$

(see [11, 15, 16]). Hence, it is appropriate to define a q -analogue of the noncentral Bell numbers through $\tilde{S}_a[n, k]_q$.

Clearly, when $q \rightarrow 1$, $\tilde{B}_{n,a}^q \rightarrow B_{n,a}$, the noncentral Bell numbers. The following theorem contains certain recurrence relations for $\tilde{B}_{n,a}^q$.

Theorem 18. The q -noncentral Bell numbers satisfy the following relation:

$$q^n \tilde{B}_{n,a}^q = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} \tilde{B}_{j,a-1}^q. \quad (86)$$

Proof. By making use of Theorem 14, with $a_1 = a - 1$ and $a_2 = 1$, we have

$$\begin{aligned} q^{-ka} S_a^*[n, k]_q &= q^{-ka} \sum_{j=k}^n (-1)^{n-j} \binom{n}{j} q^{-(n-k)} S_{a-1}^*[j, k]_q, \end{aligned} \quad (87)$$

$$\begin{aligned} \sum_{k=0}^n q^{-ka} S_a^*[n, k]_q &= \sum_{k=0}^n \sum_{j=k}^n (-1)^{n-j} \binom{n}{j} q^{-n} q^{-k(a-1)} S_{a-1}^*[j, k]_q \\ &= \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} q^{-n} \left\{ \sum_{k=0}^j q^{-k(a-1)} S_{a-1}^*[j, k]_q \right\}. \end{aligned} \quad (88)$$

Then,

$$\tilde{B}_{n,a}^q = \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} q^{-n} \tilde{B}_{j,a-1}^q, \quad (89)$$

which is exactly the desired relation. $\square$

The following corollary is a direct consequence of Theorem 18 which can be proved using the inversion formula by Riordan [2, 3].

Corollary 19. The q -noncentral Bell numbers satisfy the following relation:

$$\tilde{B}_{n,a-1}^q = \sum_{j=0}^n \binom{n}{j} q^j \tilde{B}_{j,a}^q. \quad (90)$$

To establish the Hankel transform of $\tilde{B}_{n,a}^q$, we need the concept of rising k -binomial transform by Spivey and Steil [17] as well as its property in relation to Hankel transform.

Definition 20 (Spivey-Steil [17]). The rising k -binomial transform R of a sequence $A = \{a_n\}$ is the sequence $R(A; k) = \{r_n\}$, where r_n is given by

$$r_n = \sum_{j=0}^n \binom{n}{j} k^j a_j, \quad k \neq 0. \quad (91)$$

We use $R(A, k)$ to denote the set of rising k -binomial transforms of A . That is, $R(A, k) = \{r_n\}$. Then, we have the following theorem by Spivey and Steil.

Theorem 21 (Spivey-Steil [17]). Given a sequence $A = \{a_0, a_1, \dots\}$, let $H(A) = \{h_n\}$. Then,

$$H(R(A, 0)) = \{a_0, 0, 0, \dots\}. \quad (92)$$

If $k \neq 0$,

$$H(R(A, k)) = \{k^{n(n+1)} h_n\}. \quad (93)$$

Now, we are ready to state the main result of the paper.

Theorem 22. For $a = 0, 1, 2, \dots$, the Hankel transform of the sequence of q -noncentral Bell numbers $\{\tilde{B}_{n,a}^q\}$ is given by

$$H(\tilde{B}_{n,a}^q) = q^{-an(n+1) + \binom{n+1}{3}} \prod_{j=0}^n [j]!. \quad (94)$$

Proof. We begin with the relation that $\tilde{B}_{n,0}^q = \tilde{e}_n$. Hence,

$$H(\tilde{B}_{n,0}^q) = H(\tilde{e}_n) = q^{\binom{n+1}{3}} \prod_{j=0}^n [j]!. \quad (95)$$

From Corollary 19, we see that $\tilde{B}_{n,a-1}^q$ is the binomial transform of $q^n \tilde{B}_{n,a}^q$. This means that

$$B(q^n \tilde{B}_{n,a}^q) = \tilde{B}_{n,a-1}^q. \quad (96)$$

Hence, by Layman's Theorem [1],

$$H(B(q^n \tilde{B}_{n,a}^q)) = H(q^n \tilde{B}_{n,a}^q). \quad (97)$$

That is,

$$H(\tilde{B}_{n,a-1}^q) = H(q^n \tilde{B}_{n,a}^q). \quad (98)$$

Now, Corollary 19 can also be stated as $\tilde{B}_{n,a-1}^q$ is the rising q -binomial transform of $\tilde{B}_{n,a}^q$. Using Spivey-Steil Theorem, with $A = \{\tilde{B}_{n,a}^q\}$, $h_n = H(\tilde{B}_{n,a}^q)$, and $r_n = \tilde{B}_{n,a-1}^q$, we have

$$H(\tilde{B}_{n,a-1}^q) = q^{n(n+1)} H(\tilde{B}_{n,a}^q). \quad (99)$$

We observe that when $a = 1$ and using (95), we have

$$H(\tilde{B}_{n,1}^q) = q^{-n(n+1)} H(\tilde{B}_{n,0}^q) = q^{-n(n+1) + \binom{n+1}{3}} \prod_{j=0}^n [j]!. \quad (100)$$

Also, when $a = 2$,

$$\begin{aligned} H(\tilde{B}_{n,2}^q) &= q^{-n(n+1)} H(\tilde{B}_{n,1}^q) \\ &= q^{-n(n+1)} q^{-n(n+1) + \binom{n+1}{3}} \prod_{j=0}^n [j]! \\ &= q^{-2n(n+1) + \binom{n+1}{3}} \prod_{j=0}^n [j]!. \end{aligned} \quad (101)$$

Continuing this argument, we obtain

$$H(\tilde{B}_{n,a}^q) = q^{-an(n+1) + \binom{n+1}{3}} \prod_{j=0}^n [j]!. \quad (102)$$

□

Remark 23. When $a = 0$, the Hankel transform in (94) reduces to the Hankel transform of q -exponential numbers in (85) and that in [15]. By taking $q \rightarrow 1$, we further get

$$H(B_n) = \prod_{j=0}^n j!, \quad (103)$$

the Hankel transform of the ordinary Bell numbers. The same Hankel transform is obtained when $a = -r$ and taking $q \rightarrow 1$, which is exactly the same result established by Mezö [5] for the Hankel transform of r -Bell numbers $B_r(n)$. That is,

$$H(B_r(n)) = H(B_{n,-r}) = \prod_{j=0}^n j!. \quad (104)$$

Remark 24. We can define another form of q -analogue of the noncentral Bell numbers $B_{n,a}$ as follows:

$$B_{n,a}^q = \sum_{k=0}^n S_a[n, k]_q = \sum_{k=0}^n q^{\binom{k}{2} - ra} S_a^*[n, k]_q. \quad (105)$$

In the paper by Cigler [18], certain generalization of q -Stirling numbers of the second kind, denoted by $S(n, k, r)$, was defined by means of the following recurrence relation:

$$S(n, k, r) = S(n-1, k-1, r) + [k+r]_q S(n-1, k, r) \quad (106)$$

which is equivalent to $S_a^*[n, k]_q$. That is,

$$S_a^*[n, k]_q = S(n, k, -a). \quad (107)$$

Moreover, the first form of generalized q -exponential polynomials in [18] was defined by

$$\Phi_n(x, r) = \sum_{k=0}^n S(n, k, r) q^{\binom{k}{2} + kr} x^k \quad (108)$$

whose Hankel transform is given by

$$\begin{aligned} H(\Phi_n(x, r)) &= q^{2\binom{n+1}{3} + 2r\binom{n+1}{2}} x^{\binom{n+1}{2}} \prod_{k=0}^n [k]_q! ((1-q)x; q)_k, \end{aligned} \quad (109)$$

where $(x; q)_k = \prod_{j=0}^{k-1} (1 - q^j x)$. With $x = 1$ and $r = -a$, we obtain

$$\Phi_n(1, -a) = B_{n,a}^q. \quad (110)$$

Then, by (109),

$$H(B_{n,a}^q) = q^{2\binom{n+1}{3} - 2a\binom{n+1}{2}} \prod_{k=0}^n [k]_q! (1 - q; q)_k. \quad (111)$$

Remark 25. We can also define a q -analogue of the noncentral Bell numbers $B_{n,a}^*$ as follows:

$$\widehat{B}_{n,a}^q = \sum_{k=0}^n S_a^*[n, k]_q. \quad (112)$$

Another form of generalized q -exponential polynomials in [18] was defined by

$$\varphi_n(x, r) = \sum_{k=0}^n S(n, k, r) x^k \quad (113)$$

whose Hankel transform is given by

$$H(\varphi_n(x, r)) = q^{\binom{n+1}{3}} (q^r x)^{\binom{n+1}{2}} \prod_{k=0}^n [k]_q!. \quad (114)$$

With $x = 1$ and $r = -a$, we obtain

$$\varphi_n(1, -a) = \widehat{B}_{n,a}^q. \quad (115)$$

Then, by (114),

$$H(\widehat{B}_{n,a}^q) = q^{\binom{n+1}{3} - a\binom{n+1}{2}} \prod_{k=0}^n [k]_q!. \quad (116)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This research has been partially funded by the research grant of Professor Keonhee Lee of Chungnam National University, Daejeon, South Korea.

References

- [1] J. W. Layman, "The Hankel transform and some of its properties," *Journal of Integer Sequences*, vol. 4, no. 1, 2001.
- [2] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, The Netherlands, 1974.
- [3] J. Riordan, *Combinatorial Identities*, Wiley, New York, NY, USA, 1968.
- [4] M. Aigner, "A characterization of the bell numbers," *Discrete Mathematics*, vol. 205, no. 1-3, pp. 207-210, 1999.
- [5] I. Mezö, "The r -Bell numbers," *Journal of Integer Sequences*, vol. 14, article 11.1.1, 2011.

- [6] R. B. Corcino and C. B. Corcino, "The hankel transform of generalized bell numbers and its q -analogue," *Utilitas Mathematica*, vol. 89, pp. 297-309, 2012.
- [7] R. B. Corcino, "The $(r; \beta)$ -Stirling numbers," *The Mindanao Forum*, vol. 14, no. 2, 1999.
- [8] R. B. Corcino, C. B. Corcino, and R. Aldema, "Asymptotic normality of the (r, β) -stirling numbers," *Ars Combinatoria*, vol. 81, pp. 81-96, 2006.
- [9] M. Koutras, "Non-central stirling numbers and some applications," *Discrete Mathematics*, vol. 42, no. 1, pp. 73-89, 1982.
- [10] L. Carlitz, " q -Bernoulli numbers and polynomials," *Duke Mathematical Journal*, vol. 15, no. 4, pp. 987-1000, 1948.
- [11] R. Ehrenborg, "Determinants involving q -Stirling numbers," *Advances in Applied Mathematics*, vol. 31, no. 4, pp. 630-642, 2003.
- [12] K. Conrad, "A q -analogue of Mahler expansions, I," *Advances in Mathematics*, vol. 153, no. 2, pp. 185-230, 2000.
- [13] M. S. Kim and J. W. Son, "A note on q -difference operators," *Communications of the Korean Mathematical Society*, vol. 17, no. 3, pp. 423-430, 2002.
- [14] A. De Médicis and P. Leroux, "Generalized Stirling numbers, convolution formulae and $p; q$ -analogues," *Canadian Journal of Mathematics*, vol. 47, no. 3, pp. 474-499, 1995.
- [15] J. Cigler, "Hankel determinants of q -exponential polynomials," <http://arxiv.org/abs/0901.4698>.
- [16] J. Cigler, "Eine Charakterisierung der q -Exponentialpolynome," *Sitzungsberichte der Österreichischen Akademie der Wissenschaften*, vol. 208, pp. 143-157, 1999.
- [17] M. Z. Spivey and L. L. Steil, "The k -binomial transform and the Hankel transform," *Journal of Integer Sequences*, vol. 9, article 06.1.1, 2006.
- [18] J. Cigler, "Hankel determinants of generalized q -exponential polynomials," <http://arxiv.org/abs/0909.5581>.